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RESEARCH ARTICLE

Mathematical Modeling of the Tennis Serve: Adaptive Tasks from Middle and High School to College

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Abstract

A central problem of mathematics teaching worldwide is probably the insufficient *adaptive handling of tasks*—especially in computational practice phases and modeling tasks. All students in a classroom must often work on the same tasks. In the process, the high-achieving students are often underchallenged, and the low-achieving ones are overchallenged. This publication uses different modeling of the *tennis serve* as an example to show a possible solution to the problem and develops and discusses one *adaptive task* each for middle school, high school, and college using three mathematical models of the tennis serve each time. From model to model within the task, the complexity of the modeling increases, the mathematical or physical demands on the students increase, and the new modeling leads to more realistic results. The proposed models offer the possibility to address heterogeneous learning groups by their arrangement in the surface structure of the so-called *parallel adaptive task* and to stimulate adaptive mathematics teaching on the instructional topic of mathematical modeling. Models A through C are suitable for middle school instruction, models C through E for high school, and models E through G for college. The models are classified in the specific *modeling cycle* and its extension by a digital tool model, and individual modeling steps are explained. The advantages of the presented models regarding teaching and learning mathematical modeling are elaborated. In addition, we report our first teaching experiences with the developed parallel adaptive tasks.

Keywords: differentiation in mathematics education, parallel adaptive tasks, mathematical modeling in sports, modeling cycle

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I. INTRODUCTION

Tasks have always played a central role in middle and high school mathematics education (e.g., Bromme, 1981; Hiebert et al., 2003). Chapman (2013, p. 1) even goes so far as to express that “mathematical tasks are central to the learning of mathematics”.

One can consider task selection to be one of the central practices of mathematics teachers. However, teachers have a different focus on tasks and often do not recognize the potential of a task in terms of adaptive use in the classroom during task selection (Bardy et al., 2021). Teachers often do not discover, e.g., that a task can be more open-endedly reworded so that all students at their respective ability levels can complete the task or that a task can be formulated more simply in terms of language (e.g., without the use of technical terms) or modeling assumptions can be stated explicitly, allowing easier entry into the task, especially for low-achieving students.

Overall, there are many ways to design adaptive instruction. One widely used approach is to assign different tasks to different students and/or to provide different types of support (e.g., formula sheets, check sheets, or sample tasks) for different groups of students. In this context, we speak of *closed differentiation* (e.g., Helmke, 2010; Snow, 1989). In teaching—especially in a heterogeneous class—a teacher’s discursive skills also become significant (e.g., Beck et al., 2008; Doyle, 1988). These are especially central when leading a class discussion, e.g., about a solution of a task.

In addition to the closed and discursive options described, there is also an *open* approach to differentiation in the classroom, which we focus on here. We focus specifically on a differentiation strategy that uses identical (as regards content) tasks for all students in a class. We refer to these tasks as *adaptive tasks* (Bardy et al., 2021) because adaptivity unfolds as students work on a task during practice or consolidation periods. In the literature, this type of task is also referred to as an *open-ended adaptive task*, *open-ended task*, *self-differentiating task*, or *open-ended differentiating task* (e.g., Müller & Wittmann, 1998; Sullivan, 1999). Several authors have investigated the *features* and *didactic utility* of such tasks in different contexts (e.g., Bardy et al., 2021; Boston & Smith, 2009; Leuders & Prediger, 2016; Prediger & Scherres, 2012).

An *adaptive task* is a problem that allows for multiple starting points and solutions for students of different levels so that every student—whether low- or high-achieving—can actively participate in the task. The central idea of this type of task is to cognitively activate all students at their respective ability levels by using the same unified question. Each student is then asked to complete the adaptive task at their own ability level. Adaptive tasks have so far generally been designed to be used, especially during practice periods (Müller & Wittmann, 1998; Prediger et al., 2021; Sullivan et al., 2009).

The *deep structure* of tasks (Leuders & Prediger, 2016) refers to very specific task features—such as *openness* or *difficulty* (for more features of the deep structure, see Bardy et al., 2021). The *surface structure* differs from the deep structure and is concerned with the presentation of a task, which is a kind of navigation for its completion. The surface structure of a task gives teachers and students a visual cue to the different difficulty levels of the task. Such an external cue is, e.g., the *parallel presentation* of two subtasks, one of which is easier and the other more difficult to work on (see Figure 1).

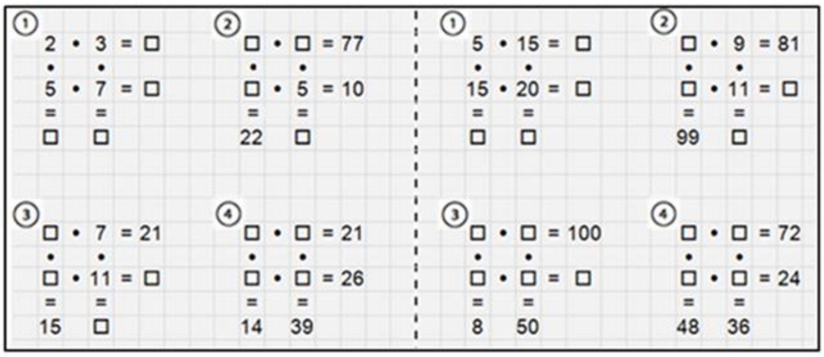

Figure 1. An adaptive task that is structured as a *parallel adaptive task* for low-achieving (left-hand side) and high-achieving (right-hand side) students (Leuders & Prediger, 2016, p. 121).

In this adaptive task (here for computational practice phases), similar mathematical content is practiced on both sides (separated by a dashed line; see surface structure), but the individual subtasks have different characteristics of the deep structure: e.g., the tasks on the left side have less complexity and smaller numbers than the tasks on the right side. In addition, the subtasks on the left side always have a unique solution. Ideally, low-achieving students in the class would work on the left side of this adaptive task, high-achieving students would work on the right side, and moderately achieving students could choose between the two sides or even switch (e.g., from (1) to (3) on the right side to (4) on the left side). We call this type of task a *parallel adaptive task* (Leuders & Prediger, 2016). The three parallel adaptive tasks we developed (one for middle school, one for high school, and one for college) have three columns each (see the Appendix).

II. RESEARCH QUESTIONS

The potential of mathematical modeling as a challenging opportunity for all learners in a heterogeneous setting has been poorly investigated. Our first research question (RQ1) derives from this research gap: *How can an adaptive modeling task on the tennis serve be designed for three levels of education (middle, high school, and college) and mapped to the specific modeling cycle, while differentiating at each level?*

Our hypothesis is that the modeling process for the tennis serve can be made easier or more difficult, particularly by selecting appropriate model assumptions, leading to an adaptive task. In the modeling cycle, adaptive tasks then require additions from the fields of physics and technology as tools.

For us, the 2nd research question (RQ2) arises in relation to the deep structure of an adaptive task: *Which task features of the deep structure are central in the design of a parallel adaptive modeling task for the tennis serve?*

Our hypothesis is that an adaptive task for mathematical modeling of the tennis serve can be developed mainly by the surface structure of a parallel adaptive task and the deep structure (with task features such as difficulty, support, or accessibility). The difficulty is determined particularly by the mathematical and physical model assumptions needed. Support for low-achieving students can be planned specifically by providing mathematical calculation methods and physical formulas.

Currently, it is still unknown how students respond to *adaptive* modeling tasks in the classroom. Our 3rd research question (RQ3) stems from this research gap: *What initial experiences and consequences can be identified when using the developed adaptive modeling tasks in a high school and a college classroom?*

Our hypothesis is that students, regardless of school level, have difficulties in selecting the appropriate column of the parallel adaptive task. Primarily, this is also because this task type is unfamiliar to them, and they choose the supposedly easy column due to convenience. Regarding the tennis serve, our hypothesis is that the students give positive feedback in motivational terms, although we are aware of the mathematical and physical difficulties.

III. MATHEMATICAL MODELS OF THE TENNIS SERVE

In this article, we present *seven* models for the tennis serve (see RQ1). In *models A* and *B*, all forces acting on the tennis ball are ignored. In *model A*, we assume that the trajectory of the tennis ball is straight and that the net over which the ball must fly is the same height everywhere ($3\ ft$, as prescribed in the middle of the court, $1\ ft = 0.3048\ m$). The straightness of the ball flight is also assumed in *model B*, but the reality of the net height is taken into account in individual play (the net stretched tightly in each case from the center to the $3.5\ ft \approx 1.07\ m$ high net supports, which are $3\ ft$ from the outline of the single field; see Figure 2). In *model C*, the reality of gravity is acknowledged, and thus the assumption of straight-line flight is abandoned. Without considering the aerodynamic drag, the trajectory of the center of the ball is a parabolic segment. *Model D* is also dedicated to aerodynamic drag. *Model E* also deals with the influence of *topspin*, a forward spin that good tennis players can impart to the ball during their strokes (including the serve). *Model F* considers *sidespin*/*slice* instead of topspin and also includes the influence of wind. Finally, *model G* deals with the combination of topspin and sidespin that often occurs in the reality of the serve.

For those who neither play tennis themselves nor are interested spectators who are familiar with the rules of tennis, here is a brief summary of the most important specifications for the *serve* (in a singles match): The serve is the stroke with which a rally begins and takes place behind the baseline (see Figure 2). After each game, the serve changes; after each point, the side of the serve changes (starting with the right side). When serving, the ball must be played into the diagonally opposite part of the opponent's *service box* (first deuce court, then advantage court; Figure 2). At the moment the ball is hit with the racket, both feet (of the person serving) must be behind the baseline without touching it. If in the first attempt it is not possible to serve in accordance with the rules, a second serve is allowed. A special case occurs when, in an otherwise rule-compliant serve, the ball touches the edge of the net and then falls into the correct court. This counts as a fault, but an additional serve attempt is allowed.

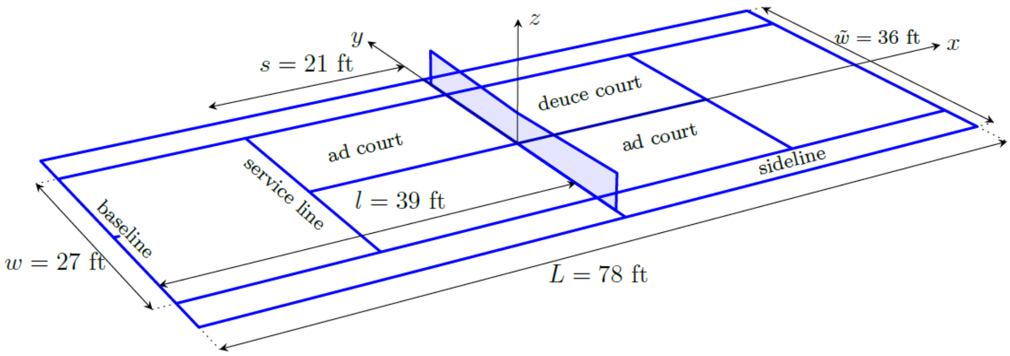

Figure 2. Coordinate system and tennis court dimensions for singles and doubles players. The height of the net in the middle is $n = 3\ ft$ and at the posts $\tilde{n} = 3.5\ ft$.

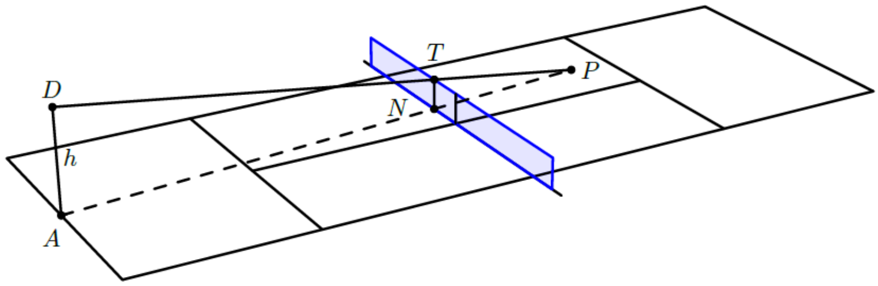

Figure 3. Serve in tennis game I.

Model A (Straight Flight of the Ball, Net Everywhere the Same Height)

Graening (1982) and Bolt (1983) discuss the question of the shape of the available service area in the service box in tennis, assuming that the trajectory of the ball is straight and the net has the same height everywhere. According to Graening (1982) and Bolt (1983), we here (and in the models B and C) do not consider both the diameter of the ball (not less than 6.54 cm and not more than 6.86 cm; according to the International Tennis Federation (ITF) regulations, 2019, p. 89; in the models D through G we calculate with $r = 0.0335\ m$) and the width of the court lines. The rules of the game demand that the ball touches at least the outer edge of the court lines. We assume that the ball leaves the racket of the person serving at a height h above the baseline (relative to the center of the ball), flies in a straight line, and lands in the correct part of the service box behind the net. According to Figure 3, the ray theorem then applies

$$\frac{h}{n} = \frac{|\overline{PA}|}{|\overline{PN}|} = \frac{|\overline{PN}| + |\overline{AN}|}{|\overline{PN}|} = 1 + \frac{|\overline{AN}|}{|\overline{PN}|} \tag{1}$$

We can determine the distance d from P to the net from Figure 4. Here, n is the height of the net and l is the half length of the court, i.e., the distance between the baseline and the net. Again, the ray theorem applies

$$\frac{|\overline{AN}|}{|\overline{PN}|} = \frac{l}{d}$$

Putting into equation (1), we get

$$\frac{h}{n} = 1 + \frac{l}{d}$$

$$\Rightarrow\ d = l \cdot \frac{n}{h-n} \tag{2}$$

This means that the distance between P and the bottom edge of the net is independent of T, the point where the ball crosses the net (see Figure 3).

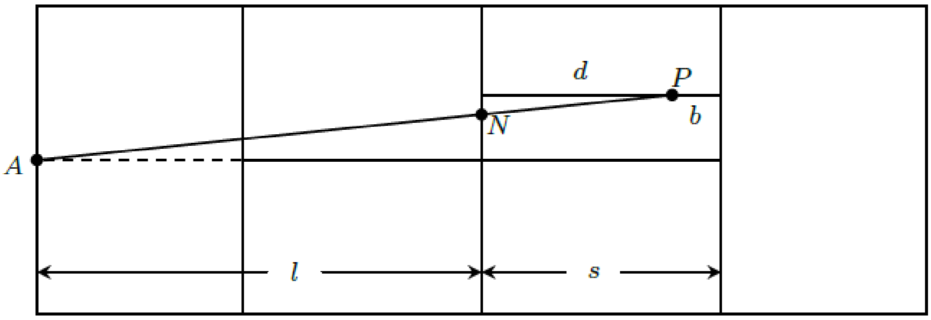

Figure 4. Serve in tennis game II.

This argument can be used in the same way when the place from which the player is serving is not—as before—in the middle of the baseline, but at any point from which a serve is allowed. If $h = 9\ ft$ ($\approx 2.74\ m$) is given, the result for the distance between the point where the ball hits the service box and the service line is $b = s - d = 1.5\ ft \approx 46\ cm$. Under the given conditions, we obtain for the region in the service box that yields a valid serve a rectangle with the side lengths $13.5\ ft$ and $1.5\ ft$. This means that the proportion of the impact area to the total area of the deuce court is (only) $b/s = 1/14$, i.e., about 7.1%.

The y-coordinate of the point P in contrast depends on the position A of the serving player. Let $T = (0 \mid y_T \mid z_T)$ and $P = (x_P \mid y_P \mid 0)$ be the coordinates of the two points T and P, respectively ($x_P = d,\ z_T = n$). Note that the horizontal coordinates of points N and T are the same, i.e., $x_T = x_N = 0$ and $y_T = y_N$. (Note also that in model B, the z-coordinate of the point T depends on the y-coordinate of T.) We have (see Figure 4) $\frac{y_P - y_T}{y_T - y_A} = \frac{y_P - y_N}{y_N - y_A} = \frac{d}{l}$, which yields together with (2)

$$y_P = y_T + (y_T - y_A) \cdot \frac{n}{h - n} \tag{3}$$

If the player serves in the middle of the baseline ($y_A = 0$), the condition for a valid serve (inside the sideline) reads $0 \leq y_P \leq \frac{w}{2} \Leftrightarrow 0 \leq y_T \leq \frac{h-n}{h} \cdot \frac{w}{2}$

For a very tall player, this means that the horizontal angle of his serve is within almost $\arctan\left(\frac{w}{L}\right) \approx 19.09°$; for a player with $h = 9.25\ ft$, we get $\arctan\left(\frac{h-n}{h} \cdot \frac{w}{L}\right) \approx 13.16°$.

Model A now raises two further interesting questions:

1. From what *height* h must the ball be hit to have a chance of landing in the service box (straight-line flight)? To answer this question, we set $d = s$ in equation (2) and get

$$h = \frac{l+s}{s} \cdot n = \frac{20}{7} \cdot n \approx 8.57\ ft \approx 2.61\ m$$

This is a height that some tennis players can barely reach. (According to Cross, 2004, p. 371, a person of the size of H usually serves the ball from a height of about $1.5 \cdot H$, so the player should be about $1.74\ m$ tall.)

2. What *serve-angle* α can a tennis player choose to achieve a valid serve, assuming he hits the ball at a height of $h = 9.25\ ft \approx 2.82\ m$ and the flight of the ball is straight along the center line (see Figure 5)? (A similar problem is addressed by Bolt, 1983, p. 7.)

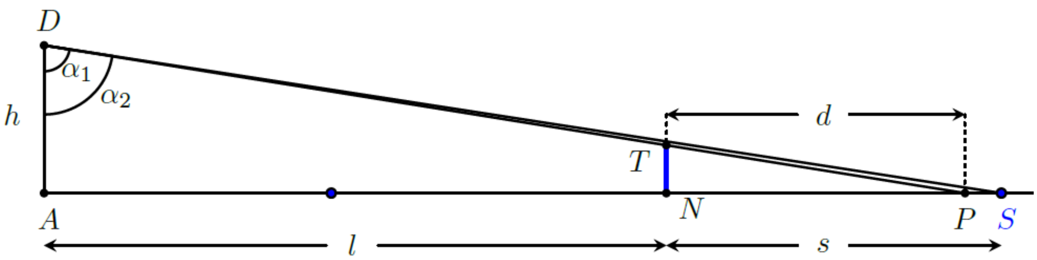

Figure 5. Smallest ($\alpha_1 = \angle ADP$) and largest ($\alpha_2 = \angle ADS$) angle measure for a valid serve.

$$\tan(\alpha_1) = \frac{l+d}{h} = \frac{l}{h-n} = \frac{39}{6.25} = 6.24 \quad \Rightarrow \quad \alpha_1 \approx 80.90°$$

$$\tan(\alpha_2) = \frac{l+s}{h} = \frac{60}{9{,}25} = 6.\overline{486} \quad \Rightarrow \quad \alpha_2 \approx 81.23°$$

So, for an angle α that leads to a valid serve, we get $80.90° \leq \alpha \leq 81.23°$. Even if model A does not fully represent reality, this calculated small *margin* for α (0.33°) makes it understandable that even world-class tennis players produce many errors on their first (very hard) serve.

Model B (Straight Flight of the Ball, Different Net Heights)

The assumption that the net is the same height everywhere does not meet reality (in the middle 3 ft, at the edge 3.5 ft; this is located 3 ft outside the sideline of the single field; see Figure 6). As in model A, we assume that the center of the ball crosses the net at the height of the net. The slope of the net edge is $\frac{0.5}{16.5} = \frac{1}{33}$.

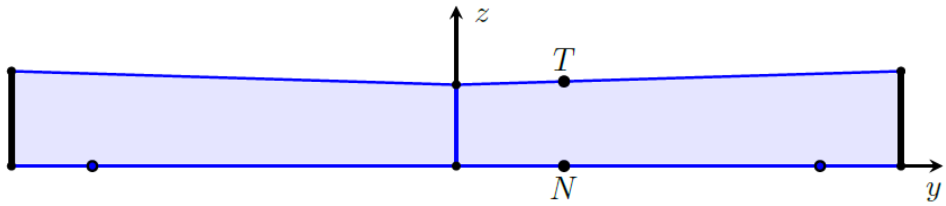

Figure 6. Different net heights (single field).

The situation is similar to Figure 4, but the height of the net now depends on the y-coordinate of the point T (see Figure 6). So let y_T be the y-coordinate of point T. The height of the net at point T is $z_T = \frac{1}{33} \cdot |y_T| + n$, where $n = 3\ ft$ is again the height of the net at the center. The x-coordinate of a point P in the deuce court or in the advantage court, i.e., the distance d of point P from the net, is given by equation (2) as

$$d = x_P = l \cdot \frac{\frac{1}{33} \cdot |y_T| + n}{h - \left(\frac{1}{33} \cdot |y_T| + n\right)} = l \cdot \frac{|y_T| + 33 \cdot n}{33 \cdot (h - n) - |y_T|}$$

The y-coordinate of the point P is (see equation (3))

$$y_P = y_T + (y_T - y_A) \cdot \frac{\frac{1}{33} \cdot |y_T| + n}{h - \left(\frac{1}{33} \cdot |y_T| + n\right)} = y_T + (y_T - y_A) \cdot \frac{|y_T| + 33 \cdot n}{33 \cdot (h - n) - |y_T|}$$

Thus, we have a parametrization of the point P with the parameter y_T. One could now resolve x_P to y_T and insert it into y_P and would get a functional equation for $x(y)$ of the points P. Of course, for $y_T > 0$ and $y_T < 0$, we get an affine function since we are dealing with the intersection of two planes. However, since the relationship between the parameter y_T and x_P or y_P is not linear, this procedure turns out to be quite challenging. Instead, we insert three values for y_T into the equations for x_P and y_P, namely $y_T = y_A$ (a valid service exists if $y_A < 0$), $y_T = -y_A$, and $y_T = 0$. We obtain the following:

$$y_T = y_A < 0 \ \Rightarrow \begin{cases} x_- = l \cdot \dfrac{33 \cdot n - y_A}{33 \cdot (h - n) + y_A} \\ y_- = y_A \end{cases}$$

$$y_T = 0 \ \Rightarrow \begin{cases} x_0 = l \cdot \dfrac{n}{h - n} \\ y_0 = -y_A \cdot \dfrac{n}{h - n} \end{cases}$$

$$y_T = -y_A > 0 \;\Rightarrow\; \begin{cases} x_+ = l \cdot \dfrac{33 \cdot n - y_A}{33 \cdot (h - n) + y_A} \\ y_+ = -y_A - 2 \cdot y_A \cdot \dfrac{33 \cdot n - y_A}{33 \cdot (h - n) + y_A} \end{cases}$$

With some algebra, we obtain the slope m of the curve $x(y)$:

$$m = \begin{cases} m_- = \dfrac{x_- - x_0}{y_- - y_0} = -\dfrac{l}{33 \cdot (h-n) + y_A} & \text{if } \; y < -y_A \cdot \dfrac{n}{h-n} \\ m_+ = \dfrac{x_+ - x_0}{y_+ - y_0} = \dfrac{l}{33 \cdot (h-n) + y_A} & \text{if } \; y > -y_A \cdot \dfrac{n}{h-n} \end{cases}$$

and, thus, finally the following functional equation:

$$x = m \cdot y + (m \cdot y_A + l) \cdot \frac{n}{h-n} \text{ where } \begin{cases} m = m_- \text{ if } \; y < -y_A \cdot \dfrac{n}{h-n} \\ m = m_+ \text{ if } \; y > -y_A \cdot \dfrac{n}{h-n} \end{cases} \tag{4}$$

Example: Let $h = 9.25\,ft$, $y_T = y_N = 3\,ft$, and $y_A = 0$. We get $d \approx 19.57\,ft$ and $b \approx 1.43\,ft$ and the distance between the point P and the service line is

$$\frac{y_P - y_N}{d} = \frac{y_N}{l} \;\Rightarrow\; y_P = \frac{l + d}{l} \cdot y_N = \frac{33 \cdot h}{33 \cdot (h - n) - y_N} \cdot y_N \approx 4.51\,ft$$

The point P is parameterized by $y_T = y_N$ and moves along a straight line. We form the relation between $(x_P - d)$ and $(y_P - 0)$ and get

$$\frac{x_P - d}{y_P} = \frac{1}{33 \cdot (h - n)} \cdot l$$

The expression is indeed independent of y_N; thus, we obtain the following straight-line equation for the points $P = (x_P \mid y_P)$:

$$x_P = \frac{l}{33 \cdot (h-n)} \cdot y_P + \frac{l \cdot n}{h-n}$$

which is the special case of equation (4) for $y_A = 0$.

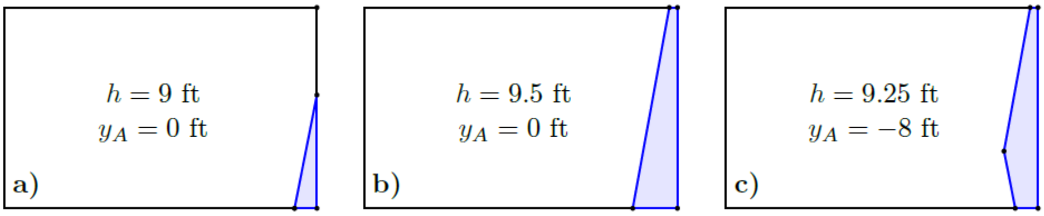

Figure 7. Impact surface (colored) in the deuce court, depending on the height h the player hits the ball and the position of the player along the service line. In panel a), the impact surface is a triangle, in panel b) a trapezium, and in panel c) a pentagon.

(If plane equations are already available in class, the last equation can, of course, also be obtained by determining the intersection line of the plane with the points $A = (-l \mid 0 \mid h)$, $B = (0 \mid 0 \mid n)$, and $C = (0 \mid p \mid \tilde{n})$ with the xy-plane. Here $p = 16.5\ ft \approx 5.03\ m$ is the y-coordinate of a post, and $\tilde{n} = 3.5\ ft \approx 1.07\ m$ is the height of a post.)

If the position of the player is further outside ($y_A < 0$), the result is (for sufficiently large h) a *pentagon* (Figure 7c). The shape of the impact area therefore depends on the serve height and the serve position.

Model C (Consideration of Gravity)

In this model, we now consider *gravity*. We use the same Cartesian system as in Figure 2.

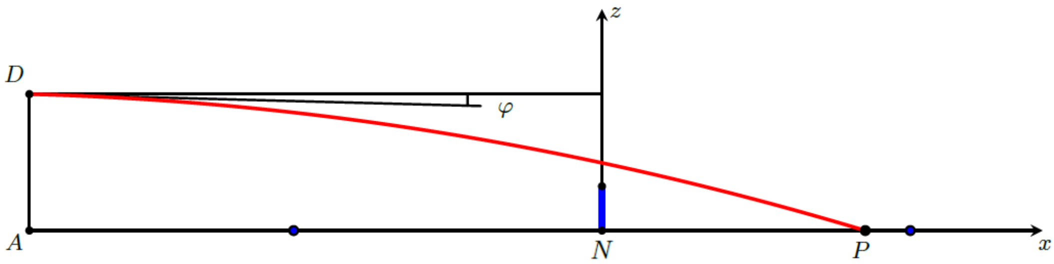

Figure 8. Serve in tennis game III.

Then, the following applies in general (oblique throw, uniform motion; v_0 serve speed):

$$\dot{\vec{s}}(t) = \vec{v} \tag{5}$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \vec{F}_G \tag{6}$$

with

$$\vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} \tag{7}$$

and initial conditions

$$\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix} \tag{8}$$

$$\vec{v}(0) = \begin{pmatrix} v_o \cdot \cos(\varphi) \\ 0 \\ v_o \cdot \sin(\varphi) \end{pmatrix} \quad \text{(Figure 8, } \varphi \text{ can be negative)} \tag{9}$$

Integrating equation (6) with initial condition (9) yields

$$v_x(t) = v_0 \cdot \cos(\varphi)$$
$$v_z(t) = v_0 \cdot \sin(\varphi) - g \cdot t$$

and equation (5) with initial condition (8) can be integrated into

$$s_x(t) = v_0 \cdot \cos(\varphi) \cdot t - l \tag{10}$$
$$s_z(t) = v_0 \cdot \sin(\varphi) \cdot t - \frac{g}{2} \cdot t^2 + h \tag{11}$$

We solve equation (10) for t and put the result into equation (11) to get an equation $z(x)$ for the trajectory of the ball:

$$z = \tan(\varphi) \cdot (x + l) - \frac{g}{2} \cdot \left(\frac{x + l}{v_0 \cdot \cos(\varphi)}\right)^2 + h \tag{12}$$

First, we want to find out what serve speed v_0 is required to hit the tennis ball from a height of h so that the angular measure is $\varphi = 0°$ and the ball passes the net at a height of n above the ground. The serve speed v_0 must therefore satisfy the following equation (note that the point T in the defined system has the coordinates $x = 0$ and $z = n$):

$$n = -\frac{g}{2} \cdot \left(\frac{l}{v_0}\right)^2 + h$$
$$\Rightarrow \quad v_0 = \sqrt{\frac{g}{2 \cdot (h - n)}} \cdot l$$

Note that the required serve speed is proportional to the length of the tennis court. We use $n = 3\,ft$, $h = 9.25\,ft$, $l = 39\,ft$, and $g = 9.81\,m/s^2 \approx 32.185\,ft/s^2$ to get $v_0 \approx 19.1\,m/s \approx 68.7\,km/h$.

Where does the ball land on the ground at this speed?

Put the term for v_0 into equation (12) with $z = 0$ and solve for x:

$$h = (h - n) \cdot \left(\frac{x + l}{l}\right)^2$$
$$\Rightarrow \quad x = \sqrt{\frac{h}{h - n}} \cdot l - l = \frac{\sqrt{h} - \sqrt{h - n}}{\sqrt{h - n}} \cdot l \approx 8.45\,ft \approx 2.57\,m \tag{13}$$

This means that the tennis ball lands $8.45\,ft$ behind the net, and there is still $12.55\,ft$ space in the service box, i.e., considerably more than the calculated $1.5\,ft$ for a straight-line flight according to model A.

The same question 1 we posed in model A: From what height h would the ball have to be hit from in order to have a chance of landing in the service box:

$$h = \frac{(s + l)^2}{(s + l)^2 - l^2} \cdot n = \frac{400}{231} \cdot n \approx 5.19\,ft \approx 1.58\,m,$$

which implies that the player should be $1.06\,m$ tall. This has to be compared with the value of $1.74\,m$ in model A.

To further compare model C with the results of models A and B, we calculate the minimum angle φ and the distance of the point P of impact from the net for the largest serve speed ever measured in an official tennis tournament. In 2012, at the ATP Challenger tournament in Busan (South Korea), Samuel Groth (AUS) reached $v_m = 263.4\ km/h \approx 73.17\ m/s \approx 240.06\ ft/s$

(https://de.wikipedia.org/wiki/Aufschlag_(Tennis)#; 06.09.2022).

The condition that the ball flies over the net is $z(0) = n$. With equation (12) and using $\frac{1}{\cos^2(\varphi)} = \tan^2(\varphi) + 1$, we get

$$n = \tan(\varphi) \cdot l - \frac{g \cdot l^2}{2 \cdot v_m^2} \cdot \left(1 + \tan^2(\varphi)\right) + h$$

The solutions of this quadratic equation in $\tan(\varphi)$ are

$$\tan(\varphi) = \frac{v_m^2}{g \cdot l} \pm \frac{v_m^2}{g \cdot l} \cdot \sqrt{1 + \frac{2 \cdot (h - n) \cdot g}{v_m^2} - \left(\frac{g \cdot l}{v_m^2}\right)^2}$$

For the given values, we calculate $\varphi_1 \approx -8.48°$ and $\varphi_2 \approx 89.38°$. Theoretically, the latter value corresponds to the situation in which the player would play a serve as a *lob*, but in practice, it is not possible to reach v_m with a *lob*. One has to compare $\varphi_1 \approx -8.48°$ with a value of $\alpha_1 - 90° \approx -9.10°$ from model A.

A ball played with the angle φ_1 and speed $73.17\ m/s$ will land $1.21\ m$ in front of the service line.

To further compare the situation with model A, we calculate how big the angle φ may be maximal so that the ball lands on the service line. For this, we solve (see equation (12))

$$0 = \tan(\varphi) \cdot (s + l) - \frac{g}{2} \cdot \left(\frac{s + l}{v_m \cdot \cos(\varphi)}\right)^2 + h$$

This quadratic equation has the following two solutions:

$$\tan(\varphi) = \frac{v_m^2}{g \cdot (s + l)} \pm \frac{v_m^2}{g \cdot (s + l)} \cdot \sqrt{1 + \frac{2 \cdot g \cdot h}{v_m^2} - \left(\frac{g \cdot (s + l)}{v_m^2}\right)^2}$$

For the given values, we calculate $\varphi_1 \approx -7.81°$ and $\varphi_2 \approx 89.05°$, and again the latter corresponds to a *lob*. Compared with model A, the allowed angles are now within 0.67° (instead of 0.33°). Compared to this model, the distance between the longest and shortest allowed serve for v_m is now $1.21\ m$ (instead of $0.46\ m$).

Model D (Consideration of Gravity and Aerodynamic Drag)

In model D, in addition to the gravitational force, we also consider the *aerodynamic drag*. In a simple physical model of aerodynamic drag, one assumes that an object moving through a gas will have some of the gas particles in its path accelerated to the speed of the body. If the entire parcel of air in the way of the object were to accelerate to the speed of the object, the following amount of aerodynamic drag force would be obtained using the conservation of energy:

$$F_D = \frac{1}{2} \cdot \rho \cdot A \cdot v^2$$

Here, $\rho = 1.167\ kg/m^3$ is the *air density*, $A = \pi \cdot r^2$ is the *cross-sectional area* of the object, and v *is* the *velocity* of the object relative to the air. To account for the dependence of the aerodynamic drag on the shape and surface properties of the object, the *drag coefficient* C_D is introduced. In vectorial form, the aerodynamic drag is then as follows:

$$\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot |\vec{v}| \cdot \vec{v} \tag{14}$$

Robinson and Robinson (2018, p. 5) obtained a velocity-dependent formula for C_D based on the test results of Goodwill et al. (2004, p. 950) (wind tunnel experiments with ball velocities of $25\ m/s$ and $50\ m/s$ using eight different types of new tennis balls) by linear interpolation:

$$C_D = 0.6204 - \frac{v - v_d}{\tilde{v}}$$

where v is in m/s, $v_d = 50\ m/s$, and $\tilde{v} = 1025\ m/s$.

However, since the tennis ball needs only about $1/3\ s$ from leaving the racket head to hitting the service area and C_D varies from ball to ball anyway, we calculate with $C_D = 0.62$. The following should always be kept in mind (Robinson & Robinson, 2018, p. 5): "C_D certainly does vary from ball to ball and also varies with the state of wear of an individual ball."

We now obtain the following system of differential equations:

$$\dot{\vec{s}}(t) = \vec{v}(t) \tag{15}$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \left(\vec{F}_G + \vec{F}_D\right) \tag{16}$$

with $m = 0.0577\ kg$ (according to the tennis rules of the ITF (2019), a tennis ball should be at least $56.0\ g$ and at most $59.4\ g$).

The consideration of aerodynamic drag requires *numerical integration* of the differential equations. In a variety of *electronic tools* (e.g., Mathematica, GeoGebra, Maple), numerical solution methods are already implemented, and one does not have to worry about the procedure (Runge–Kutta) and the choice of an optimal time step. Our calculations were performed using GeoGebra and the NSolveODE command.

Table 1. Consideration of gravity and aerodynamic drag.

Serve speed (m/s)	Minimal serve-angle φ	Distance from the net (φ minimal) (m)	Maximal serve-angle φ
45	−7.12°	4.15	−5.40°
50	−7.50°	4.35	−6.03°
55	−7.77°	4.54	−6.51°
60	−7.98°	4.68	−6.87°
65	−8.15°	4.80	−7.15°
70	−8.28°	4.90	−7.37°

For $h = 9.25\ ft$ and the serve speeds given in Table 1 and the serve-angles calculated, the solution of the system of differential equations (15) and (16) leads to the increases in the usable margin given in this table compared to the assumption of straight-line flight in model A. (In the table, minimal serve-angle means that the tennis ball crosses the net at a height of the net, and maximal serve-angle means that the center of the ball hits the service line.)

Model E (Consideration of Gravity, Aerodynamic Drag, and Topspin)

Regarding the advantages of using *topspin* in tennis serves, Brancazio (1985, p. 373) comments thus:

"Topspin is […] essential to a high-speed serve: it is virtually impossible for a ball served at a speed of $120\ \mathrm{mi/hr}$ [$\approx 193\ \mathrm{km/h} \approx 53.6\ \mathrm{m/s}$; the authors] or more to drop fast enough due to gravity alone to clear the net and still bounce in the service box. The added downward force [the so-called 'Magnus force'; the authors] contributed by topspin makes it possible."

The effect of topspin (the rotation of the ball around the axis perpendicular to the velocity vector and parallel to the horizontal plane; namely rotation in the mathematical negative sense) is illustrated in Figure 9 and explained in the following quote.

> "[...] if a ball with topspin is rising upwards at an angle to the court, the Magnus force tends to push the ball down onto the court and it pushes it forwards in a direction parallel to the surface. If a ball with topspin is falling towards the court surface, the Magnus force pushes the ball downwards and backwards. A ball with topspin therefore falls onto the court at a steeper angle than a ball without topspin." (Cross, 2004, pp. 370–371)

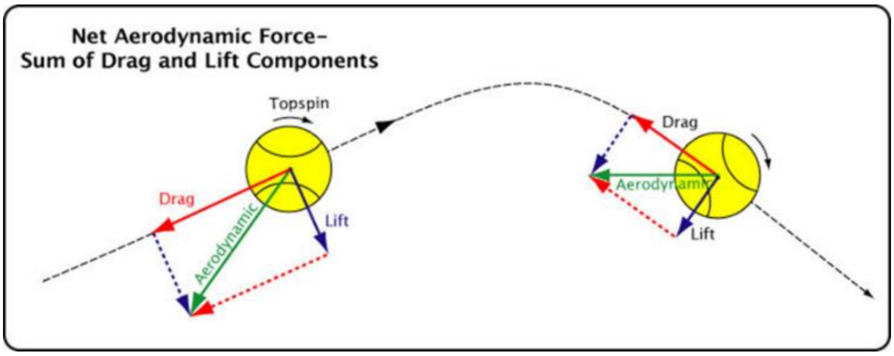

Figure 9. Total aerodynamic force during the flight of a tennis ball with a topspin (Cross & Lindsey, 2013, p. 11).

In order to set up a differential equation system for the description of the ball curve, in which the gravity, the aerodynamic drag, and the topspin are considered, we still need the term of the angular velocity, respectively, the angular velocity vector $\vec{\omega}$:

- The direction of the angular velocity vector is given by the axis of the rotation of the ball, where the screw rule gives the positive direction.
- The magnitude of the angular velocity vector, the angular velocity $\omega = |\vec{\omega}|$, corresponds to the added angular measure per unit time, where we measure the angle in radians.

The relationship between angular velocity ω and frequency f of the rotating ball, i.e., the number of revolutions per unit of time, is thus:

$$\omega = 2 \cdot \pi \cdot f$$

The *lift force* is given by

$$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot \frac{|\vec{v}|}{|\vec{\omega}|} \cdot \vec{\omega} \times \vec{v} \tag{17}$$

The dependence from $\omega = |\vec{\omega}|$ is given by

$$f(\omega) = 1 - \exp\left(-\frac{\omega}{\omega_0}\right)$$

where $\omega_0 = 403\ \mathrm{rad/s}$ and $C_L = 0.319$ (Robinson & Robinson, 2013, p. 6; Here and in the following models, we neglect the fact that angular velocity decreases somewhat during ball flight; see Robinson & Robinson, 2018, Appendix C, for more details.)

For a pure topspin $\vec{\omega} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}$ and a trajectory in the xz-plane, i.e., $\vec{v} = \begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix}$, the lift force reduces to

$$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot |\vec{v}| \cdot \begin{pmatrix} v_z \\ 0 \\ -v_x \end{pmatrix}.$$

Table 2. Consideration of gravity, aerodynamic drag, and topspin.

Serve speed (m/s)	Minimal serve-angle φ	Distance from the net (φ minimal) (m)	Maximal serve-angle φ
45	−5.38°	3.54	−2.72°
50	−5.77°	3.67	−3.37°
55	−6.04°	3.79	−3.84°
60	−6.25°	3.89	−4.20°
65	−6.42°	3.96	−4.49°
70	−6.55°	4.03	−4.71°

Now, we obtain the following system of differential equations:

$$\dot{\vec{s}}(t) = \vec{v}(t) \tag{18}$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \left(\vec{F}_G + \vec{F}_D + \vec{F}_L\right) \tag{19}$$

At $h = 9.25\ ft$ and $\omega = 300\ \mathrm{rad/s}$ as well as the serve speeds given in Table 2 and the serve-angles calculated, the solution of systems (18) and (19) leads to the increases of the usable margin given in this table compared to the assumption of straight-line flight in model A.

The results in this regard are compared with the results in Table 1 to quantify the beneficial effect of the topspin with $300\ \mathrm{rad/s}$ on the serve.

Model F (Consideration of Gravity, Aerodynamic Drag, Sidespin/Slice, and Wind)

For an arbitrary angular velocity ω and an arbitrary wind speed $\vec{W}$, we obtain the same equation of motion as in equations (15) and (16), but with aerodynamical forces where one has to replace the absolute velocity $\vec{v}$ of the ball by the relative velocity $\vec{v} - \vec{W}$ of the ball with respect to the air.

$$\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot \left|\vec{v} - \vec{W}\right| \cdot \left(\vec{v} - \vec{W}\right)$$

$$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot \frac{\left|\vec{v} - \vec{W}\right|}{|\vec{\omega}|} \cdot \vec{\omega} \times \left(\vec{v} - \vec{W}\right)$$

(Robinson & Robinson, 2013, p. 7; Robinson & Robinson, 2018, pp. 5–6).

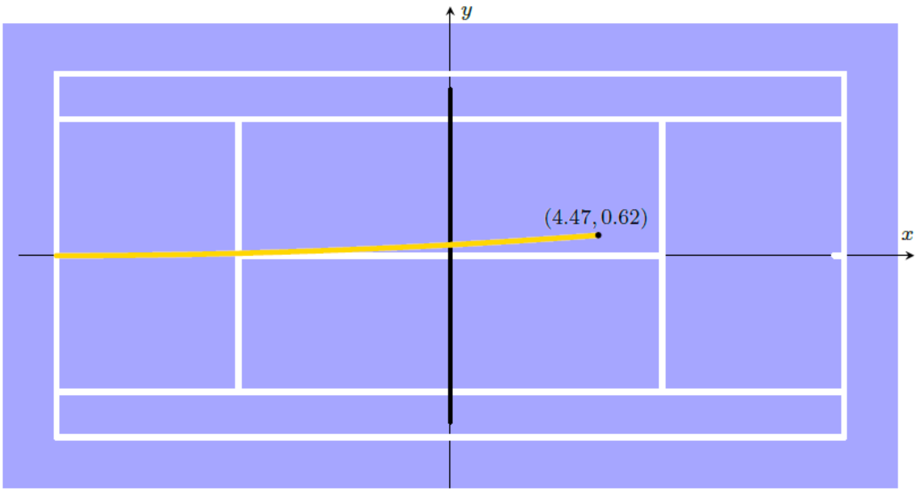

Figure 10. Projection of the trajectory on the xy-plane. One can see well the effect of the Magnus force. The ball drifts away to the left in the direction of the trajectory due to the lateral spin.

We consider the following special cases:

$$\vec{W} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(tailwind)}$$

$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ 300 \end{pmatrix} \qquad \text{(sidespin in the direction of the } y\text{-axis)}$$

We get the following results ($v_0 = 50$ m/s):

- The minimal allowed angle to cross the net is $\varphi = -7.52°$.
- The ball lands at $P = (4.47 \mid 0.62)$ (see Figure 10).

Model G (Consideration of Gravity, Drag, and a Combination of Topspin and Sidespin/Slice)

As good tennis players know, even models E and F do not fully describe reality. Robinson and Robinson (2018, p. 13) reason as follows: "In a practical situation pure side-spin (sliced) serves are probably rarely generated and […] pure topspin serves are virtually impossible to deliver. Spinning serves probably include components of side-spin, topspin and possibly axial-spin in various amounts, either by accident or intention."

In model G, we examine a combination of topspin and sidespin. We start with *pure* topspin ($\vec{\omega}_1$) and rotate this spin vector by about 15° at a time until the rotated spin vector describes a pure sidespin. Specifically, we work with the following rotation vectors (all values in rad/s):

$$\vec{\omega}_1 = \begin{pmatrix} 0 \\ 600 \\ 0 \end{pmatrix} \quad \vec{\omega}_2 = \begin{pmatrix} 0 \\ 580 \\ 155 \end{pmatrix} \quad \vec{\omega}_3 = \begin{pmatrix} 0 \\ 520 \\ 300 \end{pmatrix} \quad \vec{\omega}_4 = \begin{pmatrix} 0 \\ 424 \\ 424 \end{pmatrix}$$

$$\vec{\omega}_5 = \begin{pmatrix} 0 \\ 300 \\ 520 \end{pmatrix} \quad \vec{\omega}_6 = \begin{pmatrix} 0 \\ 155 \\ 580 \end{pmatrix} \quad \vec{\omega}_7 = \begin{pmatrix} 0 \\ 0 \\ 600 \end{pmatrix}$$

This means that in all cases $\omega_k = |\vec{\omega}_k| \approx 600$ rad/s is valid and the angle of two successive rotation vectors is approximately $15°$. In addition, $v_0 = 160\ \text{km/h} \approx 44.4\ \text{m/s}$ is assumed.

To compare model G with the models already considered, we first determine the smallest angle so that the tennis ball just sees the other side of the net. The serve is performed along the center line. We get that $\varphi = -3.60°$ and the ball lands 3.07 m behind the net.

Further calculations show that for *pure* topspin, the angle φ must be in the interval $-3.60°$ and $0.05°$ to hit a valid serve; for *pure* sidespin, this interval is $[-6.93°, -5.22°]$. The serve with a mixture of topspin and sidespin ($\vec{\omega}_4$) is valid if φ is between $-4.52°$ and $-1.48°$.

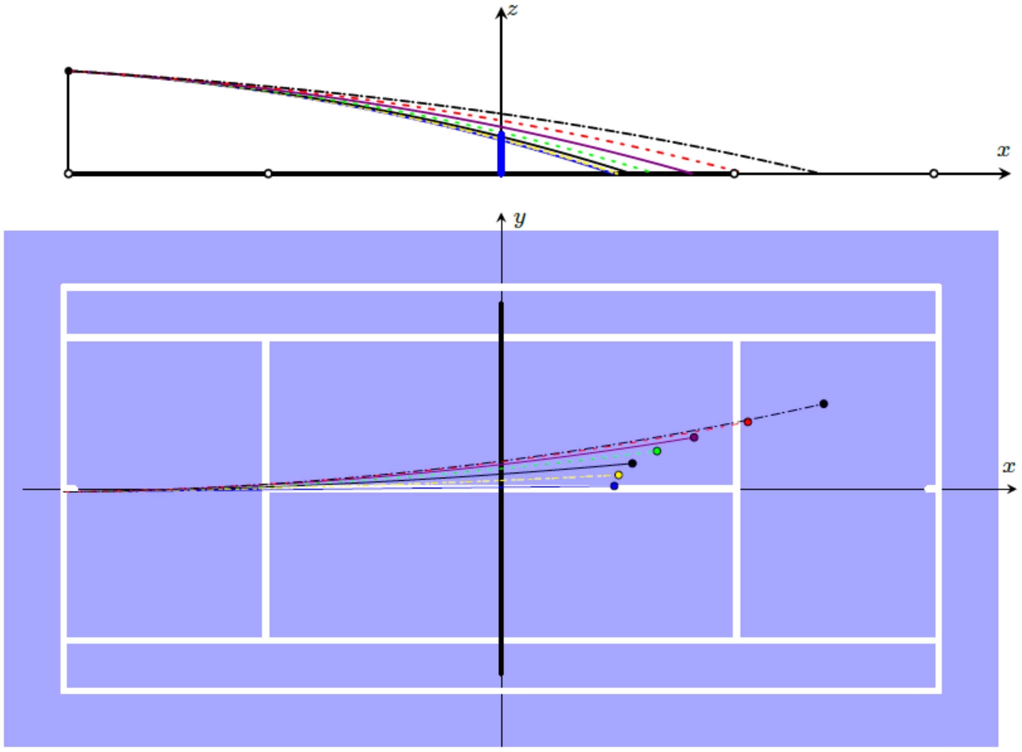

Figure 11. Serves combining topspin and sidespin (Part I). The initial projection directions were chosen so that the *pure* topspin serve ($\vec{\omega}_1$) results in the ball landing in about the middle of the deuce court. The other six serves have the same initial projection directions as the topspin serve. Only five of the seven serves are valid; see the lower panel of this figure (compare with Robinson & Robinson, 2018, p. 14).

Table 3. Consideration of gravity, aerodynamic drag, and Magnus force.

Spin vector	Minimal serve-angle φ	Distance from the net (φ minimal) (m)	Maximal serve-angle φ
$\vec{\omega}_1$	$-3.60°$	3.07	$+0.05°$
$\vec{\omega}_2$	$-3.68°$	3.12	$-0.12°$
$\vec{\omega}_3$	$-4.00°$	3.21	$-0.64°$
$\vec{\omega}_4$	$-4.52°$	3.37	$-1.48°$
$\vec{\omega}_5$	$-5.22°$	3.58	$-2.56°$
$\vec{\omega}_6$	$-6.04°$	3.86	$-3.84°$
$\vec{\omega}_7$	$-6.93°$	4.21	$-5.22°$

Figure 11 shows the calculated trajectories for all the serves with the seven spin vectors. Note the rapid increase in the x-coordinate of the landing point as the spin vector is rotated 15° at a time.

A summary of all important calculated quantities can be found in Table 3.

"This […] illustrates that it might be worth sacrificing a relatively modest amount of side-ways movement for the extra margin of error introduced by some topspin, this extra margin of error being evidenced by the difference in the net clearance for these two serves […]." (Robinson & Robinson, 2018, p. 14; see the top panel of Figure 12)

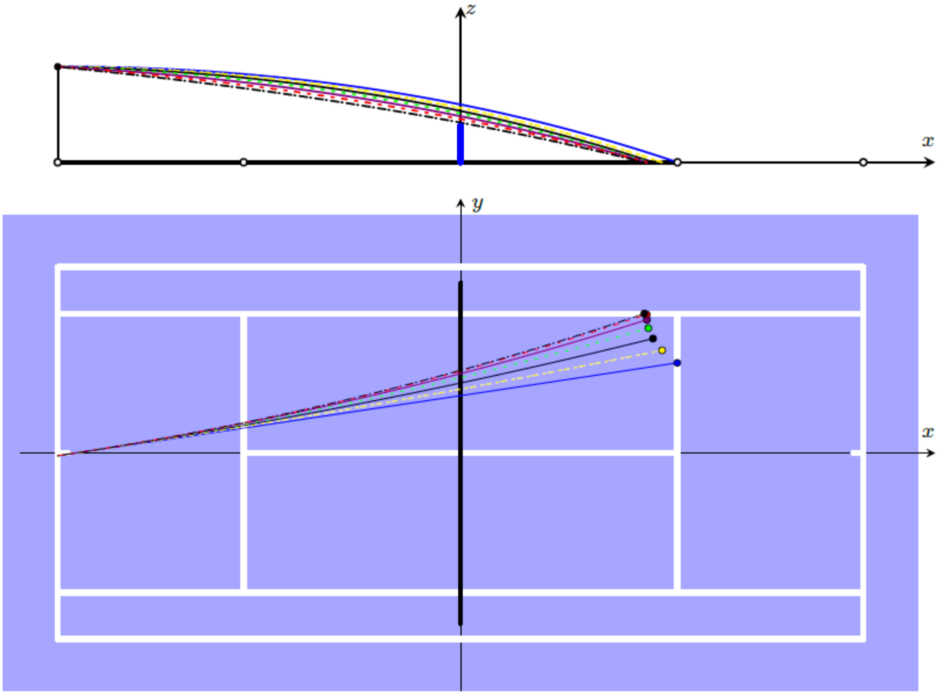

Figure 12. Serves combining topspin and sidespin (Part II). Here, the horizontal direction of the serve is constant. The angle φ was chosen so that for a *pure* topspin serve, the ball lands on the (opposite) service line and for a *pure* sidespin serve, the ball arrives on the sideline. The other five impact points are approximately equally distributed. The values for φ are, starting with *pure* topspin, $+0.19°$, $-0.38°$, $-1.11°$, $-2.01°$, $-3.06°$, $-4.27°$, and $-5.64°$ (compare with Robinson & Robinson, 2018, p. 15).

With the help of the GeoGebra app (see https://www.geogebra.org/m/md5jbrxh; author René Fehlmann), it is possible to test the different models using your own data.

In this section, an adaptive modeling task for different school levels is now available for the first time (RQ1).

IV. LOCATION OF MODELS A TO G IN THE *MODELING CYCLE*

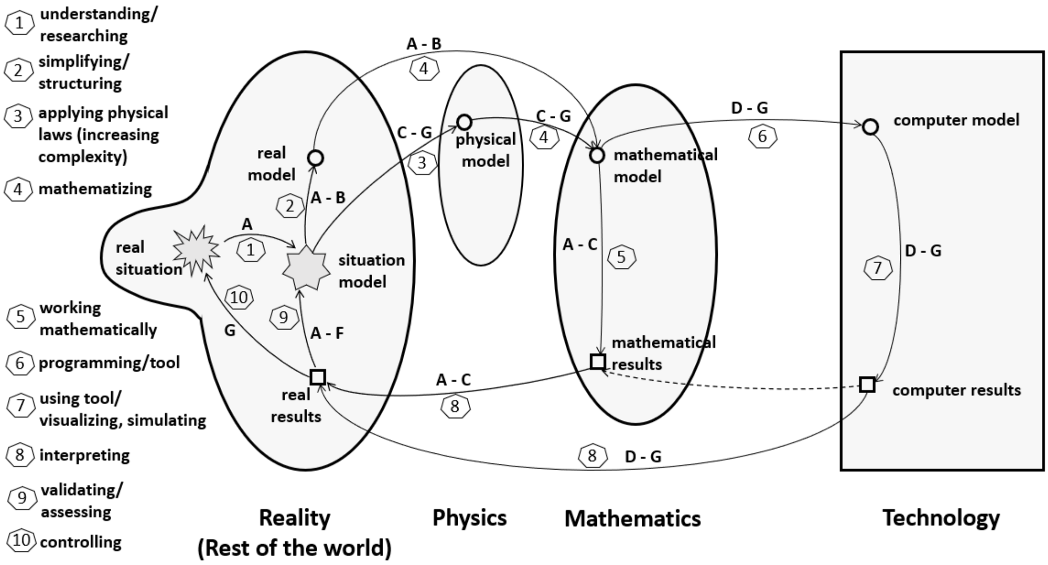

Figure 13. Location of models A to G in the *modeling cycle* of Blum and Leiss (2005, p. 19) and in the extension of Greefrath (2018, p. 52); additional inclusion of physical modeling by the authors.

The transitions from model to model each take place through a change in the situation model:

- From model A to model B: Instead of the simplifying assumption of the same net height everywhere in model A, the real situation is considered that the net is higher at the edge than in the center.
- From model B to model C: Gravity is taken into account, and thus a physical model is made before a mathematical model is developed.
- From model C to model D: In addition to gravity, aerodynamic drag now comes into play and, with it, the law of aerodynamic drag.
- From model D to model E: Now the specific situation of the serve in tennis is taken into account that the ball can be given topspin and thus another force acts on the ball, the Magnus force.
- From model E to model F: In model F, two other possible influences come into play: the Magnus force generated by sidespin, but now acting in a different direction than in topspin, and the wind that occurs, whereby any conceivable direction can be specified for it.
- From model F to model G: Here, we consider the combination of topspin and sidespin, which often occurs in tennis practice. We now assume no wind, but this should be very close to the real situation (at least if we have not made a serious error in the modeling).

Because of the great importance of physical modeling in our example, we plead for emphasizing the physical modeling in the *modeling cycle* of corresponding situations also by the fact that it appears there explicitly and is not hidden in the *rest of the world* (see Figure 13; see RQ1). In this figure, the *digital tool* used appears independently. In models D to G, the digital tool result can be interpreted directly concerning the real results without having to make a detour via mathematical results (compare with the dashed arrow in Figure 13). However, this is not necessarily the case for all models using a digital tool (see the remarks of Greefrath, 2018, p. 51).

V. MERITS OF THE PRESENTED MODELS ON TENNIS SERVE IN TERMS OF TEACHING AND LEARNING MATHEMATICAL MODELING IN MATHEMATICS EDUCATION

With regard to successful differentiation in the classroom, we pursue an *open* approach with the development and use of *parallel adaptive tasks* (Bardy et al., 2021). In order to enable this open approach within the learning area of *mathematical modeling*, we have chosen a topic from sports, which is *mostly known* to the students and which we assume to increase the motivation of the learners in particular (whereby the topic of sports—from our own teaching experience—generally already arouses *interest* in learners), but which is also well suited for a sense-making of mathematical content. According to Blum (1985), one argument for a well-suited modeling example in mathematics education is the *realism* of the example. Specifically, with the help of already-known mathematical and physical learning content, an attempt is made to describe the tennis serve with its trajectory and the landing point of the ball. This can be used to *consciously sharpen the view* of reality.

Embedding in the curriculum is provided at both school levels, e.g.:

- *Middle school* level competency: Students can mathematize, represent, calculate, interpret, and verify results in factual situations.
- *High school* level competency: Students will recognize the importance of differential equations, solve them, and be able to interpret the solutions—understand modeling and simulations using appropriate examples.

We have developed three *parallel adaptive tasks* (see the Appendix) that contain seven different mathematical models (models A to G) of the tennis serve. So far, such mathematical modeling, which allows for an ever-increasing accuracy of the description of the tennis serve, has not yet been proposed and elaborated in this form for mathematics education.

Above all, the different levels at which the individual columns of our parallel adaptive tasks can be worked on are central with regard to heterogeneous learning groups. The example of the tennis serve is accessible to all students, i.e., *not too complex*, and not too specific, but at the same time *challenging* (cf. Blum, 1985, p. 224). At the middle school level, where the focus is primarily on an *introduction to mathematical modeling*, models A through C offer modeling at different levels, with model A also showing a very descriptive approach (e.g., straight-line flight; marking the impact surfaces on the court) that helps especially low-performing students with their modeling. There is also a strong simplification of the model assumptions and structuring of the work instructions.

The GeoGebra app was initially used particularly to check the models developed. However, it can also be used as a tool to help low-achieving students find out about dependencies that influence, e.g., the impact area of the tennis ball in the field.

Specifically, our *goals* (following Maaß, 2003, p. 36) can be formulated with the use of the tennis serve example as follows:

- *Pragmatic goal*: Learners will learn about, work with, and understand a complex example of mathematical modeling by thinking about the forces affecting the flight of a tennis ball after serve, discussing model assumptions, and ultimately developing a mathematical model that is transformed into a *digital tool model* (Greefrath, 2018) using programming.
- *Methodological goal*: Modeling and reality references should provide students with competencies for applying mathematics in simple (e.g., model A) and complex situations (e.g., model C). Methodically, we use parallel adaptive tasks in order to develop the competencies at different levels. A thematization of the model-building cycle supports the students in carrying out their modeling at the respective level.
- *Cultural goal*: Modeling and reality references should provide students with a balanced view of mathematics as an overall social and cultural phenomenon. “Modeling and a worldview from a model standpoint are traits of human intellectuality.” (Blum, 1996, p. 21, translation by the authors) The example of the tennis serve shows that mathematical and physical knowledge is helpful in describing sporting activities and thus in explaining the movements of a ball. Also, through such analyses, especially in sports, tactical strategies can be revealed.
- *Pedagogical goal*: By modeling the tennis serve in a realistic way, problem-solving and reasoning skills particularly are to be promoted and trained in students. This also includes calling for creative behavior with the deliberate use of heuristic strategies. Based on the underlying calculations, the following questions can be discussed: How does the impact area change when the height of the serve is increased? How much does the serve angle influence the trajectory of the tennis ball?
- *Learning-psychological goal*: By dealing with a realistic modeling example, students should be enabled to have a more open-minded attitude toward mathematics lessons, the learners’ motivation to deal with mathematics should be increased, and the comprehension and retention of mathematical content should be supported. This provides the opportunity to “experience learned skills or mathematical knowledge as useful in meaningful contexts” (Leuders, 2003, p. 139, translation by the authors).

In the following, we summarize the merits of our example using a set of criteria from Blum (1985, p. 223): The tennis serve example

- contributes to the goals listed above,
- is close to reality,
- is familiar to the students at the respective school type, i.e., originates from their environment,
- fits the curriculum of the mathematics class (in this case, the middle and high school),
- is accessible to students, i.e., not too complex, and not too specific,
- is challenging and stimulates interest,
- is sufficiently open-ended, i.e., leaves room for student activity, and
- leads to relevant results with reasonable effort.

Blum (1985) has recommended that of the criteria mentioned, most (not necessarily all) should be met in order for an application example to be classified as *suitable for teaching mathematics*. From our point of view, our example even fulfills all mentioned criteria and, additionally, the criterion of differentiation in each addressed school type. Overall, the treatment of our example in the middle and high school mathematics classroom, along with other appropriate examples, is likely to promote "modelling competency and modelling (sub-)competencies" (see Niss & Blum, 2020, pp. 80–83).

Methodically, the modeling process with its individual facets in the modeling cycle (see Figure 13) should always be discussed with the students when working on modeling tasks in order to promote "transferable methodical qualifications" (Blum, 1985, p. 225, translation by the authors). In particular, the transition from reality to physics and mathematics is central here, each showing different qualities in our models. However, the transition from mathematics to reality should also be central. The mathematical results must be reflected in detail. "[…] the underlying 'philosophy' [must, the authors] be made conscious and discussed with students in an age-appropriate way" (Blum, 1985, p. 226, translation by the authors).

Of course, the general methodical principles also apply, such as *linking to the students' previous knowledge* or *student self-activity,* when discussing the modeling of the tennis serve. The students' *own experience* of the difficulties, hurdles, and limits of modeling as accurately as possible should also become clear to them during the process. "Only in this way can students also experience—at least to some extent—the intellectual effort involved in dealing with applications in general" (Blum, 1985, p. 226, translation by the authors).

There may also be *organizational*, *personal*, and *material barriers* to using the tennis-serve task. Teachers who think that the physical background is too complex will only use models A and B for modeling the tennis serve in class and then want to use the GeoGebra app for more precise modeling results. The time required for individual models of the tennis serve can be cited as an obstacle to its use in the classroom. In this case, a project-oriented lesson with a higher time requirement is cited as useful, but not feasible.

Challenges related to modeling in the classroom could also include personal reservations of the teacher for the demanding activity of modeling in the mathematics classroom, with the additional increased preparation effort associated with it. In particular, textbooks currently offer few real opportunities for meaningful and sense-making modeling. We have met this challenge with our elaborated models of the tennis serve.

However, the use of realistic modeling in the classroom does not have to be associated only with difficulties but also enables the formulation of *solution aids* specifically for application-related tasks. Greefrath (2018, pp. 200–205) distinguishes between different types of aids:

- Aids designed to motivate students to continue working (*motivational aids*) or to alert them that their strategies and solutions are or are not on target (*feedback aids*)
- Aids that offer concrete content-based hints for solving the task (*general and content-oriented strategic aids*, *content-based aids*)

One way of supporting students when working on modeling tasks is through the use of solution plans (*indirect aids*). These plans are often based on substeps or subskills that play an important role in modeling. Solution plans can support the processing of modeling tasks. A simplified solution plan, according to Blum (2006, p. 21), is as follows: 1) understand the task, 2) create a model, 3) use mathematics, and 4) explain the result. Our tasks presented here implicitly include such a solution plan, as they already guide students in what to do when reading the task.

VI. TEACHING EXPERIENCES WITH THE PARALLEL ADAPTIVE TASKS

In order to find out firstly, how students deal with the developed adaptive modeling tasks, secondly, how working through the modeling example on tennis serve affects learners' motivation, and thirdly, what didactic consequences should be drawn from this, we tested two of the developed parallel adaptive tasks in two learning groups (RQ3).

Our teaching experience with the presented models is based on two teaching units. In the *first group*, 17 students ($f = 6, m = 11$), who are in the 9th grade (group I), participated. They were given only models A and B to choose from. In fact, beforehand, it was not expected that a student of this group would have a level of performance that would allow the processing of model C. In the lessons prior to this unit, the group learned about centric stretching, ray theorems, and their applications. The challenge of applying the ray theorem in a horizontal and vertical view was new to the class and represented a transfer effort to be made. The *second group* consisted of 9 student teachers ($f = 3, m = 6$). Thus, these students had already completed the subject study of mathematics with a master's degree (group II). They could choose between model C, which was to be treated analytically, and models D and E, which required numerical integration.

Table 4. Excerpt from the questionnaire with frequencies.

General questions Group I (17 Students) / Group II (9 Student teachers)	fully applies	rather true	rather not true	does not apply at all
I like to deal with mathematical problems.	.35 / .78	.65 / .11	.00 / .11	.00 / .00
Through reality-based examples, I experience what mathematics can be used for.	.41 / .44	.47 / .44	.12 / .11	.00 / .00
In my math classes, we often deal with reality-based examples.	.38 / .44	.56 / .56	.06 / .00	.00 / .00
Rate the steps of the modeling cycle by difficulty.	at most difficult	less difficult	somewhat difficult	least difficult
Understanding the problem	.12 /.22	.35 / .44	.24 / .11	.29 / .22
Setting up a mathematical model	.35 / .56	.18 / .22	.35 / .11	.12 / .11
Problem solution	.35 / .11	.29 / .22	.24 / .56	.12 / .11
Comparison of the solution with reality	.18 / .22	.18 / .11	.24 / .33	.41 / .33
I have calculated tasks with different variants in class before, where I was allowed to choose the variant.	yes		no	
	.29 / .00		.71 / 1.00	
More such tasks with variants should be offered in the textbook.	yes		no	
	.63 / .71		.38 / .29	

Both groups of learners were subsequently handed out a *questionnaire* (an excerpt see Table 4). The participants had to give their opinions on the following questions, among others:

- What influences the choice of the model?
- How challenging is the task?
- How difficult have the individual steps of the modeling cycle been?
- How motivated are learners to do mathematics, and what role can modeling tasks play in their motivation?
- How important is a concrete reference to everyday life for the learners?
- How are parallel adaptive tasks perceived in mathematics lessons?

Overall, students in both groups had worked on only a few (group I: 29%) or none (group II: 100%) parallel adaptive tasks in their previous classes. Furthermore, 35% of the students in group I and 78% of the students in group II liked to deal with mathematical problems, and 88% of the students in both groups affirmed that they could experience what mathematics can be used for through real-life examples (sense-making). Also, 38% (group II: 44%) of the students stated that they often dealt with reality-based examples in mathematics lessons.

Of course, difficulties arise when working on the modeling tasks. With respect to the modeling cycle, these can be localized particularly at the following points (see also Maaß, 2004, pp. 160–167): difficulties in the whole modeling process, difficulties in model creation, difficulties in mathematical work, difficulties in interpretation, and difficulties in validation. With respect to the design of our tasks on the tennis serve, difficulties of the students can be expected, especially in model creation and mathematical work. In the case of model creation difficulties, such as using incorrect assumptions or inappropriately simplifying the real situation, the task text has a special role to play. We have tried to positively influence the understanding of the problem by clearly stating it. In addition, model simplifications are clearly described. Group I students rate this modeling step as easy by 35% (group II: 44%).

When transferring the real model into a mathematical model, problems can also occur, such as the use of incorrect formulas or the selection of wrong symbols and algorithms. Here, we offer support in the tasks by providing formulas and symbols directly. In the questionnaire for the evaluation of the four steps of the modeling cycle, 35% of the students in group I rated this step as the most difficult (group II: 56%). Problems can also occur when working in a mathematical model, especially calculation errors. Of the students of group I, 35% saw this step as the most difficult (56% of group II as somewhat difficult).

Interpreting and validating the results of the mathematical model is often not taken seriously enough by students (Greefrath, 2018, p. 193). However, students also fail to validate the results as plausible. Validation of the model also plays a role in this. Both groups found this modeling step easy (41% and 33%, respectively).

The choice between model A and model B was made exclusively in favor of model A. The reasons given for this have to do with the fact that it is important for the students to be able to successfully complete the task with certainty, and model A provides this certainty. In part, however, it was also recognized that there is a hierarchical structure in the two models, and the mathematical treatment of model B builds on the knowledge gained from model A. Therefore, there is no way around working on model A first anyway, and model B can then be studied afterward. Even though in both learning groups, most students had chosen the same model, the possibility of choice was much appreciated.

In both groups, the students gave the following reasons for their choice of model A (or C in group II): *"If I solve model A, I can also solve model B (from easy to difficult)."*; *"Because model A already looked difficult enough."*; *"I wanted to do model A first so that I understand the principle."*; *"Because model B seemed a bit too difficult to me."*; *"Because I work in sequence."*; *"Model C looked more practical."*; *"I haven't solved such a task for a while (I wanted to approach it slowly)."*. However, students also stated the following for model C, e.g., *"More exciting because there is more to consider."*.

Not unexpectedly, for the group I students, solving the mathematical problem was judged to be much more difficult than the master's degree students did. Nevertheless, even the latter judged the model with the Magnus force to be very challenging. However, this again had to do with the fact that the physical phenomenon of this force was known from everyday life, but not in a quantitative physical description.

The fact that the tasks at hand dealt with a phenomenon that they knew well from everyday life was consistently judged as positive and motivating. Regarding parallel adaptive tasks, students in both groups (63% and 71%, respectively) would like to see more elective tasks in their textbooks. This is an indication that students would like to have a processing level adapted to their level of proficiency, especially with regard to modeling tasks.

VII. CONCLUSION

The seven mathematical models of the tennis serve presented in this research show that it is possible to design models of a real situation at very different levels of competence. With our research work, an adaptive modeling task for three different school levels is now available for the first time. Using the real example of the tennis serve, it shows how an adaptive task can be set up with respect to the modeling cycle (RQ1). Thus, adaptive mathematical teaching with modeling tasks can be implemented and focused on heterogeneous learning groups in middle and high school, but also in college. In this way, different learning requirements in a classroom can be considered, and the modeling cycle can be addressed in each case (RQ1).

Students can also experience the ever-increasing accuracy (realism) of modeling. The surface structure is implemented by us via a parallel adaptive task. Central is the conception of the deep structure of the task via simplified or reduced model assumptions. In particular, we used here the task features such as difficulty, support, and accessibility (RQ2).

The use of technology also serves this purpose. The use of digital tools is also suitable to facilitate the work of students in the field of mathematics or, in particular, to enable low-achieving students to experiment with influencing variables (RQ2). Independent programming by students or interdisciplinary teaching, e.g., with the subject of computer science, is also possible. A subsequent practical implementation on the tennis court is also conceivable and offers an opportunity to implement the topic of tennis to serve as part of a larger mathematical–physical project.

One result of our teaching experience (RQ3) is that students must be intensively introduced to working with parallel adaptive tasks since they must also correctly assess their competence level in particular in order to be able to select the task that is suitable for them. This is a long-term process in which the teacher also plays an important role. Students must learn to have confidence in themselves and then be able to rely on the teacher’s help.

Also, learners should—in addition to parallel adaptive tasks—have already gained experience with mathematical modeling before working on the mathematical modeling of the tennis serve. This experience facilitates the selection of specific modeling steps and is helpful for the selection of an appropriate column of the parallel adaptive task.

Overall, the teacher should know the competence level of his class well in advance of using adaptive tasks. Of course, it can make sense to offer, e.g., a parallel adaptive task with only two columns/levels.

For the modeling example of the tennis serve, the knowledge of physical phenomena in individual adaptive tasks is a central prerequisite or can be a learning hurdle. This knowledge must be built up either in advance, e.g., in physics lessons, or in parallel to the processing in mathematics lessons. We were aware of this prior knowledge of physics when developing the parallel adaptive tasks and therefore planned them to be simple for low-achieving students (left column of the task).

Realistic examples from sports have an overall positive effect on the motivation of students to deal with mathematical models and to show a certain perseverance here (RQ3). This possibility should be used much more, and textbooks should increasingly offer tasks from the field of sports.

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APPENDIX: Tasks for Students

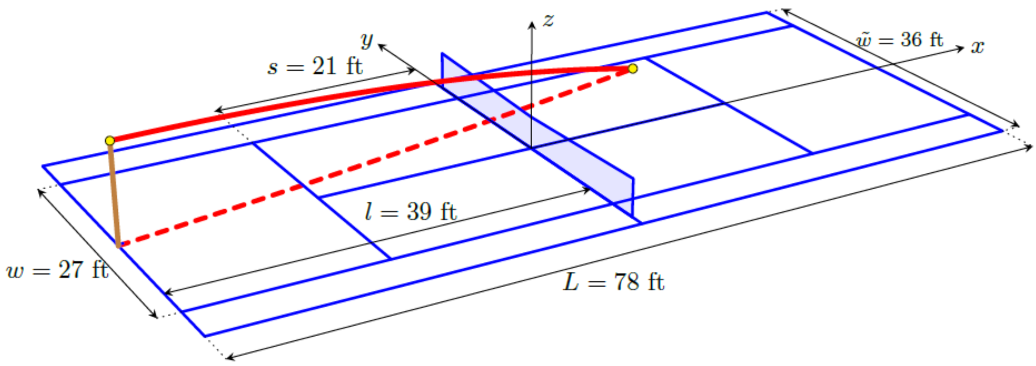
Note: The following tasks (the models) are shown here, one after the other. As a *parallel adaptive task*, they would have the following surface structure (Table A).

Table A. The seven models in the surface structure of *parallel adaptive tasks*.

Middle School			High School			College		
Parallel Adaptive Task 1			Parallel Adaptive Task 2			Parallel Adaptive Task 3		
Model A	Model B	Model C	Model C	Model D	Model E	Model E	Model F	Model G

The serve is considered one of the most difficult strokes in tennis. In this task, the aim is to find out what challenges players face when they open a rally with the serve.

The dimensions of a tennis court are given in the following figure. In singles play, the net posts are 3 ft outside the sideline of the singles court. The height of the net is 3 ft in the center and 3.5 ft at the net posts.



Choose one of the following models around the tennis serve.

Model A (Middle School)

Assume that the ball leaves the serving person's racket at height $h = 9.25\,ft$ above the baseline (relative to the center of the ball), flies in a straight line, and lands in the correct part of the service box. For simplicity, assume that the net is the same height everywhere.

First, consider the situation where the ball flies just over the net.

a) What does the path of the ball look like in the xy-plane? Make a sketch.

b) Consider the plane that is perpendicular to the xy-plane and in which the rectilinear path of the ball lies. What is the situation in this plane? Make a sketch. Also, draw the net.

c) Determine the smallest possible distance of the landing point of the ball from the net. On which parameters does this distance depend?
d) From what height h must the ball be hit at least so that there is a chance that it will land in the correct part of the service box?
e) What is the geometric shape of the possible impact surface for a straight flight and correct landing of the ball in the service box?

Model B (Middle School)

Assume that the ball leaves the serving person's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball), flies in a straight line, and lands in the correct part of the service box. Consider the variable net height.

First, consider the situation where the ball flies just over the net.

a) What is the path of the ball in the xy-plane? Create a sketch.
b) Consider the plane that is perpendicular to the xy-plane and in which the rectilinear path of the ball lies. What is the situation in this plane? Draw a sketch. Also, draw the net.
c) Determine the smallest possible distance of the landing point of the ball from the net. On which parameters does this distance depend?
d) What is the minimum height h from which the ball must be hit to have a chance of landing in the correct part of the service box?
e) What is the geometric shape of the possible impact surface for a straight flight and correct landing of the ball in the service box? Examine the situation at different heights h. From what other parameters does the shape of the impact surface depend on?

Model C (Middle School/High School)

Assume that the ball leaves the serving person's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball) and flies parabolically under the influence of gravity, landing in the right service box. For simplicity, assume that the impact occurs along the center line. (Note: $1\ ft = 0.3048\ m$.)

The physical equations for the motion of the tennis ball are as follows:

$$\dot{\vec{s}}(t) = \vec{v}(t)$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \vec{F}_G \quad \text{with } \vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix}$$

and initial conditions $\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix}$ and $\vec{v}(0) = \begin{pmatrix} v_0 \cdot \cos(\varphi) \\ 0 \\ v_0 \cdot \sin(\varphi) \end{pmatrix}$.

a) Describe the path of the center of the ball in the xz-system by function terms $x(t)$ and $z(t)$. Determine the corresponding function f for $z = f(x)$.

b) What serve speed v_0 is required to hit the ball from a height h such that the serve-angle φ between the horizontal line and $\vec{v}$ is 0° and the ball (center point) passes the net at a height of 3 ft above the ground?
c) At what distance from the net does the ball land on the ground at the speed calculated in b)?
d) The largest serve speed v_m recorded so far in a tournament is $263.4\,\text{km/h}$. Investigate which serve-angles φ between the horizontal line and $\vec{v}_m$ lead to a valid serve.

Model D (High School)

Assume that the ball leaves the serving player's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball) and flies under the influence of gravity and air resistance and lands in the right part of the service box. For simplicity, assume that the impact occurs along the center line. (Note: $1\ ft = 0.3048\ m$, mass of tennis ball: $m = 0.0577\ kg$, density of air $\rho = 1.167\ kg/m^3$, radius of tennis ball $r = 0.0335\ m$.)

The physical equations for the motion of the tennis ball are as follows:

$$\dot{\vec{s}}(t) = \vec{v}(t)$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \left(\vec{F}_G + \vec{F}_D\right)$$

with $\vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix}$ and $\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot |\vec{v}| \cdot \vec{v}$

and initial conditions $\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix}$ and $\vec{v}(0) = \begin{pmatrix} v_0 \cdot cos(\varphi) \\ 0 \\ v_0 \cdot sin(\varphi) \end{pmatrix}$.

$A = \pi \cdot r^2$ is the cross-sectional area of the ball, and $C_D = 0.62$ is the drag coefficient.

a) Solve the equations of motion numerically.
b) For the following serve speed velocities v_0 determine those serve-angles φ between the horizontal line and $\vec{v}$ that cause the ball (center point) to fly over the net in each case: $45\ \text{m/s}$, $50\ \text{m/s}$, …, $70\ \text{m/s}$.
c) For the velocities from b), determine the serve-angles φ so that the landing point of the ball is on the service line.

Model E (High School/College)

When serving, good tennis players often impart a spin to the ball. In doing so, the ball experiences a force that is perpendicular to the velocity vector $\vec{v}$ and the rotation vector $\vec{\omega}$, the so-called Magnus force (lift force).

Assume that the ball leaves the serving player's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball) and flies under the influence of gravity, air resistance, and Magnus force and lands in the correct part of the service box.

For simplicity, assume that the impact occurs along the center line. (Note: $1\ ft =$

$0.3048\ m$, mass of tennis ball: $m = 0.0577\ kg$, density of air $\rho = 1.167\ kg/m^3$, radius of tennis ball $r = 0.0335\ m$.)

The physical equations for the motion of the tennis ball are as follows:

$$\dot{\vec{s}}(t) = \vec{v}(t)$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \left(\vec{F}_G + \vec{F}_D + \vec{F}_L\right)$$

with $\vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix}$, $\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot |\vec{v}| \cdot \vec{v}$ and

$$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot \frac{|\vec{v}|}{|\vec{\omega}|} \cdot \vec{\omega} \times \vec{v}$$

and initial conditions $\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix}$ and $\vec{v}(0) = \begin{pmatrix} v_0 \cdot \cos(\varphi) \\ 0 \\ v_0 \cdot \sin(\varphi) \end{pmatrix}$.

$A = \pi \cdot r^2$ is the cross-sectional area of the ball, $C_D = 0.62$ is the drag coefficient, $\omega_0 = 403$ rad/s, and $C_L = 0.319$.

a) Write down the equations of motion component by component.
b) Solve the equations of motion for pure topspin numerically, i.e., for $\vec{\omega} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix}$.
c) For the following takeoff velocities v_0, determine those serve-angles φ between the horizontal line and $\vec{v}$ that will cause the ball (center point) to fly over the net in each case: 45 m/s, 50 m/s, …, 70 m/s. Calculate with $\omega = 300$ rad/s.
d) For the velocities from c), determine the serve-angles φ so that the landing point of the ball is on the service line.

Model F (College)

When serving, good tennis players often impart a spin to the ball. In doing so, the ball experiences a force that is perpendicular to the velocity vector $\vec{v}$ and the rotation vector $\vec{\omega}$, the so-called Magnus force (lift force).

Assume that the ball leaves the serving person's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball) and flies under the influence of gravity, air resistance, and Magnus force and lands in the correct part of the service box. For simplicity, assume that the impact occurs along the center line. Also consider any wind speed $\vec{W}$.

(Note: $1\ ft = 0.3048\ m$, mass of tennis ball: $m = 0.0577\ kg$, density of air $\rho = 1.167$ kg/m^3, radius of tennis ball $r = 0.0335\ m$.)

The physical equations for the motion of the tennis ball are as follows:

$$\dot{\vec{s}}(t) = \vec{v}(t)$$

$$\dot{\vec{v}}(t) = \frac{1}{m} \cdot \left(\vec{F}_G + \vec{F}_D + \vec{F}_L\right)$$

with $\vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix}$, $\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot |\vec{v} - \vec{W}| \cdot (\vec{v} - \vec{W})$ and

$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot \frac{|\vec{v} - \vec{W}|}{|\vec{\omega}|} \cdot \vec{\omega} \times (\vec{v} - \vec{W})$

and initial conditions $\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix}$ and $\vec{v}(0) = \begin{pmatrix} v_0 \cdot \cos(\varphi) \\ 0 \\ v_0 \cdot \sin(\varphi) \end{pmatrix}$.

$A = \pi \cdot r^2$ is the cross-sectional area of the ball, $C_D = 0.62$ is the drag coefficient, $\omega_0 = 403\ rad/s$, and $C_L = 0.319$.

a) Write down the equations of motion component by component.
b) For the following serve speeds v_0, determine the serve-angles φ between the horizontal line and $\vec{v}$ that cause the ball (center point) to fly over the net: $45\ \text{m/s}$, $50\ \text{m/s}$, …, $70\ \text{m/s}$. Calculate with $\omega = 300$ rad/s (sidespin), i.e., $\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ 300 \end{pmatrix}$, and $2\ \text{m/s}$ tail wind, i.e., $\vec{W} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.
c) For the velocities from b), determine the serve-angles φ so that the landing point of the ball is on the service line.

Model G (College)

When serving, good tennis players often give the ball a spin. In doing so, the ball experiences a force that is perpendicular to the velocity vector $\vec{v}$ and the rotation vector $\vec{\omega}$, the so-called Magnus force (lift force).

Assume that the ball leaves the serving player's racket at height $h = 9.25\ ft$ above the baseline (relative to the center of the ball) and flies under the influence of gravity, air resistance, and Magnus force and lands in the correct part of the service box.

For simplicity, assume that the impact occurs along the center line.

(Note: $1\ ft = 0.3048\ m$, mass of tennis ball: $m = 0.0577\ kg$, density of air $\rho = 1.167\ \text{kg/m}^3$, radius of tennis ball $r = 0.0335\ m$.)

The physical equations for the motion of the tennis ball are as follows:

$\dot{\vec{s}}(t) = \vec{v}(t)$

$\dot{\vec{v}}(t) = \frac{1}{m} \cdot (\vec{F}_G + \vec{F}_D + \vec{F}_L)$

with $\vec{F}_G = \begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix}$, $\vec{F}_D = -\frac{1}{2} \cdot \rho \cdot A \cdot C_D \cdot |\vec{v}| \cdot \vec{v}$ and

$\vec{F}_L = \frac{1}{2} \cdot C_L \cdot \left(1 - \exp\left(-\frac{|\vec{\omega}|}{\omega_0}\right)\right) \cdot A \cdot \rho \cdot \frac{|\vec{v}|}{|\vec{\omega}|} \cdot \vec{\omega} \times \vec{v}$

and initial conditions $\vec{s}(0) = \begin{pmatrix} -l \\ 0 \\ h \end{pmatrix}$ and $\vec{v}(0) = \begin{pmatrix} v_0 \cdot \cos(\varphi) \\ 0 \\ v_0 \cdot \sin(\varphi) \end{pmatrix}$.

$A = \pi \cdot r^2$ is the cross-sectional area of the ball, $C_D = 0.62$ is the drag coefficient, $\omega_0 = 403\ \mathrm{rad/s}$, and $C_L = 0.319$.

a) Write down the equations of motion component by component.
b) For the serve speed $v_0 = 160\ \mathrm{km/h}$, determine the serve-angles φ between the horizontal line and $\vec{v}$ that cause the ball (center point) to fly over the net.
Calculate with the following spin vectors:
$\vec{\omega}_1 = \begin{pmatrix} 0 \\ 600 \\ 0 \end{pmatrix}$ $\vec{\omega}_2 = \begin{pmatrix} 0 \\ 580 \\ 155 \end{pmatrix}$ $\vec{\omega}_3 = \begin{pmatrix} 0 \\ 520 \\ 300 \end{pmatrix}$ $\vec{\omega}_4 = \begin{pmatrix} 0 \\ 424 \\ 424 \end{pmatrix}$
$\vec{\omega}_5 = \begin{pmatrix} 0 \\ 300 \\ 520 \end{pmatrix}$ $\vec{\omega}_6 = \begin{pmatrix} 0 \\ 155 \\ 580 \end{pmatrix}$ $\vec{\omega}_7 = \begin{pmatrix} 0 \\ 0 \\ 600 \end{pmatrix}$
c) For the spin vectors from b), determine the serve-angles φ so that the landing point of the ball is on the service line.