

Towards A Dichotomy for the List Switch Homomorphism Problem for Signed Graphs

HYOBEEN KIM AND MARK SIGGERS*

Kyungpook National University Mathematics Department, Daegu, South Korea, 41566
e-mail: hbkim1029@knu.ac.kr and mhsiggers@knu.ac.kr

ABSTRACT. We make advances towards a structural characterisation of the signed graphs H for which the list switch H -colouring problem $\text{LIST-S-HOM}(H)$ can be solved in polynomial time. We conjecture two different characterisations, the second refining the first, in the case that the graph H can be switched to a graph in which every negative edge is also positive. Using a recent proof of the first characterisations for reflexive signed graphs, by Bok et. al., we prove the second characterisation for reflexive signed graphs. We also provide several tools for reducing the problem to the bipartite case, and prove a full complexity dichotomy for a related problem.

1. Introduction

The CSP-dichotomy of Bulatov [6] and Zhuk [17] tells us that the constraint satisfaction problem $\text{CSP}(\mathcal{H})$ for a core relational structure \mathcal{H} is in P if \mathcal{H} admits a WNU-polymorphism, and is otherwise in NPC . It is difficult, however, to decide if \mathcal{H} admits a WNU-polymorphism. So more tractable dichotomies, characterisations like the well known H -colouring dichotomy of [13] which says that the homomorphism problem $\text{HOM}(H)$ for a simple graph H is in P if and only if H is bipartite, are still sought. One such dichotomy arose recently in relation to signed graphs [3, 4], and in this paper we work towards a generalisation of this. The definition of WNU-polymorphisms, as well as the definitions of many other now standard algebraic terms introduced in this section without definition, can be found in Section 2.

Henceforth, *graphs* are undirected graphs in which loops — edges of the form $\{v, v\}$ — are allowed. If all vertices have loops, then the graph is *reflexive*, and if no vertices

* Corresponding Author.

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have loops, then it is *irreflexive*. We denote an edge $\{u, v\}$ of a graph simply as uv , and write $u \sim v$ to mean that uv is an edge.

A *signed graph* is a graph G together with an assignment of a sign $+$ or $-$ to each edge. Introduced by Harary in [11] in 1955, there are numerous results about signed graphs. It will be convenient to view a signed graph G as a *br-graph* (for blue-red): a pair of graphs — G^B , whose edges are called *blue edges*, and G^R , whose edges are *red edges* — on the same vertex set. Signed graphs, as defined by Harary, were irreflexive and simple. But as is done in [3, 4] we allow loops, and allow $G^B \cap G^R$ to be non-empty. An edge in $G^B \cap G^R$ is technically two edges in G , one of each colour, but it is convenient to consider it a single edge with both colours. With this view in mind, such an edge is called *bicoloured*, and an edge is *unicoloured* if it is not bicoloured. If all edges of a br-graph G are unicoloured red edges, then G is called *red*, and if all edges are bicoloured, then G is called *bicoloured*. A br-graph G is irreflexive, or bipartite, or connected or such, if the *underlying graph* $G^B \cup G^R$ is. We write $u \sim v$ to mean that uv is in $G^B \cup G^R$. Inherent to the concept of a signed graph is the operation of *switching*, introduced by Zaslavsky [16], in which, for a vertex v , one switches, with respect to G^B and G^R , the set of edges incident to v . As a loop at v can be considered to be switched twice, loops are unchanged by a switching. Bicoloured edges, being a red and a blue edge, are also unchanged by a switching. Two br-graphs are *switching-equivalent* if one can be changed to the other by a series of switchings. A unicoloured cycle is *balanced* if it has an even number of red edges, and so the term *uni-balanced* was introduced in [1] for a br-graph that is switching-equivalent to a graph in which all unicoloured cycles are balanced. It follows from a result of [16] that a br-graph is uni-balanced if and only if it is switching-equivalent to a br-graph in which all unicoloured edges are blue. As sign plays no role in this paper, any result we show for uni-balanced graphs also holds for graphs that are switching-equivalent to a br-graph in which all unicoloured edges are red.

As a *homomorphism* $\phi: G \rightarrow H$ from a graph G to a graph H is a edge preserving vertex map, a *homomorphism* from a br-graph G to another br-graph H is a vertex map $\phi: G \rightarrow H$ which preserves both red edges and blue edges. Defined in [15], a *switch-homomorphism* from G to H is a vertex map $\phi: G \rightarrow H$ such that for some graph G' that is switching-equivalent to G , the map $\phi: G' \rightarrow H$ is a homomorphism.

In [10], Foucaud and Naserasr introduced the decision problem $\text{S-HOM}(H)$ for a given br-graph H : decide for a given br-graph G if there is a switch-homomorphism to H . They addressed the problem of classifying the complexity of $\text{S-HOM}(H)$ in terms of H . A priori, $\text{S-HOM}(H)$ is not a CSP problem, but in [3] the authors showed that it is polynomially equivalent to $\text{CSP}(\mathcal{S}_H)$ for a br-graph \mathcal{S}_H called the *switching graph* of H , defined in Definition 2.3. From this, it follows by the CSP-dichotomy, that there is a complexity dichotomy. This dichotomy was characterised in [3] and [4].

The *switch-core* of a br-graph H is the unique minimal induced subgraph to which it admits a switch-homomorphism. A complexity dichotomy for the switch-homomorphism problem was conjectured in [3], and completed in [4].

Theorem 1.1 ([4]). *For a br-graph H , the problem $\text{S-HOM}(H)$ is in P if the switch-core of H has at most two edges, and is otherwise in NPC .*

Our goal is to determine, for a given br-graph H , the complexity of the list version $\text{LIST-S-HOM}(H)$ of the problem $\text{S-HOM}(H)$.

Much has already been done in [1]. Building on known ideas that we explain in more detail in the next section, the authors first observe that $\text{LIST-S-HOM}(H)$ is in P if and only if \mathcal{S}_H admits a *semi-conservative* WNU-polymorphism; this is defined at the end of Section 2. The goal then becomes to characterise the br-graphs H such that \mathcal{S}_H admits a semi-conservative WNU-polymorphism. The authors then go on to define a *special bipartite-min ordering* of the *bipartite resolution* \mathcal{B}_H of H , and define two obstructions to its existence, both of which are detectable in polynomial time: *invertible pairs* and *chains* in \mathcal{B}_H . The definition of \mathcal{B}_H is given at the beginning of Section 4.1. A restricted version of the definition of special bipartite-min ordering is given as Definitions 4.6 and 5.2. Restricted versions of the definitions of invertible pairs and chains is given as Definition 5.1.

Using these as their main tools, the authors of [1] characterise the br-trees admitting semi-conservative WNU-polymorphisms. The full characterisation is not easy to state, but for irreflexive trees, which can easily be shown to be uni-balanced, their characterisation agrees with the following conjecture.

Conjecture 1.2. *For a uni-balanced br-graph H , the following are equivalent.*

- (i) \mathcal{B}_H has no invertible pairs and no chains.
- (ii) \mathcal{B}_H has a special bipartite-min ordering.
- (iii) \mathcal{S}_H has a semi-conservative WNU-polymorphism.

The original main result of this paper was to prove Conjecture 1.2 for reflexive uni-balanced graphs. We are indebted to a reviewer who found fatal flaws in our proof of it. Our attempts to fix the flaws devolved into extensive casework, and so we abandoned it when a much more elegant proof came out in [2]. Though the approach in [2] was clearly the right approach, our approach was different than theirs and adds some valuable insight. Our proofs make significant use of symmetries in both the definition of \mathcal{S}_H and the definition of bipartite-min orderings, and using the results of [2], we get a strengthening of their result, explained below, for reflexive uni-balanced graphs.

Section 2 can be viewed as, perhaps, an expansion of this introductory section. We recall many required algebraic results of CSP theory and structural results about the list homomorphism problem for graphs.

In Section 3 we define a switch-symmetric polymorphism which commutes with the obvious symmetry in \mathcal{S}_H , and use it to observe that if a polymorphism of the red subgraph \mathcal{S}_H^R of \mathcal{S}_H is switch-symmetric, then it is a polymorphism of \mathcal{S}_H . This is a useful tool in defining polymorphisms on \mathcal{S}_H , but also yields a quick result. It allows us to quickly prove Theorem 3.2 which characterises the br-graphs H such that \mathcal{S}_H admits a conservative WNU-polymorphism. This depends on known characterisations

of graphs with conservative WNU-polymorphisms. As these do not exist for semi-conservative WNU-polymorphisms though, much work remains.

In Section 4 we make observations about a ‘reduction to the bipartite case’ for list homomorphisms of graphs that is given in [8], and show that parts of it extend transparently to br-graphs. In particular, this leads to the definition of a ‘parity-symmetric’ bipartite-min ordering, and the following strengthening of Conjecture 1.2.

Conjecture 1.3. *For a uni-balanced br-graph H , the following are equivalent.*

- (i) \mathcal{B}_H has no invertible pairs or chains.
- (ii*) \mathcal{B}_H has a parity-symmetric special bipartite-min ordering.
- (iii) \mathcal{S}_H has a semi-conservative WNU-polymorphism.

Conjecture 1.3 implies Conjecture 1.2, and indeed, it consists of Conjecture 1.2 and the following conjecture, the proof of which, for graphs, follows from non-trivial results in [8].

Conjecture 1.4. *For a uni-balanced br-graph H , \mathcal{B}_H has a special bipartite-min ordering if and only if it has a parity-symmetric one.*

Our main result is Theorem 5.5, which proves Conjecture 1.3 for reflexive uni-balanced graphs. This also proves Conjecture 1.4 for reflexive uni-balanced graphs. To be clear though, our result depends on the result of [2]. Our main contributions consist of Theorem 5.4, which gives the implication (ii*) \Rightarrow (iii) for bipartite uni-balanced graphs, and the technical Lemma 4.1, which allows us to reduce to the bipartite case: with the implication (i) \Rightarrow (ii) from [2], Lemma 4.1 allows us to show (i) \Rightarrow (ii*), and with Theorem 5.4, it gives us the the implication (ii*) \Rightarrow (iii) for all reflexive uni-balanced graphs. Lemma 4.1 is presented in some generality as it seems that it could be useful beyond the present paper.

The literature about the list homomorphism problem that we recall in Section 2 suggests many ‘polymorphism collapses’ that may be useful in finding a full characterisation of the complexity of LIST-S-HOM(H). While some have proved useful, some have been red herrings. In our final section, Section 6 we give examples showing that some of these collapses do not occur.

2. Background

In this section we recall the definitions required to properly state our problem, and make some initial observations about it. Many of these observations can also be found in [1].

2.1. Structures, CSP, and List Homomorphism

A (relational) structure \mathcal{H} consists of a set $V = V(\mathcal{H})$ of vertices with a finite ordered set of finite arity relations $R_i \subset V^k$ on V . The corresponding ordered set of the arities of the structure is called its *type*. A *homomorphism* between two structures of the same type is a vertex map that preserves each relation. The problem CSP(\mathcal{H}) for a structure \mathcal{H} is the problem of deciding if an instance structure \mathcal{G} of the same type admits a homomorphism to \mathcal{H} .

A graph H is viewed as a structure with one symmetric binary relation E , an edge uv corresponding to the pairs (u, v) and (v, u) in E . A vertex map $V(G) \rightarrow V(H)$ of graphs is a homomorphism between two graphs if and only if it is a homomorphism of the corresponding symmetric binary structures. So $\text{HOM}(H)$ is polynomially equivalent to $\text{CSP}(H)$.

In the list variation $\text{LIST-HOM}(H)$ of the homomorphism problem $\text{HOM}(H)$, an instance G comes attached with a list $L(v) \subset V(H)$ for each vertex $v \in V(G)$, and one must decide if there is a homomorphism $\phi: G \rightarrow H$ such that $\phi(v) \in L(v)$ for each vertex v of G . This is equivalent to $\text{CSP}(H_c)$ where \mathcal{H}_c is the relational structure one gets from a structure \mathcal{H} by adding a new unary relation $L_S = \{(s) \mid s \in S\}$ for each subset $S \subset V(\mathcal{H})$.

The *core* $\mathcal{C}(\mathcal{H})$ of a structure \mathcal{H} is the unique minimum induced substructure to which it admits a homomorphism. It is basic that $\text{CSP}(\mathcal{C}(\mathcal{H}))$ is polynomially equivalent to $\text{CSP}(\mathcal{H})$.

A k -ary polymorphism ϕ of \mathcal{H} is a homomorphism $\phi: \mathcal{H}^k \rightarrow \mathcal{H}$ (that is, a map preserving all relations of \mathcal{H}), where \mathcal{H}^k is the k -time categorical product of \mathcal{H} with itself. It is *idempotent* if $\phi(h, h, \dots, h) = h$ for all $h \in V(\mathcal{H})$, and it is *conservative* if $\phi(v_1, \dots, v_k) \in \{v_1, \dots, v_k\}$. An idempotent polymorphism $\phi: \mathcal{H}^k \rightarrow \mathcal{H}$, for $k \geq 3$ is *weak near unanimity* or *WNU* if for every choice of $x, y \in V(\mathcal{H})$ we have

$$\phi(x, x, \dots, x, y) = \phi(x, x, \dots, y, x) = \dots = \phi(y, x, \dots, x, x).$$

Theorem 2.1 (The CSP-dichotomy, [6, 17]). *For a relational structure \mathcal{H} the problem $\text{CSP}(\mathcal{H})$ is in P if the core of \mathcal{H} admits a WNU-polymorphism, and is otherwise in NPC.*

In general it is difficult to find the core of a structure \mathcal{H} , and tends to be difficult to give a non-algebraic description of the structures \mathcal{H} for which $\text{CSP}(\mathcal{H})$ is in P. For list colouring, it gets a little easier. For one, the structure H_c is always a core. It is easy to see that any polymorphism of H_c is conservative, so for list colouring problems we get the following corollary of Theorem 2.1 which Bulatov actually proved years earlier in [5].

Corollary 2.2. *For a relational structure \mathcal{H} the problem $\text{CSP}(\mathcal{H}_c)$ is in P if and only if the core of \mathcal{H} admits a conservative WNU-polymorphism.*

2.2. The switching graph

One might observe that $\text{S-HOM}(H)$ for a br-graph H is not a CSP, so the above tools do not apply, but one of the main tools in [3] and [4] is the following construction which allows us to view $\text{S-HOM}(H)$ as a CSP.

For a product $A \times B$ of sets or of graphs, we use π_A to denote projection map onto A and π_B the projection map onto B .

Definition 2.3. Let H be a br graph. For $i \in S := \{0, 1\}$ let H_i be the copy of H with vertex set $\{(v, i) \mid v \in V(H)\}$ for which $\pi_i((v, i)) = v$ is an isomorphism. The *switching graph* \mathcal{S}_H of H is the br-graph on the set $V(H) \times S$ that we get from the disjoint union of H_0 and H_1 by adding the red edges $(u, 0)(v, 1)$ and $(u, 1)(v, 0)$ for

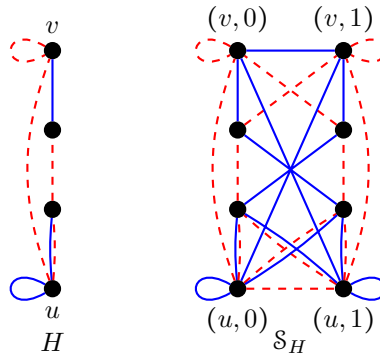


FIGURE 1. A br-graph H and its switching graph \mathcal{S}_H .

every blue edge uv of H , and the blue edges $(u, 0)(v, 1)$ and $(u, 1)(v, 0)$ for every red edge uv of H .

See Figure 1 for an example of the switching graph construction. Blue edges are solid, while red edges are dashed.

Let the *switch map* $f_s : V(\mathcal{S}_H) \rightarrow V(\mathcal{S}_H)$ be the automorphism of \mathcal{S}_H defined by $f_s((v, i)) = (v, 1 - i)$. It is evident that the edge $uf_s(v)$ is red if uv is blue, and $uf_s(v)$ is blue if uv is red.

The following proposition implies that $\text{S-HOM}(H)$ is polynomially equivalent to the homomorphism problem $\text{HOM}(\mathcal{S}_H)$ for the br-graph \mathcal{S}_H .

Proposition 2.4 ([3]). *For br-graphs G and H , G admits a switch-homomorphism to H if and only if it admits a homomorphism to \mathcal{S}_H .*

2.3. Definition of List Switch-Homomorphism

Though the problems $\text{S-HOM}(H)$ and $\text{HOM}(\mathcal{S}_H)$ are polynomially equivalent, they have different list versions. For a br-graph H we define the following two problems.

LIST-S-HOM(H)

Instance: A br-graph G with lists $L : V(G) \rightarrow 2^{V(H)}$.

Question: Does G admit a switch-homomorphism $\phi : G \rightarrow H$ preserving lists?

LIST-HOM(\mathcal{S}_H)

Instance: A br-graph G with lists $L : V(G) \rightarrow 2^{V(\mathcal{S}_H)}$.

Question: Does G admit a homomorphism $\phi : G \rightarrow \mathcal{S}_H$ preserving lists?

The second problem is clearly equivalent to the CSP problem $\text{CSP}(\mathcal{S}_H^c)$. The first problem can also be made equivalent to a CSP using Proposition 2.4. Indeed, from the proof of this proposition, it is not hard to see that it is polynomially equivalent to $\text{CSP}(\mathcal{S}_H^{sc})$ where we get \mathcal{S}_H^{sc} from \mathcal{S}_H^c by throwing away non-symmetric lists, i.e.,

throwing away the unary relation L_S for any subset $S \subset V(H)$ that is not closed under action by the switch map s from Definition 2.3.

Clearly $\text{LIST-HOM}(\mathcal{S}_H)$ encodes $\text{LIST-S-HOM}(H)$ by taking only instances with symmetric lists, and as the full lists are symmetric, $\text{LIST-S-HOM}(H)$ encodes $\text{HOM}(\mathcal{S}_H)$ by taking instances with full lists on every vertex. Thus we have the following polynomial reductions.

$$(1) \quad \text{HOM}(\mathcal{S}_H) \leq_{\text{poly}} \text{LIST-S-HOM}(H) \leq_{\text{poly}} \text{LIST-HOM}(\mathcal{S}_H)$$

Calling a polymorphism of \mathcal{S}_H *semi-conservative* if it conserves symmetric lists, Theorem 2.1 yields a nice algebraic description of the relationship between the problems $\text{LIST-S-HOM}(H)$ and $\text{LIST-HOM}(\mathcal{S}_H)$.

Fact 2.5. *For a br-graph H , $\text{LIST-S-HOM}(H)$ is in P if and only if \mathcal{S}_H admits a semi-conservative WNU-polymorphism, and $\text{LIST-HOM}(\mathcal{S}_H)$ is in P if and only if \mathcal{S}_H admits a conservative WNU-polymorphism.*

Proof. Everything would be immediate from Theorem 2.1 except that to apply it we need that the structures are cores, while the structure \mathcal{S}_H^{sc} , which we use for $\text{LIST-S-HOM}(H)$, need not be a core. So we show that if \mathcal{S}_H admits a semi-conservative WNU-polymorphism, then the core of \mathcal{S}_H^{sc} admits a WNU-polymorphism. To get the core of \mathcal{S}_H^{sc} we must identify the vertices $(v, 0)$ and $(v, 1)$ for any vertex v of H all of whose edges are bicoloured. Let r be the retraction of \mathcal{S}_H to its core that we get by making all of these identifications. It is easy to check that if $\phi : \mathcal{S}_H^k \rightarrow \mathcal{S}_H$ is a semi-conservative WNU-polymorphism, then $\phi' = r \circ \phi|_{r(\mathcal{S}_H)^k}$ is a WNU-polymorphism on the core $r(\mathcal{S}_H)$. \square

2.4. Collapses of Conservative Polymorphisms

Our goal is a nice characterisation of the br-graphs H such that \mathcal{S}_H admits a (semi-)conservative WNU-polymorphism. In the case of other graph-like structures, such characterisations often come from the fact that a structure has a conservative WNU-polymorphism if and only if it has some other nicer polymorphism such as a conservative 3-NU-polymorphism (also known as a conservative majority polymorphism).

Definition 2.6. A polymorphism $\phi : H^3 \rightarrow H$ on a graph H is 3-NU if for any $x, y \in V(H)$, we have

$$\phi(x, x, y) = \phi(x, y, x) = \phi(y, x, x) = x.$$

It is clear that a 3-NU-polymorphism is WNU, but one can find many examples of structures with WNU-polymorphisms that do not have 3-NU-polymorphisms. However, there is a collapse of these two types of polymorphisms for graphs.

Theorem 2.7 ([8]). *A graph H has a conservative WNU-polymorphism if and only if it has a conservative 3-NU-polymorphism.*

Another useful conservative WNU-polymorphism is disguised as an ordering. Recall that an ordering \leq of a set V can be viewed as a 2-ary function $\min : V^2 \rightarrow V$ such that $\min(u, v) = \min(v, u) \in \{u, v\}$ for all $u, v \in V$.

Definition 2.8. An ordering \leq of the vertices of a reflexive graph is a *min ordering* if the corresponding 2-ary function \min is a polymorphism.

A graph with a min ordering admits the induced 3-ary operation $\min(a, b, c) = \min(\min(a, b), c)$ which is a conservative WNU-polymorphism of H ; we call this the *min polymorphism* associated with the min ordering. For reflexive graphs, there is a collapse of WNU-polymorphisms to min orderings.

Theorem 2.9 ([7]). *For a reflexive graph H the following are equivalent.*

- *The problem LIST-HOM(H) is in P.*
- *H admits a conservative WNU-polymorphism.*
- *H admits a min ordering.*
- *H contains no invertible pairs.*

The existence of min orderings is known to characterise the well known class of interval graphs, so a reflexive graph has a conservative WNU-polymorphism if and only if it is an interval graph. Min orderings are also known to be characterised by the fact that they satisfy the *underbar* property:

$$(2) \quad (a \leq a', b \leq b', a \sim b', a' \sim b) \Rightarrow (a \sim b).$$

No non-empty irreflexive graph can have a min ordering, but there is a generalisation, called a bipartite-min ordering, defined in Section 4 on what we call the bipartite resolution \mathcal{B}_H of H , that exists for many irreflexive graphs.

It was shown in [9], for a graph H with a bipartite-min ordering, that the problem LIST-HOM(H) can be solved in polynomial time via the arc-consistency algorithm. In [12] the authors generalised the notion of bipartite-min orderings to digraphs, unifying many known list homomorphism characterisations.

Theorem 2.10. [12] *For a graph H the following are equivalent.*

- *The problem LIST-HOM(H) is in P.*
- *H admits a conservative WNU-polymorphism.*
- *H admits a bipartite-min ordering.*
- *\mathcal{B}_H contains no invertible pairs.*

3. Dichotomy for List \mathcal{S}_H -Colouring

Recall the switch automorphism $f_s : \mathcal{S}_H \rightarrow \mathcal{S}_H$ that maps (v, i) to $(v, 1-i)$. This is clearly also an automorphism of the red subgraph \mathcal{S}_H^R of \mathcal{S}_H . A k -ary polymorphism ϕ of \mathcal{S}_H , or of \mathcal{S}_H^R , is *switch-symmetric* if it commutes with f_s :

$$\phi(f_s(v_1), \dots, f_s(v_k)) = f_s(\phi(v_1, \dots, v_k)).$$

Lemma 3.1. *If a polymorphism of \mathcal{S}_H^R is switch-symmetric, then it is a polymorphism of \mathcal{S}_H .*

Proof. Let $\phi : (\mathcal{S}_H^R)^k \rightarrow \mathcal{S}_H^R$ be switch-symmetric. To see that it is a polymorphism of \mathcal{S}_H it is enough to observe that it is also a polymorphism of \mathcal{S}_H^B . Let $x_i \sim y_i$ in \mathcal{S}_H^B

for each $i \in [k]$. By the definition of \mathcal{S}_H we have that $x_i \sim f_s(y_i)$ in \mathcal{S}_H^R for each i . So as ϕ is a polymorphism of \mathcal{S}_H^R we get

$$\phi(x_1, \dots, x_k) \sim \phi(f_s(y_1), \dots, f_s(y_k)) = f_s(\phi(y_1, \dots, y_k))$$

is a red edge, and so $\phi(x_1, \dots, x_k) \sim \phi(y_1, \dots, y_k)$ is a blue edge. This is enough. \square

We will use this in Section 5, but for now we exhibit its utility by using it to characterise the graphs \mathcal{S}_H having conservative WNU-polymorphisms. This yields a simple complexity dichotomy for the problem LIST-HOM(\mathcal{S}_H).

Theorem 3.2. *For a br-graph H , \mathcal{S}_H admits a conservative WNU-polymorphism if and only if \mathcal{S}_H^R admits a conservative 3-NU-polymorphism.*

The proof of this theorem takes the rest of the section, but consists mostly of observations about known results. We recall these results and then give the proof formally at the end of the section. First observe the following hierarchy of polymorphisms.

$$\begin{aligned} (3) \quad \mathcal{S}_H \text{ has a conservative 3-NU} &\Rightarrow \mathcal{S}_H \text{ has a conservative WNU} \\ &\Rightarrow \mathcal{S}_H^R \text{ has a conservative WNU} \\ &\Rightarrow \mathcal{S}_H^R \text{ has a conservative 3-NU} \end{aligned}$$

The first two implications are trivial; indeed, the first is by definition, and the second is because a polymorphism of a structure is a polymorphism of any relation of the structure. We get the third implication by applying Theorem 2.7 to the graph \mathcal{S}_H^R . To prove Theorem 3.2 it is enough to show that a 3-NU-polymorphism on \mathcal{S}_H^R may be assumed to be switch-symmetric, as then by Lemma 3.1 we get that the four statements in (3) are equivalent. In fact one can assume more about the 3-NU-polymorphism on \mathcal{S}_H^R .

Definition 3.3. A conservative 3-NU-polymorphism $\phi : H^3 \rightarrow H$ is *symmetric* if it commutes with all automorphisms σ of H :

$$\phi(\sigma(a), \sigma(b), \sigma(c)) = \sigma(\phi(a, b, c)).$$

In [14], while giving a characterisation of the digraphs H for which LIST-HOM(H) is tractable, Hell and Rafiey found a useful omitted subgraph characterisation of digraphs with conservative NU-polymorphisms. Their definition is for digraphs, but we give it here for graphs.

Definition 3.4. A walk $P = u_0 \sim u_1 \sim \dots \sim u_n$ of a graph H *avoids* a walk $Q = v_0 \sim v_1 \sim \dots \sim v_n$ of H of the same length if $u_{i-1} \not\sim v_i$ for each $i \in [n]$. For a 3-tuple (a, b, c) of vertices of H , a *b-excluder* in H is a set of three walks B_a, B_b and B_c of the same length, starting at a, b and c respectively, such that B_a and B_c share the same last vertex, and B_b avoids B_a and B_c . See Figure 2. A 3-tuple (a, b, c) is a *permutable triple* if H contains an *a-excluder*, a *b-excluder*, and a *c-excluder*.

In Theorem 4.1 of [14], it was shown that a graph H has a 3-NU-polymorphism if and only if it contains no permutable triples. In the proof, it was observed that if a graph H has no permutable triples, then for any 3-tuple (x_1, x_2, x_3) in $V(H)^3$,

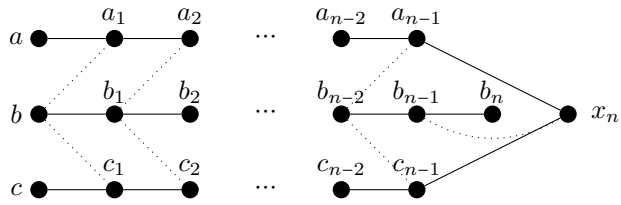


FIGURE 2. A b -excluder for (a, b, c)

there is at least one $i \in [3]$ such that there is no x_i -excluder in H for (x_1, x_2, x_3) . We isolate a fact shown within the proof of Theorem 4.1 of [14].

Lemma 3.5 ([14]). *Let H be a graph with no permutable triples. Let $\phi(x_1, x_2, x_3) = m$ if at least two of the entries are m , and otherwise let $\phi(x_1, x_2, x_3) = x_i$ where i is the minimum index such that there is no x_i -excluder in H for (x_1, x_2, x_3) . The function $\phi : H^3 \rightarrow H$ is a conservative 3-NU-polymorphism of H .*

It is clear that the existence of excluders for a triple is preserved under any automorphism s of H . The properties ‘two of x_1, x_2 and x_3 being the same’, and ‘minimum index’ are also preserved, so the conservative 3-NU-polymorphism defined in the lemma is symmetric. This allows us to prove Theorem 3.2.

Proof of Theorem 3.2. Let \mathcal{S}_H^R have a conservative 3-NU-polymorphism ϕ . By the discussion following Lemma 3.5 we may assume that ϕ is symmetric, and so, as the switch map is an automorphism of \mathcal{S}_H^R , switch-symmetric. By Lemma 3.1, ϕ is then a conservative 3-NU-polymorphism of \mathcal{S}_H , and so a conservative WNU-polymorphism. This completes the proof of the equivalence of the four polymorphisms in (3), and so proves Theorem 3.2. \square

4. Reduction to Bipartite Graphs and Bipartite-Min Orderings

In [8], Feder, Hell, and Huang, reduced the complexity of the list homomorphism problem for graphs to the bipartite (so irreflexive) case by showing that $\text{LIST-HOM}(H)$ for any graph H is polynomially equivalent to $\text{LIST-HOM}(\mathcal{B}_H)$ for a bipartite graph $\mathcal{B}_H := K_2 \times H$. Their result implies that H has a conservative WNU-polymorphism if and only if \mathcal{B}_H does. Their proof consists of two parts. Where ‘parity-symmetric’ polymorphisms on \mathcal{B}_H are defined below, the authors of [8] essentially prove the following two equivalences.

- (i) H has conservative WNU-polymorphism if and only if \mathcal{B}_H has a parity-symmetric conservative WNU-polymorphism.
- (ii) \mathcal{B}_H has a parity-symmetric conservative WNU-polymorphism if and only if it has a conservative WNU-polymorphism.

(To reconcile these statements with those of [8], the reader should observe that the graph H^{**} of [8] is simply \mathcal{B}_H augmented with a relation that forces all polymorphisms to be parity-symmetric.)

The first statement is easy. In this section we observe that it generalises and extends to similar statements about polymorphisms of br-graphs, and in particular about polymorphisms of \mathcal{S}_H . The second statement is much harder and depends on structural results.

4.1. The bipartite resolution and the constant parity subgraph

Recall that for a graph H , the categorical product $K_2 \times H$ is the graph with vertex set $\{0, 1\} \times V(H)$ in which $(i, h) \sim (i', h')$ if and only if $i \neq i'$ and $h \sim h'$.

Let B be the bicoloured K_2 on the vertex set $\{0, 1\}$. For a br-graph H , the *bipartite resolution* of H is $\mathcal{B}_H := B \times H$. It is a bipartite br-graph with partite sets $\pi_B^{-1}(0)$ and $\pi_B^{-1}(1)$, so we call $i = \pi_B((i, v))$ the *parity* of a vertex (i, v) and call the map

$$(4) \quad f_p : (i, v) \mapsto (1 - i, v),$$

which is clearly an automorphism of \mathcal{B}_H , the *parity flip* map. A polymorphism ϕ of \mathcal{B}_H is *parity-symmetric* if it commutes with the parity flip map.

Let \mathcal{P} be the set consisting of the following properties applying to polymorphisms of a graph H : k -NU, WNU, idempotent, conservative, and semi-conservative (if H is a switch graph).

For any subset $P \subseteq \mathcal{P}$, a polymorphism ϕ of H is a P -polymorphism if it has all properties in P . The main result of this section is the following.

Lemma 4.1. *For any br-graph H , and any subset P of \mathcal{P} , H has a P -polymorphism ϕ if and only if \mathcal{B}_H has a parity-symmetric P -polymorphism Φ (called the bipartite resolution of ϕ).*

Proof. This is immediate from Claims 4.2 and 4.3 given below. \square

For a vertex $v = (v_1, \dots, v_k)$ in \mathcal{B}_H^k , the *parity pattern* of v is the tuple

$$\pi_B(v) := (\pi_B(v_1), \dots, \pi_B(v_k))$$

of its coordinates' parities. Over components of \mathcal{B}_H the parity pattern is constant up to component-wise action of the parity flip map f_p . In particular the *constant parity subgraph* $\text{CP}(\mathcal{B}_H^k)$ of \mathcal{B}_H^k , which is the subgraph induced on the vertices with parity patterns $(0, 0, \dots, 0)$ or $(1, 1, \dots, 1)$, is a union of components. A coordinate of a parity pattern is *majority* if at least half of the coordinates have the same parity. A homomorphism $\Phi : \text{CP}(\mathcal{B}_H^k) \rightarrow \mathcal{B}_H$ is a P -homomorphism, for any $P \subseteq \mathcal{P}$, if it satisfies those same conditions required of a polymorphism of \mathcal{B}_H to satisfy them, and is parity-symmetric if it commutes with f_p .

Claim 4.2. *For any br-graph H and any $P \subseteq \mathcal{P}$, H has a k -ary P -polymorphism if and only if $\text{CP}(\mathcal{B}_H^k)$ has a parity-symmetric P -polymorphism.*

Proof. The details of the proof are straightforward but tedious, and so we just outline the main points.

For a polymorphism $\phi : H^k \rightarrow H$, the map $\Phi : \text{CP}(\mathcal{B}_H^k) \rightarrow \mathcal{B}_H$ defined for $v = (v_1, \dots, v_k) \in V(\text{CP}(\mathcal{B}_H^k))$ by

$$\Phi(v) = (\pi_B(v_1), \phi(\pi_H(v_1), \dots, \pi_H(v_k)))$$

is a parity-symmetric homomorphism, and for a parity-symmetric homomorphism $\Phi : \text{CP}(\mathcal{B}_H^k) \rightarrow \mathcal{B}_H$ the map $\phi : H^k \rightarrow H$ defined for $v = (v_1, \dots, v_k)$ by

$$\phi(v) = \Phi((0, v_1), \dots, (0, v_k))$$

is a polymorphism; moreover, these constructions are inverse.

That ϕ is WNU or k -NU if and only if Φ is also straightforward. To verify that ϕ is idempotent, conservative, or semi-conservative if and only if Φ is, one need only observe that Φ preserves a list $L \subset V(\mathcal{B}_H)$ if and only if ϕ preserves its intersections, $\pi_H(L \cap \pi_B^{-1}(0))$ and $\pi_H(L \cap \pi_B^{-1}(1))$, with each side of \mathcal{B}_H . \square

Claim 4.3. *For any br-graph H and any $P \subseteq \mathcal{P} \cup \{\text{parity-symmetric}\}$, any P -homomorphism $\text{CP}(\mathcal{B}_H^k) \rightarrow \mathcal{B}_H$ extends to a P -polymorphism of \mathcal{B}_H .*

Proof. Given a homomorphism $\Phi : \text{CP}(\mathcal{B}_H^k) \rightarrow \mathcal{B}_H$, extend it to a polymorphism of \mathcal{B}_H by defining it, on each component of $\text{CP}(\mathcal{B}_H^k)$, to project onto some coordinate that is majority in the parity pattern of the component.

That this maintains the properties of idempotence, conservativity, and semi-conservativity is immediate from the observation that projections preserve all lists. That parity-symmetry is maintained is immediate from the fact that projections are parity-symmetric. That the properties k -NU and WNU are maintained uses the choice of projection, but it straightforward. \square

Applying Lemma 4.1 to the br-graph \mathcal{S}_H yields this useful observation.

Proposition 4.4. *For br-graph H , \mathcal{S}_H has a semi-conservative WNU-polymorphism if and only if $\mathcal{B}_{\mathcal{S}_H}$ has a parity-symmetric semi-conservative WNU-polymorphism.*

We use this in conjunction with one more simple observation.

Fact 4.5. *For any br-graph H , $\mathcal{B}_{\mathcal{S}_H}$ and $\mathcal{S}_{\mathcal{B}_H}$ are isomorphic.*

Proof. They are both the graph with vertex set $B \times V(H) \times S$ where $(i, h, j) \sim (i', h', j')$ if and only if $i \neq i'$ and $h \sim h'$. The edge $(i, h, j)(i', h', j')$ is red if and only if hh' is red and $j = j'$ or hh' is blue and $j \neq j'$. \square

4.2. Bipartite-min orderings

As mentioned in the introduction, a graph cannot have a min ordering unless almost all of its vertices have loops. In [9], bipartite-min orderings were considered for bipartite graphs.

Definition 4.6. An ordering of the vertices of a bipartite graph H with partite sets U and V is a *bipartite-min ordering* if for any $a, b \in U$ and $a', b' \in V$, the underbar property (2) holds.

As a min-ordering \leq of H is a 2-ary function \min which defines a conservative 3-ary WNU-polymorphism \min of H , (see before and after Definition 2.8) a bipartite-min ordering of H is two functions, one on the partite set U and one on V , that defines a conservative WNU-homomorphism from $\text{CP}(\mathcal{B}_H^3)$ to \mathcal{B}_H , so by Claim 4.3 extends to a conservative WNU-polymorphism of \mathcal{B}_H . So in the same way that a min-ordering of H yields a conservative WNU-polymorphism on H , a bipartite-min ordering of \mathcal{B}_H yields a conservative WNU-polymorphism on H . If this polymorphism of \mathcal{B}_H is parity-symmetric, it gives, by Lemma 4.1, a conservative WNU-polymorphism on H .

Viewing min orderings and bipartite-min orderings as functions, we get the following case of Lemma 4.1.

Fact 4.7. *A graph H has a min ordering if and only if \mathcal{B}_H has a parity-symmetric bipartite-min polymorphism.*

5. Bipartite-Min Orderings and Uni-Balanced Graphs

As useful as min orderings are, one looks to define them for br-graphs. This was done for trees and some other classes of br-graphs in [1]. The authors defined special min orderings of reflexive br-trees, and special bipartite-min orderings of irreflexive br-trees, and then showed that for br-graphs H with these orderings the problem $\text{LIST-S-HOM}(H)$ can be solved in polynomial time.

Though explicit, the polynomial time algorithms from [1] and [2] are complicated. In this section we suggest an approach to these tractability proofs that is easier, depending on the CSP-dichotomy to sweep this complication under the rug.

We consider uni-balanced br-graphs. As mentioned in the introduction, it follows from a result of Zaslavsky in [16] that such graphs are switching-equivalent to a br-graph in which all unicoloured cycles are blue, so we will assume that our uni-balanced br-graphs have been switched to remove all unicoloured red edges. We thus talk only of blue edges, by which we mean unicoloured blue edges, and bicoloured edges. In the one place that we want to consider a bicoloured edge as blue, we say ‘not necessarily unicoloured blue’. A neighbour of a vertex is called *bicoloured* or *blue* if it is adjacent via a bicoloured or blue edge.

The following definition is from [1] but is simplified here for uni-balanced graphs. Though we do not use the definition beyond observing that an invertible pair in a br-graph H corresponds with the known definition of an invertible pair in its underlying graph, we include the definition, as invertible pairs and chains appear so prominently in the statements of our results and conjectures.

Definition 5.1. Let H be a uni-balanced br-graph. Construct a digraph \mathcal{A} on the vertex set $V(H)^2$ by letting $(b, t) \rightarrow (b', t')$ if $b \sim b', t \sim t'$, and $b \not\sim t'$. A pair (x, y) is an *invertible pair* if there are walks from (x, y) to (y, x) , and from (y, x) to (x, y) in \mathcal{A} . A *good end* is a pair (b, t) of vertices of H for which there is another vertex x that with a bicoloured edge to b and a blue edge to t . A *bad end* is a pair (b, t) for which there is another vertex y with a bicoloured edge to t and a blue edge to b . A *chain* in H is a directed walk in \mathcal{A} from a good end to a bad end.

The following definition is also a definition from [1] that has been simplified by restricting to uni-balanced graphs. Note that if H is uni-balanced, then so is \mathcal{B}_H .

Definition 5.2. Given a uni-balanced br-graph H , and a bipartite-min ordering of the underlying graph of \mathcal{B}_H , a vertex is *special* if all its bicoloured neighbours come before all its blue neighbours. The ordering is a *special bipartite-min ordering* if all vertices are special.

Our main theorem says that the existence of a special bipartite-min ordering on \mathcal{B}_H , for a reflexive uni-balanced graph H , yields a semi-conservative WNU-polymorphism on \mathcal{S}_H , so implies that $\text{LIST-S-HOM}(H)$ is in P .

A *blue component* C of a uni-balanced graph H is a subgraph induced by a component of the subgraph of unicoloured blue edges. We note that a blue component may induce bicoloured edges, but is connected via blue edges.

Lemma 5.3. *Let \mathcal{B} be a bipartite uni-balanced br-graph having a special bipartite-min ordering, and let C be a blue component of \mathcal{B} with maximum vertices u_a and u_b . Let $x_a x_b$ be a blue edge in C , and let $y_a y_b$ be an edge with $x_a \leq y_a \leq u_a$ and $x_b \leq y_b$. Then $y_a y_b$ is blue, and y_a and y_b are in C .*

Proof. As C is a blue component there is a blue path from x_a to u_a , and somewhere it must cross $y_a y_b$, so we may assume (by switching the roles of a and b if necessary) we have a blue path aba' in C with $a \leq y_a < a'$ and $b \leq y_b$. By the underbar property with the edge $a'b$ we have $y_a \sim b$, and as b is special with a blue edge to a , $y_a b$ is also blue. So y_a is in C . As $y_b > b$ we then get that $y_a y_b$ is blue by the speciality of y_a . So y_b is in C . \square

This shows that if there is a bicoloured edge $y_a y_b$ above a blue edge $x_a x_b$, then it must be above the whole blue component containing x_a and x_b .

We are now ready to prove the main result of the section.

Theorem 5.4. *Let \mathcal{B}_H be a bipartite uni-balanced graph with a special bipartite-min ordering \leq . The switch graph $\mathcal{S} := \mathcal{S}_{\mathcal{B}_H}$ has a semi-conservative WNU-polymorphism Φ . Moreover, if \leq is parity-symmetric, then so is Φ .*

Proof. For a vertex v of \mathcal{B}_H let $\text{bc}(v)$ be the *blue component* containing v . By Claim 4.2 it is enough to define $\Phi : \mathcal{S}^3 \rightarrow \mathcal{S}$ on constant parity tuples. For such tuples we set

$$\Phi((a_1, x_1), (a_2, x_2), (a_3, x_3)) = (A, X)$$

where $A = \min(a_1, a_2, a_3)$ for the bipartite-min polymorphism \min defined by \leq , and X is defined as follows. Calling (a_i, x_i) or simply a_i *relevant* if $\text{bc}(a_i) = \text{bc}(A)$, let

- $X = \text{maj}(x_1, x_2, x_3)$ if all a_i are relevant; otherwise let
- $X = x_i$ where i is the minimum i such that a_i is relevant.

The function Φ is clearly idempotent, and semi-conservative. To see that it is WNU, it is enough to show that A and X are both WNU. Certainly A is, to see that X is too, we assume that there is a repeated value in $\{(x_1, a_1), (x_2, a_2), (x_3, a_3)\}$, and so in $\{a_1, a_2, a_3\}$. We show that the value of X is chosen from among $\{x_1, x_2, x_3\}$

depending on which of the a_i is repeated, and is unchanged under permuting the indices $i = 1, 2, 3$. Indeed, if the repeated a_i is not relevant, then X is the other x_i . If the repeated a_i is relevant, while the other is not, X is the repeated x_i . If all are relevant, then X is majority, and so is the repeated x_i . In all cases, X is WNU. Observe also that Φ is switch-symmetric, as complementing the x_i does not effect A , and so does not effect which x_i is returned by X .

What remains to be shown, for the first statement of the theorem, is that Φ is a homomorphism. As it is switch-symmetric, by Lemma 3.1 it is enough to show that it preserves (not necessarily unicoloured) blue edges. Assume that $(a_i, x_i) \sim (b_i, y_i)$ is a blue or bicoloured edge for $i = 1, 2, 3$. We show that

$$(A, X) := \Phi((a_1, x_1), (a_2, x_2), (a_3, x_3)) \sim \Phi((b_1, y_1), (b_2, y_2), (b_3, y_3)) =: (B, Y),$$

is a blue or bicoloured edge.

Indeed, as \min is a polymorphism of the underlying graph $\mathcal{B}_H^B = \mathcal{B}_H^B \cup \mathcal{B}_H^R$ of \mathcal{B}_H , we have

$$A = \min(a_1, a_2, a_3) \sim \min(b_1, b_2, b_3) = B$$

in \mathcal{B}_H^B , so AB is a blue or bicoloured edge of \mathcal{B}_H . If it is bicoloured, then whatever X and Y are, we have that $(A, X)(B, Y)$ is a bicoloured edge, and we are done. We must show that if AB is blue, then $X = Y$.

Assume that AB is blue. First we claim, for all i , that if a_i is relevant, then $a_i b_i$ is blue, making $x_i = y_i$, and implying that b_i is relevant. Indeed, if a_i is relevant, we have $A \leq a_i$ and a_i is in $\text{bc}(A)$. As AB is blue we have $\text{bc}(B) = \text{bc}(A)$, and so any bicoloured neighbour of a_i must be below B by Lemma 5.3. Thus $a_i b_i$ is a blue edge, as claimed, making b_i relevant. Similarly a_i is relevant if b_i is, and this holds for all relevant arguments, so with the fact that $x_i = y_i$ for these arguments, we get $X = Y$.

For the second statement of the theorem, we have that A is parity-symmetric by definition, as \leq is parity-symmetric, so it is enough to observe that X is unchanged by replacing the a_i with $p_s(a_i)$. As A is parity-symmetric, doing so does not change which a_i are relevant, so this is immediate from the definition of X . \square

We finish off this section by showing that our theorem implies Conjecture 1.3 for reflexive uni-balanced graphs.

Theorem 5.5. *For a reflexive uni-balanced br-graph H , the following are equivalent.*

- (i) H has no invertible pairs or chains.
- (ii*) \mathcal{B}_H has a parity-symmetric special bipartite-min ordering.
- (iii) \mathcal{S}_H has a semi-conservative WNU-polymorphism.

Proof. In Theorem 4 of [2] it is shown that if H has no invertible pairs or chains, then H has a special min ordering. By Fact 4.7, a min ordering of H is exactly a parity-symmetric bipartite-min ordering of \mathcal{B}_H , and by definition, the first is special if and only if the second is, so we have the implication (i) \Rightarrow (ii*).

If \mathcal{B}_H has a parity-symmetric special bipartite-min ordering, Theorem 5.4 gives a parity-symmetric semi-conservative WNU-polymorphism of $\mathcal{S}_{\mathcal{B}_H}$. This is $\mathcal{S}_{\mathcal{B}_H}$ by

Fact 4.5, and so by Proposition 4.4, \mathcal{S}_H has a semi-conservative WNU-polymorphism. This gives us $(ii^*) \Rightarrow (iii)$.

Implication $(iii) \Rightarrow (i)$ also uses known results. Assume that \mathcal{S}_H has a semi-conservative WNU-polymorphism. Then the problem LIST-S-HOM(H) is in P by Fact 2.5, and in particular it is in P when restricting to blue instances. As the underlying graph of H is exactly the blue subgraph H^B , switching does not help for blue instances, and so LIST-S-HOM(H) for blue instances is just LIST-HOM(H^B). Thus LIST-HOM(H^B) is in P, and so H^B has no invertible pairs by Theorem 2.10. By definition, an invertible pair of H is an invertible pair of the underlying graph, which in this case is H^B , and so H has no invertible pairs. In [1] it is shown that chains in H omit semi-conservative WNU-polymorphisms on \mathcal{S}_H , so it has no chains either. \square

6. Possible and Impossible Polymorphism Collapses

The following questions arise naturally when considering the complexity dichotomy for LIST-S-HOM(H), and when observing the various polymorphism collapses we have mentioned in the paper.

Question 6.1. *Is it true for a br-graph H that*

- (i) \mathcal{S}_H has a semi-conservative WNU if and only if it has a parity-symmetric one?
- (ii) \mathcal{S}_H has a semi-conservative WNU-polymorphism if and only if it has a conservative WNU-polymorphism?
- (iii) \mathcal{S}_H has a semi-conservative WNU-polymorphism if and only if it has a semi-conservative 3-NU-polymorphism?

Part (i) is certainly tempting. Using the results and ideas of Section 4 it is the key to getting a full reduction to the bipartite case: LIST-S-HOM(H) =_{poly} LIST-S-HOM(\mathcal{B}_H). Part (ii) was tempting, as it would have reduced the whole proof of Theorem 3.2, including the part from [1], to Section 3. Part (iii) was also tempting, as it would have reduced the proof to Sections 3 and 4. We now have countless counter-examples giving negative answers to these last two questions. We give one in the proposition below.

Proposition 6.2. *Where T is the br-graph on the left of Figure 3, \mathcal{P}_T has a semi-conservative WNU-polymorphism but no conservative WNU-polymorphism or semi-conservative 3-NU-polymorphism.*

Proof. That \mathcal{P}_T has a semi-conservative WNU-polymorphism is shown in [1], and also follows from Theorem 5.4. Indeed, ordering the vertices of T according to their position in the figure as one moves up the page; except putting the top vertex in the bicoloured path below its neighbour, we have a special bipartite-min ordering of T .

It is not too hard to verify that the triple (a_2, b_1, c_1) in R is a permutable triple, showing that it and so \mathcal{P}_T^R (of which it is an induced subgraph) can have no conservative WNU-polymorphism.

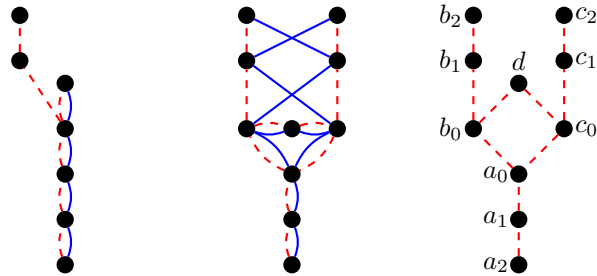


FIGURE 3. The tree T , the switch-core of \mathcal{P}_T , and its red subgraph R

We show now that R can have no semi-conservative 3-NU-polymorphisms. Note that symmetric lists are now those for which if b_i is in a list, then c_i is also, and vice-versa.

Assume, towards contradiction, that ϕ is a semi-conservative 3-NU-polymorphism of R . There is no valid image for $\phi(a_2, b_1, c_1)$. Indeed, if $\phi(a_2, b_1, c_1) = a_2$, then $\phi(a_1, b_0, c_0) = a_1$, and $\phi(a_2, d, d) = a_2$. This contradicts the fact that ϕ is 3-NU. So without loss of generality we may assume that $\phi(a_2, b_1, c_1) = b_1$. But then we see that $\phi(a_1, b_2, c_2) = b_2$, so $\phi(a_0, b_1, c_1) = b_1$, so $\phi(c_0, b_2, c_2) \in \{b_0, b_2\}$ and so $\phi(c_1, b_1, c_1) = b_1$. This again contradicts the fact that ϕ is 3-NU. \square

In [1] the authors observe that there are infinitely many trees having semi-conservative WNU-operations that contain T as an induced subgraph. The proof works for all of these.

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