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On the Tarry-Escott and Related Problems for 2×2 matrices over $\mathbb{Q}$

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Abstract. Reduced solutions of size 2 and degree n of the Tarry-Escott problem over $M_2(\mathbb{Q})$ are determined. As an application, the diophantine equation $\alpha A^n + \beta B^n = \alpha C^n + \beta D^n$, where α, β are rational numbers satisfying $\alpha + \beta \neq 0$ and $n \in \{1, 2\}$, is completely solved for $A, B, C, D \in M_2(\mathbb{Q})$.

1. Introduction

A Diophantine equation is an equation, usually with integral or rational coefficients, in which the sought-after unknowns are also integers. In 1989, Vaserstein [6] suggested solving classical problems of number theory substituting the ring $\mathbb{Z}$ by the ring $M_2(\mathbb{Z})$ of 2×2 integral matrices. Some problems become easier and some give us interesting results. The Tarry-Escott problem is a classical problem in number theory which asks one to find two distinct multisets of integers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ such that

$$\sum_{i=1}^{n} a_i^j = \sum_{i=1}^{n} b_i^j$$

for $j = 1, 2, \ldots, k$. We call n the size of the solution and k the degree. We abbreviate the above system by writing

$$\{a_1, \ldots, a_n\} =_k \{b_1, \ldots, b_n\}.$$

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Solutions with $k = n - 1$ are called ideal solutions. The Tarry-Escott problem has been extensively investigated in the literature; see for instance [1], [2] and also [5].

In 2006, Choudhry [3] introduced a matrix analog of the Tarry-Escott problem by considering the problem over $M_2(\mathbb{Z})$. The Tarry–Escott problem over $M_m(R)$ for a given ring R can be stated as follows: given $k, m, n \in \mathbb{N}$ and a ring R, two different multisets

$$A = \{A_1, A_2, \ldots, A_n\} \quad \text{and} \quad B = \{B_1, B_2, \ldots, B_n\},$$

where $A_i, B_i \in M_m(R) \setminus \{\underline{0}\}$, constitute a non-trivial solution of the Tarry–Escott problem of size n and degree k over $M_m(R)$ if

$$\sum_{i=1}^{n} A_i^j = \sum_{i=1}^{n} B_i^j \qquad (j = 1, 2, \ldots, k),$$

abbreviated as $\{A_1, \ldots, A_n\} =_k \{B_1, \ldots, B_n\}$. Choudhry [3] obtained, in parametric terms, two distinct pairs of matrices A_1, A_2 and B_1, B_2 in $M_2(\mathbb{Z})$ such that $A_1^n + A_2^n = B_1^n + B_2^n$ holds simultaneously for all integral values of n, whether positive or negative. This gives a non-trivial solution of the matrix analog of the Tarry-Escott problem of infinite degree and size 2. Using this solution, he obtained an arbitrarily long multigrade chain of matrices in $M_2(\mathbb{Z})$ such that

$$A_{11}^n + A_{12}^n = A_{21}^n + A_{22}^n = \cdots = A_{m1}^n + A_{m2}^n,$$

which also holds simultaneously for all integral values of n, whether positive or negative. Further, he obtained a parametric solution over $M_2(\mathbb{Z})$ of the equation

$$A_1^n + A_2^n + A_3^n = B_1^n + B_2^n + B_3^n,$$

for all integral values of n. This solution leads to another arbitrarily long multigrade chain of matrices in $M_2(\mathbb{Z})$.

In the present work, we present a different approach to obtain solutions of the Tarry-Escott problem over $M_2(\mathbb{Z})$; our approach also provides additional solutions different from those of Choudhry. As an application of our main result, general solutions, over $M_2(\mathbb{Q})$, are determined for the diophantine equation

$$\alpha A^n + \beta B^n = \alpha C^n + \beta D^n,$$

where $\alpha, \beta \in \mathbb{Q}$ with $\alpha + \beta \neq 0$ and $n \in \{1, 2\}$.

2. Main Results

First, we prove an auxiliary result which will be used later.

Lemma 2.1. *Let m and n be positive integers, let $A_i, B_i \in M_m(\mathbb{Q})$ $(i = 1, \ldots, n)$, and let $\alpha_i \in \mathbb{Q}$. If*

$$\sum_{i=1}^{n} \alpha_i A_i = \sum_{i=1}^{n} \alpha_i B_i \quad \textit{and} \quad \sum_{i=1}^{n} \alpha_i A_i^2 = \sum_{i=1}^{n} \alpha_i B_i^2,$$

then

$$\sum_{i=1}^{n} \alpha_i (A_i + C) = \sum_{i=1}^{n} \alpha_i (B_i + C) \quad \textit{and} \quad \sum_{i=1}^{n} \alpha_i (A_i + C)^2 = \sum_{i=1}^{n} \alpha_i (B_i + C)^2$$

for any $C \in M_m(\mathbb{Q})$.

Proof. Since $\sum_{i=1}^{n} \alpha_i A_i = \sum_{i=1}^{n} \alpha_i B_i$ and $\sum_{i=1}^{n} \alpha_i A_i^2 = \sum_{i=1}^{n} \alpha_i B_i^2$, it is easy to see that

$$\sum_{i=1}^{n} \alpha_i (A_i + C) = \sum_{i=1}^{n} \alpha_i (B_i + C)$$

and

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i (A_i + C)^2 &= \sum_{i=1}^{n} \alpha_i A_i^2 + (\sum_{i=1}^{n} \alpha_i A_i)C + C(\sum_{i=1}^{n} \alpha_i A_i) + \sum_{i=1}^{n} \alpha_i C^2 \\ &= \sum_{i=1}^{n} \alpha_i B_i^2 + (\sum_{i=1}^{n} \alpha_i B_i)C + C(\sum_{i=1}^{n} \alpha_i B_i) + \sum_{i=1}^{n} \alpha_i C^2 \\ &= \sum_{i=1}^{n} \alpha_i (B_i + C)^2. \qquad \square\end{aligned}$$

Immediate from Lemma 2.1 is

Corollary 2.2. *Let m and n be positive integers and let*

$$A = \{A_1, A_2, \ldots, A_n\} \quad \textit{and} \quad B = \{B_1, B_2, \ldots, B_n\}$$

be subsets of $M_m(\mathbb{Q})$. If $A =_2 B$, then for any matrix $C \in M_m(\mathbb{Q})$ we have

$$A + C =_2 B + C$$

where

$$A + C = \{A_1 + C, A_2 + C, \ldots, A_n + C\}, B + C = \{B_1 + C, B_2 + C, \ldots, B_n + C\}.$$

We next define equivalent solutions.

Definition 2.3. Let k, m and n be positive integers. Let

$$A = \{A_1, \ldots, A_n\}, B = \{B_1, \ldots, B_n\}, X = \{X_1, \ldots, X_n\}, Y = \{Y_1, \ldots, Y_n\}$$

be subsets of $M_m(\mathbb{Q})$. We say that $A =_k B$ and $X =_k Y$ are equivalent if there exist M and N in $M_m(\mathbb{Q})$ such that for all i,

$$X_i = MA_i + N \quad \text{and} \quad Y_i = MB_i + N.$$

Definition 2.4. Let k, m and n be positive integers. Let $A = \{A_1, \ldots, A_n\}, B = \{B_1, \ldots, B_n\}$ be subsets of $M_m(\mathbb{Q})$. Then a solution $A =_k B$ is called a reduced solution if

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = \underline{0}.$$

The concept of being reduced is useful because of the next result.

Theorem 2.5. *Let m and n be positive integers. Every solution of size n and degree 2 of the Tarry-Escott Problem over $M_m(\mathbb{Q})$ is equivalent to a reduced solution.*

Proof. Let $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ be two subsets of $M_m(\mathbb{Q})$ such that $A =_2 B$. Now let $X = \{X_1, X_2, \ldots, X_n\}$ and $Y = \{Y_1, Y_2, \ldots, Y_n\}$, where $X_i = A_i - S$, $Y_i = B_i - S$ for $i = 1, \ldots, n$ and $S = (A_1 + \cdots + A_n)/n$. It is easy to see that

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i = \underline{0}.$$

Thus $X =_2 Y$ is a reduced solution. Since $A =_2 B$ and $X =_2 Y$, by Lemma 2.1, $A =_2 B$ is equivalent to a reduced solution $X =_2 Y$. $\square$

We now consider the so-called symmetric solutions of the Tarry-Escott Problem over $M_2(\mathbb{Q})$; these are integral matrices X and Y satisfying

$$X^n + (-X)^n = Y^n + (-Y)^n,$$

for all positive integers n. It suffices to show that $X^2 = Y^2$. We first recall a result from [4].

Theorem 2.6. *Let $c \in \mathbb{C}$. Suppose that X and Y are two elements in $M_2(\mathbb{C})$ such that $XY \neq YX$. If $X^2 + Y^2 = cI$ where I is the identity matrix, then* $\operatorname{tr}(X) = \operatorname{tr}(Y) = 0$ *and* $\det X + \det Y = -c$.

We prove now another auxiliary result.

Lemma 2.7. *Suppose X and Y are nonzero elements in $M_2(\mathbb{Q})$ and $X \neq Y$. Then $X^2 = Y^2$ and $XY = YX$ if and only if there exist nonzero matrices $A, B \in M_2(\mathbb{Q})$ such that*

$$AB = BA = \underline{0}, X = \frac{A+B}{2} \quad \text{and} \quad Y = \frac{A-B}{2}.$$

Proof. Suppose $X^2 = Y^2$ and $XY = YX$. Next, we let $A = X+Y$ and $B = X-Y$. Then the results follows easily. For the converse, we suppose that $X = (A+B)/2$ and $Y = (A-B)/2$ where $AB = BA = \underline{0}$. Then it is easy to see that $XY = YX$ and $X^2 = Y^2$. Hence the converse holds as desired. □

From Lemma 2.7, in order to find commutative solutions of $X^n + (-X)^n = Y^n + (-Y)^n$ for all positive integer n, it suffices to solve for matrices A and B such that $AB = BA = \underline{0}$, and this leads us to our first main result.

Theorem 2.8. *Suppose X and Y are nonzero elements in $M_2(\mathbb{Q})$. Then $\{X,-X\} =_2 \{Y,-Y\}$ if and only if X, Y belong to one of the following two classes.*

1. $XY \neq YX$, $X^2 = Y^2$, $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *and* $Y = \begin{pmatrix} w & x \\ y & -w \end{pmatrix}$ *where* a,b,c,w,x,y *are rationals such that* $a^2+bc = w^2+xy$ *and* ($bx \neq cx$ *or* $ax \neq bw$ *or* $ay \neq cw$).
2. $XY = YX$, $X^2 = Y^2$, *and there exist nonzero matrices* $A, B \in M_2(\mathbb{Q})$ *such that*
$$AB = BA = \underline{0}, X = \frac{A+B}{2} \quad \text{and} \quad Y = \frac{A-B}{2}.$$
where A and B are of the following forms:
   (a) $A = \begin{pmatrix} a & ma \\ c & mc \end{pmatrix}$ *and* $B = \begin{pmatrix} w & x \\ -\frac{w}{m} & -\frac{x}{m} \end{pmatrix}$ *where* $acmw \neq 0$ *and* $aw + xc = 0$,
   (b) $A = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ *and* $B = \begin{pmatrix} w & 0 \\ -\frac{cw}{d} & 0 \end{pmatrix}$ *where* $cdw \neq 0$,
   (c) $A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ *and* $B = \begin{pmatrix} w & -\frac{bw}{d} \\ 0 & 0 \end{pmatrix}$ *where* $bdw \neq 0$,
   (d) $A = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ *and* $B = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$ *where* $dw \neq 0$,
   (e) $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ *where* $cy \neq 0$,
   (f) $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ *where* $bx \neq 0$.

Proof. Suppose $X^2 = Y^2$.
Case 1: $XY \neq YX$. Then by Theorem 2.6, $\operatorname{tr}(X) = \operatorname{tr}(Y) = 0$ and $\det X + \det iY = 0$. Let $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, Y = \begin{pmatrix} w & x \\ y & -w \end{pmatrix}$ where $a,b,c,w,x,y \in \mathbb{Q}$. Since $\det X + \det iY = 0$, $\det X = \det Y$. This implies that $a^2+bc = w^2+xy$ as desired. Now note that $XY = \begin{pmatrix} aw+by & ax-bw \\ cw-ay & cx+aw \end{pmatrix}$ and $YX = \begin{pmatrix} aw+cx & bw-ax \\ ay-cw & by+aw \end{pmatrix}$. Thus we have $by \neq cx$ or $ax \neq bw$ or $ay \neq cw$.

Case 2: $XY = YX$. By Lemma 2.7, there exist $A, B \in M_2(\mathbb{Q})$ such that $AB = BA = \underline{0}$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Thus we have the following system of equations:

$$aw + by = aw + cx = 0 \tag{2.1}$$

$$ax + bz = bw + xd = 0 \tag{2.2}$$

$$cw + dy = ay + cz = 0 \tag{2.3}$$

$$cx + dz = by + dz = 0 \tag{2.4}$$

Since $AB = \underline{0}$, this implies that $\det A = 0$ or $\det B = 0$. We may assume that $\det A = 0$. Thus $ad - bc = 0$.

Case 2.1: $abcd \neq 0$. Since $ad - bc = 0$, $a/b = c/d$. Let $m = b/a$. By (2.1), $y = -w/m$. By (2.2), $z = -x/m$. Thus $A = \begin{pmatrix} a & ma \\ c & mc \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ -\frac{w}{m} & -\frac{x}{m} \end{pmatrix}$ as desired.

Case 2.2: $ad = bc = 0$. Thus there are 4 cases to consider.

Case 2.2.1: $a = b = 0$. Then (2.1)-(2.4) imply that

$$cx = xd = cw + dy = cz = cx + dz = dz = 0.$$

Since at least one of c and d is nonzero, we have $x = z = 0$. If $cd \neq 0$ then $cw + dy = 0$. Then $y = -cw/d$. Thus we obtain solutions of the form $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ and $B = \begin{pmatrix} w & 0 \\ -\frac{cw}{d} & d \end{pmatrix}$ where $cdw \neq 0$.

Case 2.2.1(i): $c = 0$. Then $y = 0$. The fact that A, B are nonzero matrices implies $dw \neq 0$. Thus we obtain solutions of the form $A = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$ where $dw \neq 0$.

Case 2.2.1(ii): $d = 0$. Then $w = 0$. Again since A, B are nonzero matrices, we have $cy \neq 0$. So we obtain solutions of the form $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ where $cy \neq 0$.

Case 2.2.2: $a = c = 0$. Then (2.1)-(2.4) imply that

$$by = bz = bw + dx = dy = dz = by + dz = 0.$$

Since at least one of b and d is nonzero, we have $y = z = 0$.

Case 2.2.2(i): $bd \neq 0$. Then $bw + dx = 0$ and $x = -bw/d$. Thus we obtain solutions of the form $A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} w & -\frac{bw}{d} \\ 0 & 0 \end{pmatrix}$ where $bdw \neq 0$.

Case 2.2.2(ii): $b = 0$. Then $x = 0$. The fact that A, B are nonzero matrices implies $dw \neq 0$. Thus we obtain solutions of the form $A = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ and $B = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}$ where $dw \neq 0$.

Case 2.2.2(iii): $d = 0$. Then $w = 0$. Again since A and B are nonzero matrices, we have $bx \neq 0$. So we obtain solutions of the form $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ where $bx \neq 0$.

For the case $b = d = 0$ and $c = d = 0$, we proceed similarly and obtain solutions as shown in the previous cases. The converse is easily checked. □

We next provide an example.

Example 2.9. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, B = \begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix}$. It is easy to see that $AB = BA = \underline{0}$. Next, we let $X = (A+B)/2$ and $Y = (A-B)/2$. Then

$$X = \begin{pmatrix} 5/2 & 3 \\ -3/2 & -2 \end{pmatrix}, Y = \begin{pmatrix} -3/2 & -1 \\ 1/2 & 0 \end{pmatrix}.$$

By Lemma 2.7, $XY = YX$ and $X^2 = Y^2$. Thus $\{X, -X\} =_n \{Y, -Y\}$ for any positive integer n. Moreover, since X and Y are nonsingular matrices, $\{X, -X\} =_n \{Y, -Y\}$ for all integer n.

We proceed now to our final task which is to solve

$$\alpha_1 X_1^n + \alpha_2 X_2^n = \alpha_1 Y_1^n + \alpha_2 Y_2^n$$

for $n = 1$ and 2 over $M_2(\mathbb{Q})$. We need two more auxiliary results.

Lemma 2.10. *Let m be a positve integer. Let A_1, A_2, B_1, B_2 be matrices in $M_m(\mathbb{Q})$. Let α_1 and α_2 be nonzero rational numbers such that $\alpha_1 + \alpha_2 \neq 0$. If*

$$\alpha_1 A_1^n + \alpha_2 A_2^n = \alpha_1 B_1^n + \alpha_2 B_2^n$$

for $n = 1, 2$ then there exist A_i' and B_i' in $M_m(\mathbb{Q})$ such that

$$\alpha_1 A_1' + \alpha_2 A_2' = \underline{0} = \alpha_1 B_1' + \alpha_2 B_2' \quad \text{and} \quad \alpha_1 A_1'^2 + \alpha_2 A_2'^2 = \alpha_1 B_1'^2 + \alpha_2 B_2'^2.$$

Proof. Suppose $\alpha_1 A_1^n + \alpha_2 A_2^n = \alpha_1 B_1^n + \alpha_2 B_2^n$ for $n = 1, 2$ and $\alpha_1\alpha_2(\alpha_1 + \alpha_2) \neq 0$. For $i = 1, 2$, we define

$$S = \alpha_1 A_1 + \alpha_2 A_2, A_i' = A_i - \frac{S}{\alpha_1 + \alpha_2}, B_i' = B_i - \frac{S}{\alpha_1 + \alpha_2}.$$

Then $\alpha_1 A_1' + \alpha_2 A_2' = \alpha_1 B_1' + \alpha_2 B_2' = 0$. Since $\alpha_1 A_1^n + \alpha_2 A_2^n = \alpha_1 B_1^n + \alpha_2 B_2^n$ for $n = 1, 2$, by Lemma 2.1, we have $\alpha_1 A_1'^2 + \alpha_2 A_2'^2 = \alpha_1 B_1'^2 + \alpha_2 B_2'^2$. □

Lemma 2.11. *Let m be a positive integer. Let α_1 and α_2 be nonzero rational numbers such that $\alpha_1 + \alpha_2 \neq 0$. There exist $A_1, A_2, B_1, B_2 \in M_m(\mathbb{Q})$ such that*

$\alpha_1 A_1 + \alpha_2 A_2 = \alpha_1 B_1 + \alpha_2 B_2 = \underline{0}$ and $\alpha_1 A_1^2 + \alpha_2 A_2^2 = \alpha_1 B_1^2 + \alpha_2 B_2^2$ if and only if $A_1^2 = B_1^2$.

Proof. Suppose $\alpha_1 A_1 + \alpha_2 A_2 = \alpha_1 B_1 + \alpha_2 B_2 = \underline{0}$, $A_2 = (-\alpha_1 A_1)/\alpha_2$ and $B_2 = (-\alpha_1 B_1)/\alpha_2$. Then

$$\left(\alpha_1 + \alpha_1^2/\alpha_2\right) A_1^2 = \alpha_1 A_1^2 + \alpha_2 A_2^2 = \alpha_1 B_1^2 + \alpha_2 B_2^2 = \left(\alpha_1 + \alpha_1^2/\alpha_2\right) B_1^2.$$

Thus $A_1^2 = B_1^2$.

For the converse, suppose $A_2 = (-\alpha_1 A_1)/\alpha_2$ and $B_2 = (-\alpha_1 B_1)/\alpha_2$. Then

$$\alpha_2 A_2 + \alpha_1 A_1 = \alpha_1 B_1 + \alpha_2 B_2 = \underline{0}.$$

Since $A_1^2 = B_1^2$, we obtain that

$$\alpha_1 A_1^2 + \alpha_2 A_2^2 = \alpha_1 B_1^2 + \alpha_2 B_2^2.$$

□

Combining the results of Lemmas 2.10 and 2.11, we arrive at our final main result.

Theorem 2.12. *Let α_1 and α_2 be nonzero rational numbers such that $\alpha_1 + \alpha_2 \neq 0$. Then all solutions of the diophantine equation*

$$\alpha_1 X_1^n + \alpha_2 X_2^n = \alpha_1 Y_1^n + \alpha_2 Y_2^n$$

for $n = 1$ and 2 over $M_2(\mathbb{Q})$ are the form $X_1 = A_1 + C$, $X_2 = (-\alpha_1 A_1)/\alpha_2 + C$, $Y_1 = B_1 + C$, $Y_2 = (-\alpha_1 B_1)/\alpha_2 + C$ where $A_1^2 = B_1^2$ and $C \in M_2(\mathbb{Q})$.

We end this paper with an example.

Example 2.13. We show how to solve the following system of equations

$$2X_1^n + 3X_2^n = 2Y_1^n + 3Y_2^n \qquad (1 \le n \le 2)$$

over $M_2(\mathbb{Q})$.

By Lemma 2.10, we can restrict our attention to find solutions A_1, A_2, B_1, B_2 in $M_2(\mathbb{Q})$ such that

$$2A_1 + 3A_2 = 2B_1 + 3B_2 = \underline{0} \quad \text{and} \quad 2A_1^2 + 3A_2^2 = 2B_1^2 + 3B_2^2.$$

By Theorem 2.8, it suffices to find $A_1, B_1 \in M_2(\mathbb{Q})$ such that $A_1^2 = B_1^2$. We work out some solutions for two different cases.

Case 1: $A_1 B_1 = B_1 A_1$. Let $A_1 = \begin{pmatrix} 5/2 & 3 \\ -3/2 & -2 \end{pmatrix}$ and $B_1 = \begin{pmatrix} -3/2 & -1 \\ 1/2 & 0 \end{pmatrix}$. Then we have

$$A_2 = \frac{-2}{3} A_1 = \begin{pmatrix} -5/3 & -2 \\ 1 & 4/3 \end{pmatrix} \text{ and } B_2 = \frac{-2}{3} B_1 \begin{pmatrix} 1 & 2/3 \\ -1/3 & 0 \end{pmatrix}.$$

Case 2: $A_1B_1 \neq B_1A_1$. Let $A_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 3 & 1 \\ -9 & -3 \end{pmatrix}$. Then

$$A_2 = \frac{-2}{3}A_1 = \begin{pmatrix} -4/3 & -2/3 \\ 8/3 & 4/3 \end{pmatrix} \text{ and } B_2 = \frac{-2}{3}B_1 = \begin{pmatrix} -2 & -2/3 \\ 6 & 2 \end{pmatrix}.$$

Therefore,

$$2(A_1 + C)^n + 3(A_2 + C)^n = 2(B_1 + C)^n + 3(B_2 + C)^n$$

for $n = 1, 2$ where $C \in M_2(\mathbb{Q})$.

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