KYUNGPOOK Math. J. 63(2023), 333-344
https://doi.org/10.5666/KMJ.2023.63.3.333
pISSN 1225-6951 eISSN 0454-8124

Weak u-S-flat Modules and Dimensions

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Abstract. In this paper, we generalize the notions uniformly S-flat, briefly u-S-flat, modules and dimensions. We introduce and study the notions of weak u-S-flat modules. An R-module M is said to be weak u-S-flat if $\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion for any ideal I of R. This new class of modules will be used to characterize u-S-von Neumann regular rings. Hence, we introduce the weak u-S-flat dimensions of modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed.

1. Introduction

Throughout this article, all rings considered are commutative with unity, all modules are unital and S always is a multiplicative subset of R, that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let R be a ring and M an R-module. Recall from Zhang, [3],that an R-module M is said to be uniformly S-torsion if $sT = 0$ for some $s \in S$. The abbreviateion u- will always stand for 'uniformly'. An R-module M is S-finite if and only if M/F is u-S-torsion for some finitely generated submodule F of M. In the same way, Zhang defined an R-sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ to be u-S-exact (at N) provided that there is an element $s \in S$ such that $s\,\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\,\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. We say a long R-sequence $\ldots \longrightarrow A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \longrightarrow \ldots$ is u-S-exact, if for any n there is an element $s \in S$ such that $s\,\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n)$ and $s\,\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1})$. A u-S-

* Corresponding Author.
Received August 21, 2022; revised November 17, 2022; accepted November 28, 2022.
2020 Mathematics Subject Classification: 13D05, 13D07, 13H05.
Key words and phrases: flat module, u-S-flat module, weak u-S-flat module, u-S-torsion, u-S-exact sequence, u-S-von Neumann regular ring.

exact sequence $0 \to A \to B \to C \to 0$ is called a short u-S-exact sequence. An R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. It is easy to verify an R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{CoKer}(f)$, both $\mathrm{Ker}(f)$ and $\mathrm{CoKer}(f)$) is a u-S-torsion module.
In [3], Zhang introduced the class of u-S-flat modules F for which the functor $F \otimes_R -$ preserves u-S-exact sequences. The class of u-S-flat modules can be seen as a uniform generalization of that of flat modules, since an R-module F is u-S-flat if and only if $\mathrm{Tor}_1^R(F, M)$ is u-S-torsion for any R-module M. The class of u-S-flat modules has the following u-S-hereditary property: let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence, if B and C are u-S-flat so is A (see [[3], Proposition 3.4]).
In [5], the author introduced the u-S-flat dimensions of modules and rings. Let R be a ring, S a multiplicative subset of R and n be a positive integer. We say that an R-module has a u-S-flat dimension less than or equal to n, $u\text{-}S\text{-fd}_R(M) \le n$, if $\mathrm{Tor}_{n+1}^R(M, N)$ is u-S-torsion R-module for all R-modules N. Hence, the u-S-weak global dimension of R is defined to be

$$u\text{-}S\text{-w.gl.dim}(R) = \sup\{u\text{-}S\text{-fd}_R(M) \mid \ M \text{ is an } R\text{-module}\}.$$

As in [4], a u-S-exact sequence of R-modules $0 \to A \to B \to C \to 0$ is said to be u-S-pure provided that for any R-module M, the induced sequence $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is also u-S-exact, and a submodule A of B is called a u-S-pure submodule if the exact sequence $0 \to A \to B \to B/A \to 0$ is u-S-pure exact.
In [3], Zhang defined the u-S-von Neumann regular ring as follows: Let R be a ring and S a multiplicative subset of R. R is called a u-S-von Neumann regular ring provided there exists an element $s \in S$ such that for any $a \in R$ there exists $r \in R$ with $sa = ra^2$. Thus by [[3], Theorem 3.13], R is a u-S-von Neumann regular ring if and only if every R-module is u-S-flat.

In Section 2, we introduce the concept of w-u-S-flat modules and we study some characterization of w-u-S-flat modules. Hence, we prove that a ring R is u-S-von Neumann regular if and only if every R-module is w-u-S-flat. We prove also, if an R-module F is w-u-S-flat, then F_S is flat over R_S. A new local characterization of flat modules also is given. Section 3 deals with the w-u-S-flat dimension of modules and rings. After a routine study of these dimensions, we prove that R is a u-S-von Neumann regular ring if and only if $w\text{-}u\text{-}S\text{-w.gl.dim}(R) = 0$ if and only if every R/I is w-u-S-flat for any ideal I of R.

2. Weak u-S-flat Modules

In this section, we introduce a class of modules called weak u-S-flat modules and we study their properties and give their characterizations. The abbreviation w- always stands for ‘weak’. We start with the following definition.

Definition 2.1. An R-module M is said to be w-u-S-flat if $\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion for any ideal I of R.

Obviously, every u-S-flat module is w-u-S-flat. If S is consist of units, then w-u-S-flat modules and u-S-flat modules coincide.

Remark 2.2. *Let $R = \mathbb{Z}$ the ring of integers, p a prime in $\mathbb{Z}$ and $S = \{p^n | n \geq 0\}$. Let $M = \mathbb{Z}_{(p)}/\mathbb{Z}$ be a $\mathbb{Z}$-module where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at S. By Exapmle [[3], Example 3.3], we have M is w-u-S-flat but not u-S-flat.*

Recall from [[2], Theorem 2.5.6], that an R-module M is flat if and only if for any (finitely generated) ideal I of R, $0 \to I \otimes_R M \to R \otimes_R M$ is exact if and only if for any (finitely generated) ideal I of R, the natural homomorphism $0 \to I \otimes_R M \to IM$ is an isomorphism. We give a u-S-analogue of this result.

Proposition 2.3. *Let R be a ring, S be a multiplicative subset of R, and M be an R-module. The following are equivalent:*

1. *M is w-u-S-flat.*
2. *$\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion for any finitely generated ideal I of R.*
3. *The natural homomorphism $I \otimes_R M \to R \otimes_R M$ is a u-S-monomorphism, for any ideal I of R,.*
4. *The natural homomorphism $I \otimes_R M \to R \otimes_R M$ is a u-S-monomorphism, for any finitely generated ideal I of R.*
5. *The natural homomorphism $\mu_I : I \otimes_R M \to IM$ is a u-S-isomorphism, for any ideal I of R.*
6. *The natural homomorphism $\mu_I : I \otimes_R M \to IM$ is a u-S-isomorphism, for any finitely generated ideal I of R.*

Proof. The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) are obvious.
(1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4). Let I be a (finitely generated) ideal of R. Then we have a long exact sequence:

$$0 \to \mathrm{Tor}_1^R(R/I, M) \to I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0$$

Consequently, $\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion if and only if $I \otimes_R M \to R \otimes_R M$ is a u-S-monomorphism.
(3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6). Let I be a (finitely generated) ideal of R. Then we have the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I \otimes_R M & \longrightarrow & R \otimes_R M \\ & & \downarrow{\scriptstyle \mu_I} & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & IM & \longrightarrow & M \end{array}$$

Then, μ_I is a u-S-monomorphism. Since the multiplicative map μ_I is an epimorphism, μ_I is a u-S-isomorphism.
(6) $\Rightarrow$ (3). Let I be an ideal of R. We just need to show $\mathrm{Ker}(\mu_I)$ is u-S-torsion. Suppose that $\mu_I(\Sigma_{i=1}^n(a_i \otimes x_i)) = \Sigma_{i=1}^n a_i x_i = 0$, $a_i \in I$, $x_i \in M$. Let $I_0 = Ra_1 + \cdots + Ra_n$. Hence, $I_0 \subseteq I$. Consider the following commutative diagram:

$$\begin{array}{ccc} I_0 \otimes_R M & \xrightarrow{g} & I \otimes_R M \\ \downarrow{\scriptstyle \mu_{I_0}} & & \downarrow{\scriptstyle \mu_I} \\ I_0 M & \xrightarrow{h} & IM \end{array}$$

By (6), μ_{I_0} is u-S-isomorphism. Thus, there exists $s \in S$ such that $s\Sigma_{i=1}^n a_i \otimes x_i = 0$ in $I_0 \otimes_R M$. Since h is a monomorphism, g is a u-S-monomorphism. Hence, there exists $s' \in S$ such that $s'\Sigma_{i=1}^n a_i \otimes x_i = 0$ in $I \otimes_R M$, which implies that $\mathrm{Ker}(\mu_I)$ is u-S-torsion. □

Corollary 2.4. *Let R be a ring, S be a multiplicative subset of R and M be an R-module. The class of w-u-S-flat R-modules is closed under u-S-isomorphisms.*
Proof. Let $f : M \to N$ be a u-S-isomorphisms, and I be an ideal of R. There exists two exact sequence $0 \to T_1 \to M \to L \to 0$ and $0 \to L \to N \to T_2 \to 0$ with T_1 and T_2 u-S-torsion. Consider the induced two long exact sequence, $\mathrm{Tor}_1^R(R/I, T_1) \to \mathrm{Tor}_1^R(R/I, M) \to \mathrm{Tor}_1^R(R/I, L) \to R/I \otimes_R T_1$ and $\mathrm{Tor}_2^R(R/I, T_2) \to \mathrm{Tor}_1^R(R/I, L) \to \mathrm{Tor}_1^R(R/I, N) \to \mathrm{Tor}_1^R(R/I, T_2)$. By [[3], Corollary 2.6], M is w-u-S-flat if and only if N is w-u-S-flat. □

Proposition 2.5. *Let R be a ring, S be a multiplicative subset of R. R is u-S-von Neumann regular ring if and only if every R-module of R is w-u-S-flat.*

Proof. $\Rightarrow$. By [[3], Theorem 3.13].
$\Leftarrow$. Let I and J be ideals of R. We have $\mathrm{Tor}_1^R(R/I, R/J)$ u-S-torsion since R/J is w-u-S-flat. Thus, there exsits $s \in S$ such that $s\,\mathrm{Tor}_1^R(R/I, R/J) = 0$. So, R is u-S-von Neumann regular by [[3], Theorem 3.13]. □

Remark 2.6. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$ be a semi-simple ring and $s = (1, 0) \in T$. Then any element $a \in T$ satisfies $a^2 = a$ and $2a = 0$. Let $R = T[x]/\langle sx, x^2\rangle$ with x the indeterminate and $S = \{1, s\}$ be a multiplicative subset of R. By [[3], Example 3.18], R is u-S-von Neumann regular and not von Neumann regular, so there exsits an R-module which is w-u-S-flat but not flat (see, Proposition 2.5).

Recall that an R-module M is said to be an S-torsion-free module if $sx = 0$, for $s \in S$ and $x \in M$, implies $x = 0$.

Lemma 2.7. *Let R be a ring, S be a multiplicative subset of R, and M be an R-module. If M is a w-u-S-flat, then* $\mathrm{Hom}_R(M, E)$ *is injective for any injective S-torsion-free R-module E.*

Proof. Let I be an ideal of R and E be an injective S-torsion-free. By [[2], Theorem 3.4.11] we have the isomorphism

$$\mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, M), E).$$

Since, M is w-u-S-flat, we have that $\mathrm{Tor}_1^R(R/I, M)$ is u-S-torsion and by [[3], Proposition 2.5] we have $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, M), E) = 0$. Thus, $\mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(M, E)) = 0$ which implies that $\mathrm{Hom}_R(M, E)$ is injective. □

Proposition 2.8. *Let R be a ring, S be a multiplicative subset of R. Then the following statements hold.*

1. *The class of all w-u-S-flat modules is closed under pure submodules and pure quotients.*
2. *Any finite direct sum of w-u-S-flat modules is w-u-S-flat.*
3. *Let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence. If A is u-S-torsion. Then B is w-u-S-flat if and only if C is w-u-S-flat.*
4. *Let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence. If C is w-u-S-flat with u-S-$fd_R(C) \leq 1$. Then A is w-u-S-flat if and only if B is w-u-S-flat.*

Proof. (1). Let I be an ideal of R. Suppose $0 \to M \to N \to L \to 0$ is a pure exact sequence. We have the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_R I & \longrightarrow & N \otimes_R I & \longrightarrow & L \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & M \otimes_R R & \longrightarrow & N \otimes_R R & \longrightarrow & L \otimes_R R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R R/I & \longrightarrow & N \otimes_R R/I & \longrightarrow & L \otimes_R R/I & \longrightarrow & 0
\end{array}$$

By the S-analogue of the Five Lemma (see[[5], Theorem 1.3]), the natural homomorphism $f : M \otimes_R I \to M \otimes_R R$ and $g : L \otimes_R I \to L \otimes_R R$ are all u-S-monomorphisms. Consequently, M and L are all w-u-S-flat by Proposition 2.3.
(2). Let $F_1, \ldots, F_n$ be a w-u-S-flat modules and I be an ideal of R. Then, there exists $s_i \in S$ such that $s_i \mathrm{Tor}_1^R(R/I, F_i) = 0$. Set $s = s_1 \ldots s_n$. Thus,

$$s \,\mathrm{Tor}_1^R(R/I, \bigoplus_{i=1}^{n} F_i) = \bigoplus_{i=1}^{n} s \,\mathrm{Tor}_1^R(R/I, F_i) = 0,$$

which implies that $\bigoplus_{i=1}^n F_i$ is w-u-S-flat.

(3). Let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence and I be an ideal of R. By [[5], Theorem 1.5], we have the following u-S-exact sequence

$$\mathrm{Tor}_1^R(R/I, A) \to \mathrm{Tor}_1^R(R/I, B) \to \mathrm{Tor}_1^R(R/I, C) \to R/I \otimes_R A.$$

Since A is u-S-torsion, we get that $\mathrm{Tor}_1^R(R/I,A)$ and $R/I\otimes_R A$ are u-S-torsion by [[3], Corollary 2.6]. Hence, $\mathrm{Tor}_1^R(R/I,B)$ u-S-torsion if and only if $\mathrm{Tor}_1^R(R/I,C)$ u-S-torsion, which implies that B is w-u-S-flat if and only if C is w-u-S-flat.
(4). Let $0\to A\to B\to C\to 0$ be a u-S-exact sequence and I be an ideal of R. By [[5], Theorem 1.5], we have the following u-S-exact sequence

$$\mathrm{Tor}_2^R(R/I,C)\to \mathrm{Tor}_1^R(R/I,A)\to \mathrm{Tor}_1^R(R/I,B)\to \mathrm{Tor}_1^R(R/I,C).$$

The left term is u-S-torsion by [[5], Proposition 2.3] and the right term is u-S-torsion since C is w-u-S-flat. Hence, $\mathrm{Tor}_1^R(R/I,A)$ u-S-torsion if and only if $\mathrm{Tor}_1^R(R/I,B)$ u-S-torsion, which implies that A is w-u-S-flat if and only if B is w-u-S-flat. □

Lemma 2.9. *Let R be a ring and S a multiplicative subset of R. If A is a flat R-module and B a w-u-S-flat R-module, then, $A\otimes_R B$ is w-u-S-flat R-module.*

Proof. Let I be an ideal of R. By [[2], Theorem 3.4.10] we have the isomorphism

$$\mathrm{Tor}_1^R(R/I,A\otimes_R B)\cong A\otimes_R \mathrm{Tor}_1^R(R/I,B).$$

For any $s\in S$ we have

$$s\,\mathrm{Tor}_1^R(R/I,A\otimes_R B)\cong s(A\otimes_R \mathrm{Tor}_1^R(R/I,B)) = A\otimes_R s\,\mathrm{Tor}_1^R(R/I,B).$$

Since B is a w-u-S-flat, $\mathrm{Tor}_1^R(R/I,B)$ is a u-S-torsion with respect to, say s. So $s\,\mathrm{Tor}_1^R(R/I,B)=0$. Thus,

$$s\,\mathrm{Tor}_1^R(R/I,A\otimes_R B)\cong A\otimes_R 0.$$

Hence, $\mathrm{Tor}_1^R(R/I,A\otimes_R B)$ is u-S-torsion. Then, $A\otimes_R B$ is a w-u-S-flat. □

Proposition 2.10. *Let R be a ring, S be a multiplicative subset of R. If M is w-u-S-flat over a ring R, then M_S is flat over R_S. The converse holds if S consists of finite elements.*

Proof. Let I_S be an ideal of R_S, where I is an ideal of R. Then there exists $s\in S$ such that $s\,\mathrm{Tor}_1^R(R/I,M)=0$. Hence, by [[2], Theorem 3.4.12], we have $0=\mathrm{Tor}_1^R(R/I,M)_S\cong \mathrm{Tor}_1^{R_S}(R_S/I_S,M_S)$. So M_S is flat over R_S. For the converse, let I be an ideal of R. By [[2], Theorem 3.4.12] again, we have $\mathrm{Tor}_1^R(R/I,M)_S=0$ which implies that $\mathrm{Tor}_1^R(R/I,M)$ is S-torsion by [[2], Example 1.6.13]. Hence, $\mathrm{Tor}_1^R(R/I,M)$ is u-S-torsion by [[3], Proposition 2.3] and so M is w-u-S-flat. □

By Proposition 2.10 and [[3], Proposition 3.8] we have the following corollary.

Corollary 2.11. *Let R be a ring, S be a multiplicative subset of R consisting of finite elements. Then, every w-u-S-flat R-module is u-S-flat.*

Let $\mathfrak{p}$ be a prime ideal of R. We say an R-module M is w-u-$\mathfrak{p}$-flat shortly provided that M is w-u-$(R-\mathfrak{p})$-flat.

Proposition 2.12. *Let R be a ring and M an R-module. Then the following statements are equivalent:*

1. *M is flat.*
2. *M is w-u-$\mathfrak{p}$-flat for any $\mathfrak{p} \in \mathrm{Spec}(R)$.*
3. *M is w-u-$\mathfrak{m}$-flat for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). These are trivial.
(3) $\Rightarrow$ (1). Let I be an ideal of R. Hence, $\mathrm{Tor}_1^R(R/I, M)$ is u-$(R - \mathfrak{m})$-torsion. Then, for any $\mathfrak{m} \in \mathrm{Max}(R)$, there exists $s_{\mathfrak{m}} \in S$ such that $s_{\mathfrak{m}} \mathrm{Tor}_1^R(R/I, M) = 0$. Since the ideal generated by all $s_{\mathfrak{m}}$ is R, $\mathrm{Tor}_1^R(R/I, M) = 0$. So M is flat. □

Let R be a ring and M an R-module. $R[x]$ denotes the polynomial ring with one indeterminate, where all coefficients are in R. Set $M[x] = M \otimes_R R[x]$, then $M[x]$ can be seen as an $R[x]$-module naturally.

Proposition 2.13. *Let R be a ring, S be a multiplicative subset of R and M is an $R[x]$-module. If M is w-u-S-flat over $R[x]$, then M is w-u-S-flat over R.*

Proof. Suppose that M is a w-u-S-flat $R[x]$-module. Then it is easy to verify that $M[x]$ is also a w-u-S-flat $R[x]$-module. By [[1], Theorem 1.3.11], $\mathrm{Tor}_1^R(R/I, M)[x] \cong \mathrm{Tor}_1^{R[x]}((R/I)[x], M[x]) = \mathrm{Tor}_1^{R[x]}(R[x]/I[x], M[x])$ is u-S-torsion. Hence, there exists an element $s \in S$ such that $s \,\mathrm{Tor}_1^R(R/I, M)[x] = 0$. Thus, $s \,\mathrm{Tor}_1^R(R/I, M) = 0$. It follows that M is a w-u-S-flat R-module. □

3. The Weak u-S-flat Dimension of Modules and Rings

Let R be a ring. The flat dimension of an R-module M is defined as the shortest flat resolution of M. In this section, we introduce and investigate the notion of weak u-S-flat dimension of modules and rings as follows.

Defenition 3.1. If M is an R-module, then w-u-S-$\mathrm{fd}_R(M)$ (w-u-S-fd abbreviates weak u-S-flat dimension) if there is a u-S-exact sequence of R-modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad (*)$$

where each F_i is a u-S-flat $(i = 0, \cdots, n-1)$ and F_n is w-u-S-flat. The u-S-exact sequence $(*)$ is called a w-u-S-flat u-S-resolution of length n of M. If no such finite w-u-S-flat u-S-resolution exists, then w-u-S-$\mathrm{fd}_R(M) = \infty$; otherwise, define w-u-S-$\mathrm{fd}_R(M) = n$ if n is the length of a shortest w-u-S-flat u-S-resolution of M.
The weak u-S-flat dimension of R is defined by:

$$w\text{-}u\text{-}S\text{-w.gl.dim}(R) = \sup\{w\text{-}u\text{-}S\text{-fd}_R(M) : M \text{ is an } R\text{-module}\}.$$

Obviously, w-u-S-$\mathrm{fd}_R(M) \le u$-S-$\mathrm{fd}_R(M) \le \mathrm{fd}_R(M)$, with equality when S is composed of units. However, this inequality may be strict (see, Remarks 2.2 and 2.6). It is also obvious that an R-module M is w-u-S-flat if and only if w-u-S-$\mathrm{fd}_R(M) = 0$. Also, w-u-S-w.gl.dim$(R) \le u$-S-w.gl.dim$(R) \le$ w.gl.dim(R), with equality when S

is composed of units, and this inequality may be strict (see, Proposition 2.5 and [[3], Example 3.18]).

By Corollary 2.4, we have the following Lemma.

Lemma 3.2. *Let R be a ring, S a multiplicative subset of R. If A is u-S-isomorphic to B, then w-u-S-$\mathrm{fd}_R(A) = w$-u-S-$\mathrm{fd}_R(B)$.*

In the next result, we give a description of the w-u-S-flat dimension of modules.

Proposition 3.3. *Let R be a ring and S be a multiplicative subset of R. The following statements are equivalent for an R-module M.*

1. *w-u-S-$\mathrm{fd}_R(M) \leqslant n$.*
2. *$\mathrm{Tor}^R_{n+1}(R/I, M)$ is u-S-torsion for any ideal I of R.*
3. *$\mathrm{Tor}^R_{n+1}(R/I, M)$ is u-S-torsion for any finitely genrated ideal I of R.*
4. *If the sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ is an exact with $F_0, \cdots, F_{n-1}$ are flat R-modules, then F_n is w-u-S-flat.*
5. *If the sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ is a u-S-exact with $F_0, \cdots, F_{n-1}$ are u-S-flat R-modules, then F_n is w-u-S-flat.*
6. *If the sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ is an exact with $F_0, \cdots, F_{n-1}$ are u-S-flat R-modules, then F_n is w-u-S-flat.*
7. *If the sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ is a u-S-exact with $F_0, \cdots, F_{n-1}$ are flat R-modules, then F_n is w-u-S-flat.*
8. *There exists a u-S-exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, where $F_0, \cdots, F_{n-1}$ are flat R-modules and F_n is w-u-S-flat.*
9. *There exists an exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, where $F_0, \cdots, F_{n-1}$ are flat R-modules and F_n is w-u-S-flat.*
10. *There exists an exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, where $F_0, \cdots, F_n$ are w-u-S-flat.*

Proof. (1) $\Rightarrow$ (2). We prove (2) by induction on n. For the case $n = 0$, (2) holds by Proposition 2.3 as M is a w-u-S-flat module. If $n > 0$, then there is a u-S-exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ with all F_i u-S-flat ($i = 0, \cdots, n-1$) and F_n is w-u-S-flat. Let $K_0 = \ker(F_0 \to M)$. We have two u-S-exact sequences $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \ldots \to F_1 \to K_0 \to 0$. We note that w-u-S-$\mathrm{fd}_R(K_0) \leqslant n-1$. Hence, by induction we have, $\mathrm{Tor}^R_n(R/I, K_0)$ is u-S-torsion for any ideal I of R. Thus, it follows from [[5], Corollary 1.6], that $\mathrm{Tor}^R_n(R/I, M))$ is u-S-torsion.
(2) $\Rightarrow$ (3). This is obvious.
(3) $\Rightarrow$ (4). Let $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ be an exact sequence.

Set $K_0 = \ker(F_0 \to M)$ and $K_i = \ker(F_i \to F_{i-1})$, where $(i = 1, \dots, n-1)$. Since all $F_0, F_1, \dots, F_{i-1}$ are flat, $\mathrm{Tor}_1^R(R/I, F_n) \cong \mathrm{Tor}_{n+1}^R(R/I, M)$ is u-S-torsion for all finitely generated ideal I of R. Thus, F_n is a w-u-S-flat module by Proposition 2.3.
(4) ⇒ (1). Trivial.
(3) ⇒ (5). Let $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ be a u-S-exact sequence. Set $L_n = F_n$ and $L_i = \mathrm{Im}(F_i \to F_{i-1})$, where $(i = 1, \dots, n-1)$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are u-S-exact sequences. By using [[5], Corollary 1.6] repeatedly, we can obtain that $\mathrm{Tor}_1^R(F_n, R/I)$ is u-S-torsion for all finitely generated ideal I of R, which implies that F_n is w-u-S-flat by Proposition 2.3.
(5) ⇒ (6) ⇒ (4) and (5) ⇒ (7) ⇒ (4). These implications are trivial.
(4) ⇒ (9). Let $\dots \to P_n \to P_{n-1} \xrightarrow{f} P_{n-2} \dots \to P_0 \to M \to 0$ be a projective resolution of M. Set $F_n = \mathrm{Ker}(f)$. Then we have an exact sequence $0 \to F_n \to P_{n-1} \xrightarrow{f} P_{n-2} \dots \to P_0 \to M \to 0$. By (4), F_n is w-u-S-flat. So (9) holds.
(9) ⇒ (10) ⇒ (1) and (9) ⇒ (8) ⇒ (1). These are obvious. □

Corollary 3.4. *Let R be a ring and $S' \subseteq S$ multiplicative subsets of R. Suppose M is an R-module, then w-u-S-$\mathrm{fd}_R(M) \leq w$-u-S'-$\mathrm{fd}_R(M)$.*

Proof. Suppose $S' \subseteq S$ are multiplicative subsets of R. Let M be an R-modules and I be an ideal of R. If $\mathrm{Tor}_{n+1}^R(R/I, M)$ is u-S'-torsion, then $\mathrm{Tor}_{n+1}^R(R/I, M)$ is u-S-torsion. Hence, by Proposition 3.3., we have the result. □

Corollary 3.5. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then, $\mathrm{fd}_{R_S}(M_S) \leq w$-$u$-$S$-$\mathrm{fd}_R(M)$. Moreover, if S is composed of finite elements, then w-u-S-$\mathrm{fd}_R(M) = \mathrm{fd}_{R_S}(M_S)$.*

Proof. Let $0 \to F_n \to F_{n-1} \to \dots \to F_1 \to F_0 \to M \to 0$ be an exact sequence, where $F_0, F_1, \dots, F_{n-1}$ are flat R-modules. By localizing at S, we get an exact sequence of R_S-modules, $0 \to (F_n)_S \to (F_{n-1})_S \to \dots \to (F_1)_S \to (F_0)_S \to (M)_S \to 0$. By Proposition 2.10, if F_n is w-u-S-flat, so $(F_n)_S$ is flat over R_S, and the converse if S composed of finite elements. Hence, the desired result follows.

The proof of the next proposition is standard homological algebra. Thus we omit its proof.

Proposition 3.6. *Let R be a ring, S be a multiplicative subset of R, and $0 \to M'' \to M' \to M \to 0$ be an exact sequence of R-modules. If two of w-u-S-$\mathrm{fd}_R(M'')$, w-u-S-$\mathrm{fd}_R(M')$ and w-u-S-$\mathrm{fd}_R(M)$ are finite, so is the third. Moreover*

1. *w-u-S-$\mathrm{fd}_R(M'') \leq \max\{w$-$u$-$S$-$\mathrm{fd}_R(M'),\ w$-$u$-$S$-$\mathrm{fd}_R(M) - 1\}$.*
2. *w-u-S-$\mathrm{fd}_R(M') \leq \max\{w$-$u$-$S$-$\mathrm{fd}_R(M''),\ w$-$u$-$S$-$\mathrm{fd}_R(M)\}$.*
3. *w-u-S-$\mathrm{fd}_R(M) \leq \max\{w$-$u$-$S$-$\mathrm{fd}_R(M'),\ w$-$u$-$S$-$\mathrm{fd}_R(M'') + 1\}$.*

Corollary 3.7. *Let R be a ring, S be a multiplicative subset of R, and $0 \to M'' \to M' \to M \to 0$ be an exact sequence of R-modules. If M' is w-u-S-flat and w-u-S-$\mathrm{fd}_R(M) > 0$, then w-u-S-$\mathrm{fd}_R(M) = w$-u-S-$\mathrm{fd}_R(M'') + 1$.*

Proposition 3.8. *Let R be a ring, S be a multiplicative subset of R, and $\{M_i\}$ be a finite family of R-modules. Then* w-u-S-$\mathrm{fd}_R(\oplus_i M_i) = \sup_i\{w$-$u$-$S$-$\mathrm{fd}_R(M_i)\}$.

Proof. The proof is straightforward. □

Proposition 3.9. *Let R be a ring, S be a multiplicative subset of R, and $n \geq 0$ be a an integer. Then the following statements are equivalent:*

1. w-u-S-w.gl.dim$(R) \leq n$.
2. w-u-S-$\mathrm{fd}_R(M) \leqslant n$ *for all R-modules M.*
3. w-u-S-$\mathrm{fd}_R(R/J) \leqslant n$ *for all ideals J of R.*
4. $\mathrm{Tor}^R_{n+1}(R/I, M)$ *is u-S-torsion for any R-module M and any ideal I of R.*
5. $\mathrm{Tor}^R_{n+1}(R/I, M)$ *is u-S-torsion for any R-module M and any finitely generated ideal I of R.*

Consequently, we have

$$w\text{-}u\text{-}S\text{-w.gl.dim}(R) = \sup\{w\text{-}u\text{-}S\text{-fd}_R(R/J) \mid J \text{ is an ideal of } R\}$$

Proof. (1) ⇔ (2) ⇒ (3) and (4) ⇒ (5). The are obvious.
(2) ⇒ (4) and (5) ⇒ (2). These are immediate from Proposition 3.3.
(3) ⇒ (1). Let J be an ideal of R, so w-u-S-$\mathrm{fd}_R(R/J) \leqslant n$ by (3). By Proposition 3.3, $\mathrm{Tor}^R_{n+1}(R/I, R/J)$ is u-S-torsion for any ideal I of R. Thus, there exists $s \in S$ such that $s\,\mathrm{Tor}^R_{n+1}(R/I, R/J) = 0$ and so by [[5], Proposition 3.2], we have u-S-w.gl.dim$(R) \leq n$ for any R-module M. Thus, w-u-S-w.gl.dim$(R) \leq n$. □

Next, we show that rings R with w-u-S-w.gl.dim$(R) = 0$ are exactly u-S-von Neumann regular rings.

Proposition 3.10. *Let R be a ring, S be a multiplicative subset of R. The following are equivalent:*

1. w-u-S-w.gl.dim$(R) = 0$.
2. *Every R-module is w-u-S-flat.*
3. *R/I is w-u-S-flat for any ideal I of R.*
4. *R is a u-S-von Neumann regular ring.*

Proof. The equivalence of (1), (2), and (3), follows from Proposition 3.9.
(2) ⇔ (4). Follows from Proposition 2.5. □

The proof of the follwing Proposition flows from Proposition 3.9. Thus, we omit its proof.

Proposition 3.11. *Let R be a ring, S be a multiplicative subset of R. Then the following are equivalent:*

1. w-u-S-w.gl.dim$(R) \leq 1$.
2. *Every submodule of w-u-S-flat R-module is w-u-S-flat.*
3. *Every submodule of flat R-module is w-u-S-flat.*
4. *Every ideal of R is w-u-S-flat.*

Let $\theta : R \to T$ be a ring homomorphism. Suppose S is a multiplicative subset of R, then $\theta(S) = \{\theta(s) | s \in S\}$ is a multiplicative subset of T.

Lemma 3.12. *Let $\theta : R \to T$ be a ring homomorphism, S a multiplicative subset of R. Suppose L is a w-u-$\theta(S)$-flat T-module. Then for any ideal I of R and any $n \geq 0$, $\mathrm{Tor}_n^R(R/I, L)$ is u-S-isomorphic to $\mathrm{Tor}_n^R(R/I, T) \otimes_T L$. Consequently, w-u-S-$\mathrm{fd}_R(L) \leq w$-u-S-$\mathrm{fd}_R(T)$.*

Proof. Similar to proof [[5], Lemma 4.1]. □

Proposition 3.13. *Let $\theta : R \to T$ be a ring homomorphism, S a multiplicative subset of R. Suppose M is an T-module. Then*

$$w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(M) \leq w\text{-}u\text{-}\theta(S)\text{-}\mathrm{fd}_T(M) + w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(T).$$

Proof. Suppose that w-u-$\theta(S)$-$\mathrm{fd}_T(M) = n < \infty$. If $n = 0$, then M is w-u-$\theta(S)$-flat over T. By Lemma 3.12, w-u-S-$\mathrm{fd}_R(M) \leq n + w$-$u$-$S$-$\mathrm{fd}_R(T)$.

Now we assume $n > 0$. Let $0 \to A \to F \to M \to 0$ be an exact sequence of T-modules, where F is a free T-module. Then w-u-$\theta(S)$-$\mathrm{fd}_T(A) = n - 1$ by Corollary 3.7. By induction, w-u-S-$\mathrm{fd}_R(A) \leq n - 1 + w$-$u$-$S$-$\mathrm{fd}_R(T)$. Note that w-u-S-$\mathrm{fd}_R(T) = w$-u-S-$\mathrm{fd}_R(F)$. By Proposition 3.6, we have

$$\begin{aligned} w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(M) &\leqslant \max\{w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(F), w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(A) + 1\} \\ &\leqslant n + w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(T) \\ &= w\text{-}u\text{-}\theta(S)\text{-}\mathrm{fd}_T(M) + w\text{-}u\text{-}S\text{-}\mathrm{fd}_R(T) \end{aligned}$$

□

Proposition 3.14. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then, w-u-S-$\mathrm{fd}_{R[x]}(M[x]) = w$-$u$-$S$-$\mathrm{fd}_R(M)$.*

Proof. Suppose that w-u-S-$\mathrm{fd}_R(M) \leq n$. Then $\mathrm{Tor}_{n+1}^R(R/I, M)$ is u-S-torsion for any ideal I of R. Let $I[x]$ be an ideal of $R[x]$. By [[1], Theorem 1.3.11], we have $\mathrm{Tor}_{n+1}^{R[x]}((R/I)[x], M[x]) \cong \mathrm{Tor}_{n+1}^R(R/I, M) \otimes_R R[x]$. And by [[3], Corollary 2.6], we have $\mathrm{Tor}_{n+1}^R(R/I, M) \otimes_R R[x]$ is u-S-torsion since $\mathrm{Tor}_{n+1}^R(R/I, M)$ is u-S-torsion. Thus, $\mathrm{Tor}_{n+1}^{R[x]}((R/J)[x], M[x])$ is u-S-torsion, which implies that, w-u-S-$\mathrm{fd}_{R[x]}(M[x]) \leq n$ by Proposition 3.3.

Conversely, Let $0 \to F_n \to \ldots \to F_1 \to F_0 \to M[x] \to 0$ be an exact sequence with each F_i u-S-flat over $R[x]$ $(1 \leq i \leq n-1)$ and F_n w-u-S-flat over $R[x]$. Hence, it is also a w-u-S-flat resolution of $M[x]$ over R by Proposition 2.13. Then, by Proposition 3.3, we have $\mathrm{Tor}_{n+1}^R(R/I, M[x])$ is u-S-torsion for any ideal I of R. It follows that $s\,\mathrm{Tor}^R_{n+1}(R/I, M[x]) = s\bigoplus_{n=1}^{\infty}\mathrm{Tor}^R_{n+1}(R/I, M) = 0$. Hence, $\mathrm{Tor}^R_{n+1}(R/I, M)$ is u-S-torsion. Consequently, w-u-S-$\mathrm{fd}_R(M) \leq w$-u-S-$\mathrm{fd}_{R[x]}(M[x])$ by Proposition 3.3 again. □

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