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Annihilating Conditions of Generalized Skew Derivations on Lie Ideals

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Abstract. Let $\mathfrak{A}$ be a prime ring of $char(\mathfrak{A}) \neq 2$, $\mathscr{L}$ a non-central Lie ideal of $\mathfrak{A}, \mathscr{F}$ a generalized skew derivation of $\mathfrak{A}$ and $p \in \mathfrak{A}$, a nonzero fixed element. If $p\mathscr{F}(\eta)\eta \in C$ for any $\eta \in \mathscr{L}$, then $\mathfrak{A}$ satisfies S_4.

1. Introduction

Throughout this article, $\mathfrak{A}$ is a prime ring with center $Z(\mathfrak{A})$, right Martindale quotient ring $\mathscr{Q}$, extended centroid C and $p \in \mathfrak{A}$, a nonzero fixed element. Any information about definitions and main properties can be found in [3]. The standard polynomial identity S_4 in four variables is defined as $S_4(\eta_1, \eta_2, \eta_4, \eta_4) = \sum(-1)^\sigma \eta_{\sigma(1)}\eta_{\sigma(2)}\eta_{\sigma(3)}\eta_{\sigma(4)}$, where $(-1)^\sigma$ is $+1$ when σ is an even permutation, and $(-1)^\sigma$ is (-1), when σ is an odd permutation in the symmetric group S_4. An additive map $d : \mathfrak{A} \to \mathfrak{A}$ is a skew derivation of $\mathfrak{A}$ if $d(\eta\omega) = d(\eta)\omega + \alpha(\eta)d(\omega)$ for all $\eta, \omega \in \mathfrak{A}$, where α is associated automorphism of d. If α is an identity automorphism of $\mathfrak{A}$ then d is derivation of $\mathfrak{A}$. In particular, for a fixed $a \in \mathfrak{A}$, the mapping $I_a(\eta) = [\eta, a]$ is derivation called an inner derivation of $\mathfrak{A}$. An additive map $\mathscr{F} : \mathfrak{A} \to \mathfrak{A}$ is a generalized skew derivation if $\mathscr{F}(\eta\omega) = \mathscr{F}(\eta)\omega + \alpha(\eta)d(\omega)$ for all $\eta, \omega \in \mathfrak{A}$, where d is an skew derivation of $\mathfrak{A}$ with associated automorphism α. Further, a generalized skew derivation $\mathscr{F} : \mathfrak{A} \to \mathfrak{A}$ is called X - inner if there exist elements $a, b \in \mathscr{Q}$ and an automorphism α of $\mathfrak{A}$ such that $\mathscr{F}(\eta) = a\eta + \alpha(\eta)b$ for all

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$\eta \in \mathfrak{A}$, otherwise it is outer. Similarly, a skew derivation $d : \mathfrak{A} \to \mathfrak{A}$ is called X- inner if there is element $a \in \mathscr{Q}$ and an automorphism α of $\mathfrak{A}$ such that $d(\eta) = a\eta - \alpha(\eta)a$ for all $\eta \in \mathfrak{A}$, otherwise, it is X-outer. The notion of a generalized skew derivation is combination of the notions of skew derivation and generalized derivation.

In [13], Sharma and Dhara proved that if $\mathfrak{A}$ is a prime ring with non-zero derivation d, $\mathscr{L}$ a non-central Lie ideal and $a \in \mathfrak{A}$ such that $a\eta^n d(\eta)^m = 0$ for all $\eta \in \mathscr{L}$, where $n \geq 1$ and $m \geq 1$ are fixed integers, then one of the following holds:

(1) $a = 0$ or $d(\mathscr{L}) = 0$ if $char(\mathfrak{A}) \neq 2$.

(2) $a = 0$ or $d(\mathfrak{A}) = 0$ if $[\mathscr{L}, \mathscr{L}] \neq 0$ and $\mathfrak{A} \neq M_2(F)$.

In [8], Dhara and De Filippis considered generalized derivations. They proved for a prime ring $\mathfrak{A}$ that if H is a generalized derivation of $\mathfrak{A}$ and $\mathscr{L}$ a non-commutative Lie ideal of $\mathfrak{A}$ such that $\eta^s\ H(\eta)\ \eta^t = 0$ for all $\eta \in \mathscr{L}$, where $s \geq 0, t \geq 0$ are fixed integers, then $H(\eta) = 0$, for all $\eta \in \mathfrak{A}$, unless $Char(\mathfrak{A}) = 2$ and $\mathfrak{A}$ satisfies S_4.

In [9], Du and Wang demonstrated the following result for generalized derivations. Let $\mathfrak{A}$ be a prime ring, U be its Utumi ring of quotients, H a nonzero generalized derivation of $\mathfrak{A}$, $\mathscr{L}$ a non-central Lie ideal of $\mathfrak{A}$ and $0 \neq a \in \mathfrak{A}$. Suppose that $a\eta^s H(\eta)\eta^t = 0$ for all $\eta \in \mathscr{L}$, where $s, t \geq 0$ and $n \geq 1$ are fixed integers. Then either $s = 0$ and there exists $b \in U$ such that $H(\eta) = b\eta$ for all $\eta \in \mathfrak{A}$ with $ab = 0$ or $\mathfrak{A}$ satisfies S_4.

Inspired by the above outcomes, in the present paper, we prove the following result about generalized skew derivations with an annihilating condition on the non-central Lie ideal.

Theorem 1.1. *Let $\mathfrak{A}$ be a prime ring of $Char(\mathfrak{A}) \neq 2$, $\mathscr{L}$ a non-central Lie ideal of $\mathfrak{A}$, $\mathscr{F}$ a generalized skew derivation of $\mathfrak{A}$ and $p \in \mathfrak{A}$, a nonzero fixed element. If $p\mathscr{F}(\eta)\eta \in C$ for any $\eta \in \mathscr{L}$, then $\mathfrak{A}$ satisfies S_4.*

2. Preliminaries

The following facts are often referenced in the proofs of our results:

Fact 2.1. Let $\mathfrak{A}$ be a prime ring and $\mathscr{I}$ be a two sided ideal of $\mathfrak{A}$. Then $\mathscr{I}, \mathfrak{A}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identity with coefficients in $\mathscr{Q}$ [5]. Furthermore, $\mathscr{I}$, $\mathfrak{A}$ and $\mathscr{Q}$ satisfy the same generalized polynomial identity with automorphisms [6].

Fact 2.2. ([14, Lemma 2.1]) Let $\mathfrak{A}$ be a prime ring with extended centroid C. Then the following conditions are equivalent:

(1) $dim_{\mathscr{C}}\mathfrak{A}\mathscr{C} \leq 4$.

(2) $\mathfrak{A}$ satisfies S_4.

(3) $\mathfrak{A}$ is commutative or $\mathfrak{A}$ embeds in $M_2(F)$, for F a field.

(4) $\mathfrak{A}$ is algebraic of bounded degree 2 over $\mathscr{C}$.

(5) $\mathfrak{A}$ satisfies $[[a^2, b], [a, b]] = 0$.

Fact 2.3. ([7, Theorem 1]) Let $\mathfrak{A}$ be a prime ring, D be an X-outer skew derivation of $\mathfrak{A}$ and α be an X- outer automorphism of $\mathfrak{A}$. If $(\psi(a_i,), D(a_i), \alpha(a_i))$ is a generalized polynomial identity for $\mathfrak{A}$, then $\mathfrak{A}$ also satisfies the generalized polynomial identity $\psi(a_i, b_i, c_i)$, where a_i, b_i, c_i are distinct indeterminates.

Fact 2.4. Let $\mathfrak{A}$ be a prime ring and $\mathscr{L}$ a be non-central Lie ideal of $\mathfrak{A}$. If $char(\mathfrak{A}) \neq 2$, by [3, Lemma 1] there exists a nonzero ideal $\mathscr{I}$ of $\mathfrak{A}$ such that $0 \neq [\mathscr{I}, \mathfrak{A}] \subseteq \mathscr{L}$. If $char(\mathfrak{A}) = 2$ and $dim_{\mathscr{C}}\mathfrak{A}\mathscr{C} > 4$, i.e., $char(\mathfrak{A}) = 2$ and $\mathfrak{A}$ does not satisfy S_4, then by [11, Theorem 13] there exists a nonzero ideal $\mathscr{I}$ of $\mathfrak{A}$ such that $0 \neq [\mathscr{I}, \mathfrak{A}] \subseteq \mathscr{L}$. Thus, if either $char(\mathfrak{A}) \neq 2$ or $\mathfrak{A}$ does not satisfy S_4, then we may conclude that there exists a nonzero ideal $\mathscr{I}$ of $\mathfrak{A}$ such that $[\mathscr{I}, \mathscr{I}] \subseteq \mathscr{L}$.

Fact 2.5. ([2, Lemma 7.1]) Let V_D be a vector space over a division ring D with $dimV_D \geq 2$ and $T \in End(V)$. If $\mathfrak{s}$ and $T\mathfrak{s}$ are D-dependent for every $\mathfrak{s} \in V$, then there exists $\chi \in D$ such that $T\mathfrak{s} = \chi\mathfrak{s}$ for every $\mathfrak{s} \in \mathscr{V}$.

3. Some Important Results

We start with the following lemma and proposition; they are required for the development of our theorems:

Lemma 3.1. *Suppose* $\mathfrak{A}$ *is a primitive ring isomorphic to a dense ring of linear transformations on some vector space* V *over a division ring* $D, dim_D V \geq 2, f \in End(V)$ *and* $a \in \mathfrak{A}$. *If* $a\mathfrak{s} = 0$, *for any* $\mathfrak{s} \in V$ *such that* $\{\mathfrak{s}, f(\mathfrak{s})\}$ *is linearly* D*-independent, then* $a = 0$, *unless* $dim_D V = 2$ *and* $char(\mathfrak{A}) = 2$.

Proof. A vector $\mathfrak{s} \in V$ is fixed such that $\{\mathfrak{s}, f(\mathfrak{s})\}$ is linearly D-independent, then $a\mathfrak{s} = 0$. Let $\mathfrak{r} \in V$ such that $\{\mathfrak{r}, \mathfrak{s}\}$ is linearly D-dependent. Then both $a\mathfrak{r} = 0$ and $\mathfrak{r} \in$ span $\{\mathfrak{s}, f(\mathfrak{s})\}$ are trivial.

Now, let $\mathfrak{r} \in V$ such that $\{\mathfrak{r}, \mathfrak{s}\}$ is linearly D-independent and $a\mathfrak{r} \neq 0$. By the hypothesis, we have $\{\mathfrak{r}, f(\mathfrak{r})\}$ is linearly D-dependent, as are $\{\mathfrak{r}+\mathfrak{s}, f(\mathfrak{r}+\mathfrak{s})\}$ and $\{\mathfrak{r}-\mathfrak{s}, f(\mathfrak{r}-\mathfrak{s})\}$. Thus, there exists $\kappa_{\mathfrak{r}}, \kappa_{\mathfrak{r}+\mathfrak{s}}, \kappa_{\mathfrak{r}-\mathfrak{s}} \in D$ such that

$$f(\mathfrak{r}) = \mathfrak{r}\kappa_{\mathfrak{r}},\ f(\mathfrak{r}+\mathfrak{s}) = (\mathfrak{r}+\mathfrak{s})\kappa_{\mathfrak{r}+\mathfrak{s}},\ f(\mathfrak{r}-\mathfrak{s}) = (\mathfrak{r}-\mathfrak{s})\kappa_{\mathfrak{r}-\mathfrak{s}}.$$

In other words, we have

$$\mathfrak{r}\kappa_{\mathfrak{r}} + f(\mathfrak{s}) = \mathfrak{r}\kappa_{\mathfrak{r}+\mathfrak{s}} + \mathfrak{s}\kappa_{\mathfrak{r}+\mathfrak{s}} \tag{3.1}$$

and

$$\mathfrak{r}\kappa_{\mathfrak{r}} - f(\mathfrak{s}) = \mathfrak{r}\kappa_{\mathfrak{r}-\mathfrak{s}} - \mathfrak{s}\kappa_{\mathfrak{r}-\mathfrak{s}}. \tag{3.2}$$

Suppose $dim_D V \geq 3$. It is obvious that $\mathfrak{r} \in Span\{\mathfrak{s}, f(\mathfrak{s})\}$, otherwise equation (3.1) is contradicted. Thus for any $\mathfrak{r} \in V$, we have $\mathfrak{r} \in Span\{\mathfrak{s}, f(\mathfrak{s})\}$, that is $V = Span\{\mathfrak{s}, f(\mathfrak{s})\}$, a contradiction.

To complete the proof, suppose $dim_D V = 2$ and suppose $char(\mathfrak{A}) \neq 2$, if not we are done.

By equating (3.1) and (3.2), we get both

$$\mathfrak{r}(2\kappa_{\mathfrak{r}} - \kappa_{\mathfrak{r}+\mathfrak{s}} - \kappa_{\mathfrak{r}-\mathfrak{s}}) + \mathfrak{s}(\kappa_{\mathfrak{r}-\mathfrak{s}} - \kappa_{\mathfrak{r}+\mathfrak{s}}) = 0 \tag{3.3}$$

and

$$2f(\mathfrak{s}) = \mathfrak{r}(\kappa_{\mathfrak{r}+\mathfrak{s}} - \kappa_{\mathfrak{r}-\mathfrak{s}}) + \mathfrak{s}(\kappa_{\mathfrak{r}+\mathfrak{s}} + \kappa_{\mathfrak{r}-_{\mathfrak{s}}}). \tag{3.4}$$

By (3.3) and since $\{\mathfrak{r}, \mathfrak{s}\}$ is D-independent and $char(\mathfrak{A}) \neq 2$, we have $\kappa_{\mathfrak{r}} = \kappa_{\mathfrak{r}+\mathfrak{s}} = \kappa_{\mathfrak{r}-\mathfrak{s}}$. Thus, $2f(\mathfrak{s}) = 2\mathfrak{s}\kappa_{\mathfrak{r}}$ by (3.4). Since $\{\mathfrak{s}, f(\mathfrak{s})\}$ is D-independent, the conclusion $\kappa_{\mathfrak{r}} = \kappa_{\mathfrak{r}+\mathfrak{s}} = 0$ follows, that is $f(\mathfrak{r}) = 0$ and $char(\mathfrak{A}) \neq 2$, it follows that $a\mathfrak{r} = 0$, for any $\mathfrak{r} \in V$, i,e. $aV = (0)$. Hence, $a = 0$ follows. □

Lemma 3.2. *Let $\mathfrak{A}$ be a non-commutative prime ring, $a \in \mathfrak{A}, I$ a nonzero two-sided ideal of $\mathfrak{A}$ such that $[pa[\eta_1, \eta_2]^2, [\omega_1, \omega_2]] = 0$ for all $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. Then $\mathfrak{A}$ satisfies S_4.*

Proof. By hypothesis, I satisfies $[pa[\eta_1, \eta_2]^2, [\omega, \omega_2]] = 0$ for all $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. In particular $\eta_1 = b$, we get $[pa[b, \eta_2]^2, [\omega_1, \omega_2]] = 0$. Then by [1, Lemma 2.2], we get $pa[b, \eta_2]^2 \in C$. Since $[b, \eta_2]$ is an nonzero inner derivation of $\mathfrak{A}$ then $pa[b, \eta_2]^2$ is a central DI for I. Thus by [4, Lemma 2], $dim_C \mathfrak{A}C \leq 4$. Hence, by Fact 2.2, $\mathfrak{A}$ satisfies S_4. □

Proposition 3.3. *Let $\mathfrak{A}$ be a prime ring, $a, b \in \mathfrak{A}, I$ be a nonzero two-sided ideal of $\mathfrak{A}$ such that $[p(a[\eta_1, \eta_2] + [\eta_1, \eta_2]b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$ for any $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. Then $\mathfrak{A}$ satisfies S_4.*

Proof. Suppose $b \in C$, then by given hyphothesis $[p(a + b)[\eta_1, \eta_2]]^2, [\omega_1, \omega_2]] = 0$ for all $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. Hence, by Lemma 3.2 the required conclusion follows. In case $b \notin C$, assume $dim_C V \geq 3$ then $[p(a[\eta_1, \eta_2] + [\eta_1, \eta_2]b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$ is a non-trivial generalized polynomial identity (GPI) for I. Then by Fact 2.1 $[p(a[\eta_1, \eta_2] + [\eta_1, \eta_2]b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$ is a non-trivial GPI for $\mathfrak{A}$ and $\mathscr{Q}$ also. By Martindale Theorem [12], $\mathscr{Q}$ is a primitive ring having non-zero socle and its associated division ring is finite dimensional over C. Hence, by Jacobson Theorem in [10, Page 75], $\mathscr{Q}$ is isomorphic to a dense ring of linear transformation on some vector space V over C. Assume that $\mathfrak{s} \in V$ exists such that $\{\mathfrak{s}, b\mathfrak{s}\}$ are linearly C-independent. Since $dim_C V \geq 3$, then there exists $w \in V$ such that $\{\mathfrak{s}, b\mathfrak{s}, w\}$ are linearly C-independent. By the density of $\mathscr{Q}$, there exists $h_1, h_2, k_1, k_2 \in \mathscr{Q}$ such that

$$h_1\mathfrak{s} = \mathfrak{s},\ h_2\mathfrak{s} = 0,\ k_1 w = w,\ k_2\mathfrak{s} = w,\ k_1\mathfrak{s} = 0,$$

$$h_1 w = 0,\ h_2 w = \mathfrak{s},\ h_1 b\mathfrak{s} = 0,\ h_2 b\mathfrak{s} = \mathfrak{s}.$$

This gives, $0 = \big(p(a[h_1, h_2] + [h_1, h_2]b)[h_1, h_2], [k_1, k_2]]\big)\mathfrak{s} = p\mathfrak{s}$. Hence, we have proved $p\mathfrak{s} = 0$ for any vector $\mathfrak{s} \in V$ such that $\{\mathfrak{s}, b\mathfrak{s}\}$ are linearly C-independent. By

Lemma 3.1, $p = 0$ follows a contradiction. Thus, $\{\mathfrak{s}, b\mathfrak{s}\}$ are linearly C-dependent for all $\mathfrak{s} \in V$. Then by Fact 2.5, there exists $\kappa \in C$ such that $b\mathfrak{s} = \kappa\mathfrak{s}$ for any $\mathfrak{s} \in V$. For any $r \in \mathfrak{A}$, we have that $[b, r]\mathfrak{s} = b(r\mathfrak{s}) - rb\mathfrak{s} = \kappa r\mathfrak{s} - \kappa r\mathfrak{s} = 0$, that is, $[b, r]V = 0$. Hence $[b, r] = 0$ for any $r \in \mathfrak{A}$, which implies that $b \in C$. This is a contradiction. Thus, $dim_C V \leq 2$. Hence, $\mathfrak{A}$ satisfies S_4. □

4. Case of inner generalized skew derivation

In this case, we have $a, b \in \mathscr{Q}$ such that $F(\eta) = a\eta + \alpha(\eta)b$ for all $\eta \in \mathfrak{A}$, where $\alpha \in Aut(\mathscr{Q})$.

Lemma 4.1. *Let $\mathfrak{A}$ be a prime ring of $Char(\mathfrak{A}) \neq 2$ and $a, b, q \in \mathscr{Q}$. If q is an invertible element of $\mathscr{Q}$ and I be a nonzero two-sided ideal of $\mathfrak{A}$ such that*

$$[p(a[\eta_1\eta_2] + q[\eta_1, \eta_2]q^{-1}b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0 \tag{4.1}$$

for any $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. Then $\mathfrak{A}$ satisfies S_4.

Proof. Suppose $q^{-1}b \in C$, then by (4.1), we have $[p(a + b)[\eta_1, \eta_2]^2, [\omega_1, \omega_2]] = 0$, for all $\eta_1, \eta_2, \omega_1, \omega_2 \in I$. Hence by Lemma 3.2, $\mathfrak{A}$ satisfies S_4. Furthermore, in case $q \in C$, we have $[p(a[\eta_1, \eta_2] + [\eta_1, \eta_2]b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$ for any $\eta_1, \eta_2, \omega_1, \omega_2 \in I$, and hence by Proposition 3.3, we get the conclusion. From now, we have $q^{-1}b \notin C$ and $q \notin C$. So, (4.1) is a non-trivial GPI for I. Then by Fact 2.1, (4.1) is a non-trivial GPI for $\mathfrak{A}$ and $\mathscr{Q}$ also. By Martindale Theorem in [12], $\mathscr{Q}$ is a primitive ring having non-zero socle and its associated division ring is finite dimensional over C. Hence, by Jacobson Theorem in [10, page 75], $\mathscr{Q}$ is isomorphic to a dense ring of linear transformation on some vector space V over C. Assume $dim_C V \geq 3$ and suppose that there exists $\mathfrak{s} \in V$ such that $\{\mathfrak{s}, q^{-1}b\mathfrak{s}\}$ are linearly C-independent. Since $dim_C V \geq 3$, then there exists $w \in V$ such that $\{\mathfrak{s}, q^{-1}b\mathfrak{s}, w\}$ are linearly C-independent. By the density of $\mathscr{Q}$, there is h_1, h_2, k_1, k_2 such that

$$h_1\mathfrak{s} = \mathfrak{s},\ h_2\mathfrak{s} = 0,\ k_1\mathfrak{s} = 0,\ k_2\mathfrak{s} = w,\ k_1w = w$$

$$h_1w = 0,\ h_2w = \mathfrak{s},\ h_1q^{-1}b\mathfrak{s} = 0,\ h_2q^{-1}b\mathfrak{s} = \mathfrak{s}$$

This gives, $0 = \big([p(a[h_1, h_2] + q[h_1, h_2]q^{-1}b)[h_1, h_2], [k_1, k_2]]\big)\mathfrak{s} = pq\mathfrak{s}$. Hence, we have proved $pq\mathfrak{s} = 0$ for any vector $\mathfrak{s} \in V$ such that $\{\mathfrak{s}, q^{-1}b\mathfrak{s}\}$ are linearly C-independent. By Lemma 3.1, $pq = 0$ follows a contradiction. Thus, $\{\mathfrak{s}, q^{-1}b\mathfrak{s}\}$ are linearly C-dependent for all $\mathfrak{s} \in V$. Then by Fact 2.5, there exists $\kappa \in C$ such that $q^{-1}b\mathfrak{s} = \kappa\mathfrak{s}$ for any $\mathfrak{s} \in V$. For any $r \in \mathfrak{A}$, we have that $[q^{-1}b, r]\mathfrak{s} = q^{-1}b(r\mathfrak{s}) - rq^{-1}b\mathfrak{s} = \kappa r\mathfrak{s} - \kappa r\mathfrak{s} = 0$, that is,$[q^{-1}b, r]V = 0$. Hence $[q^{-1}b, r] = 0$ for any $r \in \mathfrak{A}$, which implies that $q^{-1}b \in C$. This is a contradiction. Thus, $dim_C V \leq 2$. Hence, $\mathfrak{A}$ satisfies S_4. □

Lemma 4.2. *Let $\mathfrak{A}$ be a prime ring of $char(\mathfrak{A}) \neq 2$, $\alpha : \mathfrak{A} \to \mathfrak{A}$ be an outer automorphism of $\mathfrak{A}$. suppose that $a, b \in \mathfrak{A}$ such that*

$$[p(a[\eta_1, \eta_2] + \alpha([\eta_1, \eta_2])b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0 \tag{4.2}$$

for all $\eta_1, \eta_2, \omega_1, \omega_2 \in \mathfrak{A}$. *Then* $\mathfrak{A}$ *satisfies* S_4.

Proof. Assume $\alpha \neq I$, otherwise the conclusion directly follows from Proposition 3.3. By Fact 2.1, $\mathfrak{A}$ is a GPI-ring and Q is also a GPI-ring. By Martindale Theorem in [12], $\mathscr{Q}$ is a primitive ring having non-zero socle and its associated division ring is finite dimensional over C. Hence, by Jacobson Theorem in [10, page 79], $\mathscr{Q}$ is isomorphic to a dense ring of linear transformation on some vector space V over C. By [13, page 79], there exists a semi-linear transformation $T \in End(V)$ such that $\alpha(\eta) = T\eta T^{-1}$ for all $\eta \in \mathscr{Q}$. Assume that $\mathfrak{s}$ and $T^{-1}b\mathfrak{s}$ are linearly C-dependent for all $\mathfrak{s} \in V$. By Fact 2.5, there exists $\kappa \in C$ such that $T^{-1}b\mathfrak{s} = \kappa\mathfrak{s}$ for all $\mathfrak{s} \in V$. In this case for all $\eta \in \mathscr{Q}$, $(a\eta + \alpha(\eta)b)\mathfrak{s} = (a\eta + T\eta T^{-1}b)\mathfrak{s} = a\eta\mathfrak{s} + T\eta T^{-1}b\mathfrak{s} = a\eta\mathfrak{s} + T(\kappa\eta\mathfrak{s}) = a\eta\mathfrak{s} + T(\kappa(\eta\mathfrak{s})) = a\eta\mathfrak{s} + T(T^{-1}b)(\eta\mathfrak{s}) = (a+b)\eta\mathfrak{s}$. This means that $(a\eta + \alpha(\eta)b)\mathfrak{s} = (a+b)\eta\mathfrak{s}$ for all $\eta \in \mathscr{Q}$ and $\mathfrak{s} \in V$, since V is faithful, it follows that $(a\eta + \alpha(\eta)b) = (a+b)\eta$. Thus, (4.2) reduces to $[p(a+b)[\eta_1, \eta_2]^2, [\omega_1, \omega_2]] = 0$. Hence, by Lemma 3.2, $\mathfrak{A}$ satisfies S_4. Now, Suppose there exists $\mathfrak{s} \in V$ such that $\{\mathfrak{s}, T^{-1}b\mathfrak{s}\}$ are linearly C-independent. Assume $dim_C V \geq 3$, then there exists $\mathfrak{w} \in V$ such that $\{\mathfrak{s}, T^{-1}b\mathfrak{s}, \mathfrak{w}\}$ are linearly C-independent. By the density of $\mathscr{Q}$, there exists h_1, h_2, k_1, k_2 such that

$$h_1 T^{-1}b\mathfrak{s} = 0,\ h_2 T^{-1}b\mathfrak{s} = \mathfrak{s},\ k_1\mathfrak{s} = 0,\ k_2\mathfrak{s} = \mathfrak{w},\ k_1\mathfrak{w} = \mathfrak{w}$$

$$h_1\mathfrak{s} = \mathfrak{s},\ h_2\mathfrak{s} = 0,\ h_1\mathfrak{w} = 0,\ h_2\mathfrak{w} = \mathfrak{s}$$

$0 = \big([p(a[h_1, h_2] + T[h_1, h_2]T^{-1}b)[h_1, h_2], [k_1, k_2]]\big)\mathfrak{s} = pT(\mathfrak{s})$. Hence, we have proved that $pT(\mathfrak{s}) = 0$ for every $\mathfrak{s} \in V$ such that $\{\mathfrak{s}, T^{-1}b\mathfrak{s}\}$ are linearly C-independent. By Lemma 3.1, $p = 0$ follows a contradiction. Thus $dim_C V \leq 2$. Hence $\mathfrak{A}$ satisfies S_4. □

Proof of Theorem 1.1. In view of the Fact 2.4, a nonzero two-sided ideal I exists such that $0 \neq [I, \mathfrak{A}] \subseteq L$. Therefore , I satisfies $[pF([\eta_1, \eta_2])[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$. we known there exists a skew derivation d of $\mathfrak{A}$ and an element $a \in \mathscr{Q}$ such that $F(\eta) = a\eta + d(\eta)$ for all $\eta \in \mathfrak{A}$.

Case 1: If d is inner, then $d(\eta) = b\eta - \alpha(\eta)b$ for some $b \in \mathscr{Q}$ for all $\eta \in \mathscr{Q}$, So that $F(\eta) = (a+b)\eta - \alpha(\eta)b$. Then, we have
$[p((a+b)[\eta_1, \eta_2] - \alpha([\eta_1, \eta_2])b)[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$ for all $\eta_1, \eta_2, \omega_1, \omega_2 \in I$

Subcase 1: When α is an identity map, then the conclusion follows from Proposition 3.3.

Subcase 2: When α is inner , then there is $q \in \mathscr{Q} - C$, such that $\alpha(\eta) = q\eta q^{-1}$ for all $\eta \in \mathscr{Q}$, then the conclusion follows from Lemma 4.1.

Subcase 3: When α is outer, then the conclusion follows from Lemma 4.2.

Case 2: When d is outer, then we get

$$[p(a[\eta_1, \eta_2] + d([\eta_1, \eta_2])[\eta_1, \eta_2], [\omega_1, \omega_2]] = 0$$

for all $\eta_1, \eta_2, \omega_1, \omega_2 \in \mathscr{Q}$.

This implies that

$$[p(a[\eta_1,\eta_2] + d(\eta_1)\eta_2 + \alpha(\eta_1)d(\eta_2) - d(\eta_2)\eta_1 - \alpha(\eta_2)d(\eta_1))[\eta_1,\eta_2],[\omega_1,\omega_2]] = 0$$

for all $\eta_1, \eta_2, \omega_1, \omega_2 \in \mathscr{Q}$.

Here d is not inner, by applying Fact 2.3, $\mathfrak{A}$ satisfies

$$[p(a[\eta_1,\eta_2] + t_1\eta_2 + \alpha(\eta_1)t_2 - t_2\eta_1 - \alpha(\eta_2)t_1)[\eta_1,\eta_2],[\omega_1,\omega_2]] = 0$$

In particular, $t_1 = t_2 = 0$, we get $[pa[\eta_1,\eta_2]^2,[\omega_1,\omega_2]] = 0$, then again by Lemma 3.2, the given conclusion follows.

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