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RESEARCH ARTICLE

Mathematics Inquiring Based on Pattern Similarity

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Abstract

Mathematics is a science of pattern. Mathematics is a subject of inquiring which aims at discovering the models hidden behind the world. Pattern is abstraction and generalization of the model. Mathematical pattern is a higher level of mathematical model. Mathematics patterns are often hidden in pattern similarity. Creation of mathematics lies largely in discovering the pattern similarity among the various components of mathematics. Inquiring is the core and soul of mathematics teaching. It is very important for students to study mathematics like mathematicians' exploring and discovering mathematics based on pattern similarity. The author describes an example about how to guide students to carry out mathematics inquiring based on pattern similarity in classroom.

Keywords: mathematics patterns, pattern similarity, mathematics inquiring

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I. INTRODUCTION

According to Wikipedia in English, in the March 2014 version, the answer to "What is Mathematics?" is Mathematics is the abstract study of topics such as quantity (numbers), structure, space, and change. There is a range of views among mathematicians and philosophers as to the exact scope and definition of mathematics. Mathematicians seek out patterns (Highland & Highland, 1961) and use them to formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. When mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature. For mathematics, there is no commonly accepted definition; today it is usually described as a science that investigates abstract structures that it created itself by logical definitions using logic for their properties and patterns. Mathematics is a science of pattern. What mathematics studies is not the real world, but only mathematical model of the real world, that is, a fictitious and simplified version of the real world. Modeling is a basic concept in mathematics. It is at the heart of all mathematical applications and at the core of some of the most abstract pure mathematics. Mathematics comes from the abstract modeling of the real world. The development of mathematics is a process of abstract modeling. It is the whole history of mathematics which is continuously to construct mathematical models and gradually develop these models. All kinds of different levels of mathematical models and their corresponding research constitute mathematics science we are facing now. Pattern is abstraction and generalization of the model. Mathematics pattern is a higher level of mathematical model. Mathematics pattern is often hidden in pattern similarity of mathematics theories, methods and problems.

Creation of mathematics lies largely in discovering the pattern similarity among the various components of mathematics. The modelling perspective is increasingly influential in mathematics education research (Kaiser et al., 2011; Lesh, 2006; Lesh & Doerr, 2003; Lesh & Lehrer, 2003). Gravemeijer (1999) distinguishes four different levels of activities involved in the transition from the status of model-of to the status of model-for: situational, referential, general and formal. The use of model-of to develop model-for indicated a similar systemic relation in that initial drafts of the models that mostly built-in informal ways of thinking are used to develop more formal ways of mathematical thinking. Model development is the progression from model-of to model-for and results in the development of a new and formal mathematical insight, and therefore demonstrates bi-directional breeding between modeling and conceptual development (Gravemeijer, 2004; Gravemeijer & Stephan, 2002). Sevinc (2022) mentions in this respect, the work of Lesh (Lesh & Harel, 2003) on model-eliciting activities, where the activity of the students is not so much that of applying mathematical ideas but of developing new mathematical ideas. The emergent modeling approach taps into the same potential, but with a focus on long-term learning processes, in which a model develops from an informal, situated model into a more sophisticated model. (Gravemeijer, 2007, p. 138). Modeling constitutes an important process in the development of mathematical thinking.

Mathematics is a process of inquiring which aims at discovering the models hidden behind the world. Inquiring is the core and soul of mathematics teaching. It is an important goal of mathematics instruction to cultivate students' ability of inquiring. It is very important for students to learn to inquire mathematics like mathematicians' exploring and discovering mathematics based on pattern similarity. In the process of inquiring mathematics, the students need to try their best to find a pattern of revealing the essence of problems and further pose new problems based on pattern similarity. The ideas in this article help to answer the questions "What is mathematics inquiring based on pattern similarity?" and "How can we carry out mathematics inquiring teaching based on pattern similarity in classrooms?" Many teachers do not know how to carry out mathematics inquiry teaching in practice. In this paper, the author will take one plane geometry problem as an example to discuss how to carry out mathematics inquiring teaching based on pattern similarity.

II. RELATED LITERATURE

Mathematics as Science of Pattern

Mathematics is described as "a science of pattern" (Steen, 1988; Van De Walle, 1998). Generally speaking, pattern refers to abstract mathematical theory, that is, the commonly known mathematical structure. Specifically, if a mathematical proposition is based on an abstract mathematical theory or is accepted as a part of an abstract mathematical theory, we can also call it a mathematical pattern. Therefore, in terms of modern research in mathematics, we can say that the mathematical object refers to quantitative patterns (Xu & Zheng, 1990). Pattern is the essence of mathematics. Pattern and modelling are a fundamental concept in mathematics. It is at the heart of all mathematical applications and also at the core of some of the most abstract pure mathematics. As a creative discipline, mathematics operates in three basic steps: (i) experience a problem and find a pattern from it; (ii) define a symbol system to express the pattern; (iii) organize symbol system into a systematic language (Kapur, 1973). In modern mathematics, Mathematicians extract abstract structures and patterns from special cases and study these structures. The advantage of this research is that the knowledge obtained by this way can be applied not only to the special situation of generating knowledge, but also to all other systems with the same pattern. Mathematics has its own structure, which can be found through generalization of patterns. Pattern is an abstract structure, which is a kind of relationships reflecting the essence of prototypes. Mathematical pattern is the mathematical representation of the prototype structure. Structural mathematics essentially supports universality of mathematical patterns. Virtually all mathematics is based on pattern and structure. As Warren (2005) asserts, "The power of mathematics lies in relations and transformations which give rise to patterns and generalizations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning" (p. 305). Mathematicians seek patterns in number, in space, in computers, in science, and even in imagination. Mathematicians are people who put together certain kinds of patterns. Keith Devlin goes as far as to describe mathematics as the science of patterns: "It was only within the last twenty years or so that a definition of mathematics emerged on which most mathematicians now agree: mathematics is the science of patterns.” (Devlin, 2003, p. 3)

Mathematics is a formal science of pattern. Both concepts and propositions in mathematics, or problems and methods in mathematics, should be regarded as a universal pattern. The essential characteristic of mathematics is to study patterns in the process of abstracting from different problems. As a science of patterns, many mathematicians are concerned about finding and analyzing new patterns in the world, constructing rules to describe patterns and promote the further research. Whether it is exploring, discovering new patterns, interpreting the meaning of each pattern, or creating new patterns similar to known patterns, it is increasing the content of mathematics (Sawyer, 2005, p. 32). As a science of pattern, its significance is measured by the degree of connection among patterns in one field and patterns in other fields. The most explanatory and subtle patterns are the most profound results, which form the basis of all branches of mathematics. It is a goal pursued by mathematicians to discover and explore such a pattern and making it meaningful. Mathematicians study various patterns that exist in the real world and human experience. Mathematicians always try to find patterns that are widely used, and connect one kind of pattern with another, and constantly discover new mathematics patterns. Many philosophers, mathematicians and mathematics educators believe that pattern is important in the study of mathematics (Davis, 1984; Reys et al., 1984). 'In mathematics which is permeated by patterns pupils must be encouraged to look for them' (Department of Education and Science [DES], 1987, p. 3)

In modern mathematics, we extract abstract structures and patterns from special cases and study these structures. The advantage of this kind of research is that the knowledge thus obtained can be applied not only to the special number system from which we start, but also to all other systems showing the same pattern (Kapur, 1973). According to its own logic, mathematics starts from the scientific pattern and ends by adding all the derived patterns (Steen, 1988). Mathematicians pay more attention to the development of general patterns that are widely used in the study of special situations. In the history of the development of mathematics, there are lots of examples that mathematicians construct mathematics patterns. For example, The Elements of Euclid edited by Euclid in 300 BC is a recognized mathematical classic. Euclid constructed a very effective mathematics pattern for the spatial form of the real world. On the basis of the pattern, a whole set of geometry and mathematics research methods with deductive reasoning as the core have been developed, which has been playing a great role today. In this sense, Euclidean geometry has always been the treasure of mathematical science.

Pattern is the product of mathematical abstract thinking. We define the way a mathematical pattern is organized as its structure. Mathematical structure is most often expressed in the form of a generalization (Mulligan & Mitchelmore, 2009). When there are common structural characteristics among many different problems according to mathematicians’ intuition and experience, mathematicians may systematically clarify and explore the basic structural characteristics in an accurate and objective form of pattern. The development of mathematics depends largely on mathematicians’ finding the appropriate formal structure (Deng, 1990). Giving patterns is a very important thing in mathematical thinking. The established patterns quickly become mathematical objects, and these objects are adapted to further patterns, which are recognized as objects after being familiar with, and so on (Kapur, 1973). Where there is a pattern, it must have significant meaning. No matter what kind of mathematical work, if an amazing pattern repeats, it means that mathematicians should study why it happens. There must be some meaning in it, and mathematicians should master its core idea (Sawyer, 2005, p. 42).

Mathematics is science of patterns. These patterns can be found anywhere you want to find them. Moreover, the pattern can often inspire new patterns and produce patterns of patterns. It is an extremely important to discover patterns of mathematical thinking. The established patterns quickly become objects of mathematics, and these objects can be adapted to further patterns, which, after becoming familiar, are recognized as objects again, and so on. Studying mathematics must have the ability of recognizing patterns, appreciating patterns, discovering patterns, establishing patterns, expanding patterns and applying patterns. Studying mathematics must be immersed in discovering the charm patterns and their logical classification. Hardy (1992, p. 13) wrote: “mathematicians, like painters or poets, are makers of patterns.” It is a popular view among mathematics educators that the use and understanding of patterns is central to mathematical activities (Steinbring, 2005). The reason might be that patterns are seen more as a process component of mathematics methodology (Vogel, 2005). Devlin (2003) also pointed out: “It is for most laymen to do math means that learning a lot of unrelated rules and techniques to solve all kinds of problems. When you meet a mathematician who says to you, 'Oh, it's obvious that you do it, do it again, and then the answer comes out.' Most people think that doing math requires a special brain. In fact, that is not the case. The main reason mathematicians know what to do in this case is that they see a potential structure of the problem. If you can see this structure, you will know what to do next. This structure is actually a pattern.

Mathematics Inquiring Based on Pattern Similarity

Mathematical inquiry is a cornerstone of mathematical practice (Lakatos, 1976). Inquiry is also central to mathematics. The process of doing mathematics is far more than just calculation or deduction; it involves observation of patterns, testing of conjectures, and estimation of results. Mathematics reveals hidden patterns that help us understand the world around us. Mathematical inquiry is inquiry-based teaching and learning specifically related to the development of mathematical knowledge and understanding (Richards, 1991; von Glaserfeld, 1991). It is an approach to mathematics education in which students learn by engaging in mathematical discussions, listening to mathematical arguments, proposing conjectures, asking mathematical questions, and solving unfamiliar problems (Richards, 1991). Mathematical inquiry focuses on building knowledge and deep conceptual understanding through carefully selected tasks chosen by the teacher (NCTM, 2000). The goal is to “develop a repertoire of general heuristics and approaches that can be applied in many different situations” by engaging in habits such as looking for patterns, experimenting, describing one’s work, visualizing, conjecturing, guessing, thinking about the big picture, thinking about the specific details of a particular case, seeing multiple points of view, using mathematical language, and utilizing inductive and deductive reasoning (Cuoco et al., 1996, p. 378). Inquiry is also at the heart of problem posing (Singer et al., 2013).

Mathematics is an organic and unified science. There are always countless connections among mathematics theories, methods and problems. On the surface, there are very different among the various components of mathematics. But there often shows amazing pattern similarity among various components of mathematics. Mathematics patterns are often hidden in pattern similarity. Creation of mathematics lies largely in discovering the similarities among various components of mathematics. It is essential that learner should have the intuition of finding the similarity among the patterns. As Banach, one of the founders of functional analysis, once pointed out, a mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies (Beziau, 2018). The most satisfying moment in the history of mathematics is the discovery of two fields that have always been regarded as distant and irrelevant, but basically different disguises of the same thing (Sawyer, 2005). The essence of mathematics is thinking creatively (Dreyfus & Eisenberg, 1966; Ginsburg, 1996). The talent of mathematical thinking lies in the acuity of wisdom of discovering pattern similarity between two seemingly unrelated problems. Analogical reasoning, the ability to perceive and operate on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar, is also deemed an essential part of the human capacity to adapt to novel contexts (Holyoak & Thagard, 1995).

There are full of patterns and relationships in mathematics. It is one of the most important topics for learning mathematics to explore patterns and relationships in mathematics (Hargreaves, 1998). Students are encouraged to create patterns that are applicable to a range of similarly structured situations, and as a result, they can generalize and extend their solutions (Doerr & English, 2003; English, 2006). Generalizing refers to a disposition to secure knowledge by seeking the broader structures or patterns that govern the generation of any particular case or set of cases (Lehrer et al., 2013). Generalization is also used to refer to a statement made about the regularities or properties of a pattern, or to be more precise the regularities or properties of relationships within the pattern. This means that students should try to generalize the results obtained and the methods used, and connect them in order to progressively develop mathematical concepts and structures (Maaß & Artigue, 2013). Pattern and making generalizations are thought to be very important in the study of mathematics (Hargreaves, 1998). Engaging in this kind of pattern building is not seen as finding a solution to a given problem but rather as developing generalizations that a learner can use and reuse to find solutions (Bransford et al., 1996; Lehrer & Schauble, 2000). Generalizing and reusing patterns are central activities in learning mathematics. Students should be good at looking for the similarity among problems and methods based on the pattern structure of problems. In particular, they should be good at posing new mathematics problems based on pattern similarity. As Descartes pointed out, when I have intuitively understood several simple theorems... It will be very useful if I can understand these theorems through continuous and uninterrupted thinking activities, and understand the relationships among them, and imagine several of them as clearly as possible at the same time (Pólya, 1954).

Pólya (1962) once pointed out that one method of solving problem, whether obtained by yourself, learned or heard, can become a pattern for you as long as it passes through your own experience. When you encounter similar mathematics problems again, it is the pattern you can follow. Descartes also said: "every problem we solve will become an example to solve other problems" (Pólya, 1962). There are thousands of mathematics problems. We can't finish them one by one. However, many mathematics problems, whether it's problem setting, conclusion, overall structure, visual image or method of solving problem, show or imply a certain mathematics pattern. If students are good at observing, identifying and capturing the characteristics of pattern, they can often quickly obtain the way of solving these problems. Moreover, students should often be encouraged to find the internal relationship among different problems based on pattern similarity, so as to extend the original problem and pose new mathematics problems. Of course, it is often not easy to identify the pattern hidden in the problems and pose new mathematics problems based on pattern similarity. Students need a certain talent, but also need intuitive insight and flexible, natural and graceful imagination hidden in thinking. Teachers should consciously guide, educate and cultivate students to improve students' mathematics ability of inquiring based on pattern similarity.

Mason (1996) believes that the roots of mathematical thinking lie in detecting sameness and difference, in making distinctions, in classifying and labeling, or simply in algorithm seeking. Mathematical inquiry requires looking for mathematical similarities and differences within and between patterns are likely to develop an understanding of the structure of those patterns. Moreover, students engaged in mathematical inquiry will also tend to look for similarities and differences in new patterns and broaden their structural understanding accordingly. It needs to constantly connect similar problems based on pattern similarity in order to run through the theorems and integrate them. Of course, it is not easy to find the hidden pattern and structure of problems and the internal relationship or similarity among different problems. In practice, if the structure of objects studied is too similar to each other, it is a little less interesting to extend them. However, for objects with different faces, as long as their common nature and structure can be identified, it is very valuable to extend them. Through generalizations of pattern, mathematicians can often get some more profound mathematics. Teachers should be good at guiding students to extend original problems and pose new mathematics problems based on pattern similarity. In the case of school children, the development of elementary, but powerful mathematical patterns, should be considered to be among the most important goals of mathematics instruction (Lesh & Lehrer, 2003).

Strategy of Mathematics Inquiring Teaching Based on Pattern Similarity

Discovering pattern behind mathematics problems in classroom provides a platform for students to improve their understanding. With the construction of patterns, students can develop scientific attitude and methods, and gradually develop innovative ability. Although teachers recognize the value of guiding their students to discover mathematics patterns, few have had opportunities to experience the process of discovering mathematics patterns, and many teachers feel unsure of how to do it. The key of mathematics inquiring teaching based on pattern similarity is to develop students' ability of pattern. Pattern ability mainly includes pattern recognition, description, communication and extension. Among them, pattern recognition is the foundation. Pattern recognition is to obtain the structure of the pattern, that is, to identify what the basic units of pattern are, what the constituent elements of pattern units are, and what the relationships among the units of pattern are. Pattern recognition is actually a process of organizing and processing information. In pattern recognition, we must first carefully examine the elements of pattern and their relationships, identify the characteristics that can reflect the essence of problems from the complex representation, classify the elements of pattern according to these characteristics, and then summarize the structure of pattern and its regular relations. Therefore, pattern recognition is very important, which is the core of understanding mathematics. Through pattern recognition, some seemingly unrelated information in mathematics problems can be combined into a whole, and the intrinsically stable regular relationship can be obtained.

Patterns can be explored, identified, extended, reproduced, compared, varied, represented, described and created. Description, communication and extension of patterns are the ability of pattern application based on pattern recognition. Pattern description refers to the general representation of pattern structure and its regular relations. Through communication of pattern, students can exchange their understanding, interpretation and representation of pattern, which helps them to comprehensively grasp the characteristics of each constituent element of pattern, and enables them to exclude the influence of some extrinsic characteristics when summarizing the pattern, so as to obtain a more accurate generalization of pattern structure. Pattern extension is based on pattern recognition to analyze the overall structure of pattern and its regular relations, and apply the pattern to other situations or problems (Hargreaves et al., 1998). Therefore, in mathematics inquiring teaching based on pattern similarity, Teachers should fully respect and protect students' inquiry thinking of pattern recognition, description, communication and extension, and stimulate students to discover the hidden patterns of problems. Teachers should pay attention to the process of students' thinking, and promote their mathematical exploration based on pattern similarity. Teachers should give students ample opportunities to express their opinions and encourage them to express their idea and the process of exploring. The teachers' role is not to stand back and expect students to discover everything for themselves; it is rather to scaffold the processes of inquiry through the use of carefully designed tasks and structured lessons (Artigue & Blomhoej, 2013).

In order to effectively implement mathematical inquiry teaching based on pattern similarity in class, model-eliciting activities (MEAs) were initially created in the mid 1970s by mathematics educators (Lesh et al., 2000; Lesh & Lamon, 1992). MEAs encourage students to create mathematical models to solve complex problems, just as mathematicians do mathematics. To create MEAs, researchers in mathematics education follow specific guidelines. These guidelines are referred to as the model construction principle, construct shareability and reusability principle for designing MEAs. The model construction principle states that a successful response to the problem demands the creation of a model. A model is a system that consists of elements, relationships among those elements, operations that describe how those elements interact, and patterns or rules that apply to the relationships and operations. A model is evident when one system describes another system. The construct shareability and reusability principle states that the product should be able to be used in a parallel situation. If the model developed can be generalized to other situations requiring a similar model, then the response is a successful one (Chamberlin & Moon, 2005).

Teaching mathematics inquiring based on pattern similarity is challenging, especially for teachers who are new to the process of mathematics inquiring. It begins by selecting or developing a task. The discovery of models or pattern similarity by students is one of the most powerful mathematical activities in which a student may engage. Discovering mathematical models or pattern similarity serves to illustrate the interconnectedness of mathematics. Glas (2002) lists four educational outcomes achieved by modeling in the mathematics classroom. Models and modeling help students (a) recognize the interconnectedness inside and outside of mathematics, (b) recognize various perspectives on a domain of knowledge, (c) be creative in mathematical thought, and (d) view mathematics in a practical and applicable way. When teaching mathematics inquiring based on pattern similarity, teachers might ask themselves the following questions: What kinds of problem-solving strategies are students likely to use? How will the students identify the hidden pattern of the problem? How can I help students pose new problems based on pattern similarity? The teacher might summarize the major mathematical ideas that students used in their solutions. This step can also serve as an opportunity of guiding students to carry out pattern recognition, description, communication and extension. Teachers might ask students to reflect on the process of pattern recognition, and comment on strategies that helped them succeed in pattern extension. Revisiting is an ideal opportunity for discussing how the problem could be changed or extended and whether the students' solutions are still viable in these new situations (Hernández et al., 2017).

Mathematics inquiring teaching based on pattern similarity in classrooms should be open. Students should be required to try their best to identify the hidden patterns of the problems and pose new problems based on the pattern similarity. The mutual discussion among group members can enable students to open their minds and collide with new and unique ideas, which are very important for mathematics inquiring teaching based on pattern similarity, and also a necessary condition for mathematics inquiring teaching based on pattern similarity. Teachers should be able to create a mathematics classroom that allows inquiring to ferment, which depends on the successful organization and effective implementation of teachers with high mathematics quality. Appropriate scaffolding is required to increase students' ability of identifying and extending patterns in an inquiring classroom. An inquiring classroom based on pattern similarity with appropriate scaffolding supports students to engage in complex tasks beyond their current capacity through inquiry. For such scaffolding to occur, the instructor needs to have proper educational knowledge related to mathematical content (Speer & Wagner, 2009).

One could even go so far as to say that identifying and describing patterns is elementary for mathematics (cf. Devlin, 2003). Practicing good interacting with patterns supports not only the active learning of mathematics but also a deeper understanding of the world in general. To explore the possibilities of sustained inquiry, Lehrer et al. (2013) designed instruction to support middle school students (ages 11, 12) to initiate questions, formulate conjectures in light of their questions, conduct investigations that informed their questions and conjectures, draw conclusions from these investigations, and pose new questions. Their goal was to sustain a cycle of inquiry in which students would be agents of mathematical learning. Vogel (2005) provides some examples of pattern utilization and detailed analyses thereof. These ideas serve as “hooks” to encourage the good use of patterns to facilitate active learning processes in mathematics. Menezes et al. (2015) developed a framework with the goal of describing practices of inquiry-based mathematics teaching. The framework synthesizes the teacher’s instructional actions and the main intentions behind those actions in each of the four phases of the lesson. In order to better understand the real meaning of mathematics inquiring according to pattern similarity, the author will take the solutions and extensions of a plane geometry problem as an example to illustrate how to carry out mathematics inquiring teaching based on pattern similarity in classroom.

III. METHODS

Mathematics Inquiring Teaching Based on Pattern Similarity in Classroom

Problem. As shown in figure 1, ΔABC is the inscribed triangle of $\odot O$. $AB = AC$. Take any point D on the arc BC opposite to $\angle BAC$. Link AD, BD, CD. If $\angle BAC = 120^{\circ}$, what is quantitative relationship between $BD + CD$ and AD?

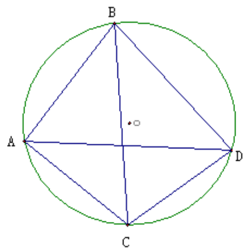

Figure 1. A plane geometry problem

When students first start working on this problem, they don't have any ideas. Teacher guide students to analyze the conditions of the problem and the conclusions that need to be answered.

Teacher: Have you ever solved similar problems before?

Student: I have solved similar problems before. When exploring the quantitative relationship between $BD + CD$ and AD, we often first connect BD and CD into one line segment, or divide AD into two line segments related to BD and CD, then we will seek the quantitative relationship between $BD + CD$ and AD.

Teacher: Can you solve this problem by this way?

Student: It seems difficult to solve this problem by this way.

Teacher: Can you get any other ways of solving this problem?

The students are actively thinking, but at the moment, no student has found a solution to this problem. The teacher had to continue to guide the students to look for ways of solving this problem from another perspective.

Teacher: Let's take a look at the known conditions of the problem again. What idea can you generate?

Student: Because of $AB = AC$, we can obtain ∠ADB = ∠ADC. As shown in figure 2, make vertical lines to both sides of $\angle BDC$ through point A, and the vertical foot are E, F respectively. Then AE = AF. Because $AB = AC$,$\angle BAC = 120^{\circ}$ and ΔABC is the inscribed triangle of $\odot\, O$, we can obtain $\angle ADB = \angle ADC = 30^{\circ}$.

In $Rt\Delta AEB$ and $Rt\Delta AFC$, $\begin{cases} AB = AC \\ AE = AF \end{cases}$, then $\Delta AEB \cong \Delta AFC$ (HL). Then $BD + CD = 2DE = \sqrt{3}AD$.

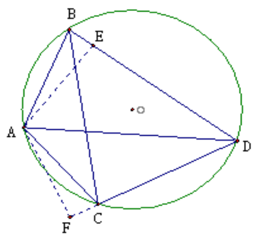

Figure 2

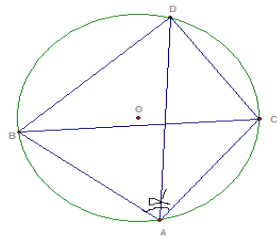

Figure 3

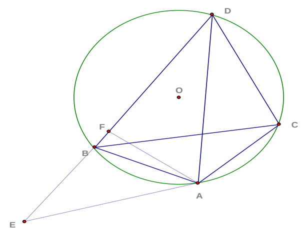

Figure 4

Teacher: Great, how do you get this solution?

Student: Because of obtaining ∠ADB = ∠ADC, we can get this solution by applying the properties of the angular bisector. It is a commonly method of making auxiliary lines about angular bisectors.

Teacher: Can it be seen as a general method or pattern for solving such problems?

Student: Yes! Through pattern recognition, we can find a basic pattern hidden in the problem. Based on this basic pattern, we can get this solution.

Teacher: Great, many students often believe that it is over after solving problem, and do not further explore the deep pattern hidden in the problem. Obviously, they will lose a good opportunity of improving their own mathematics literacy.

Teacher: We should not only be satisfied with answering this question, but should further explore the essence of this problem. For example, can we propose a more general mathematical problem based on pattern similarity?

Student: As shown in figure 3, if $\angle BAC = \alpha$, what is the quantitative relationship between $BD + CD$ and AD?

Teacher: Can you solve this general problem?

Student: Yes, As shown in figure 4, Do $\angle EAD = \angle BAC = \alpha$,which intersects the extension line of DB at point E. Then $\angle EAB = \angle DAC$. By using the four-point common circle theorem, we can get $\angle EBA = \angle DCA$.

In ΔEAB and ΔDAC, $\begin{cases} \angle EAB = \angle DAC \\ AB = AC \\ \angle EBA = \angle DCA \end{cases}$, then $\Delta EAB \cong \Delta DAC$ (ASA). So, $BE = CD, AE = AD$.

Do $AF \perp DE$, the vertical foot is a point F. Then $\angle FAD = \frac{\alpha}{2}$. So, $BD + CD = DE = 2DF = 2AD\ sin\frac{\alpha}{2}$.

Teacher: How do you obtain this solution?

Student: Because I find that this general problem is essentially the same as the original problem in terms of pattern structure, I can solve it by using the same method based on pattern similarity.

Teacher: Good, after careful analysis of the two problems, we can find that the two problems have common similar pattern. That is, the key conditions of two problems are that (1) AD is angular bisector of $\angle BDC$, (2) Four points A, B, C and D are circular. (Namely, $\angle BAC + \angle BDC = 180^{\circ}$). Therefore, based on pattern similarity embodied in two problems, we can further explore the two problems deeply. For example, if we remove the condition of circle from the original problem and only retain the equivalent condition of diagonal complementary, then, we can obtain a new problem (i.e. extended problem 1).

Extended Problem 1. As shown in figure 5, $\angle AOB = \angle DCE = 90^{\circ}$. OC is angular bisector of $\angle AOB$.what is quantitative relationship between $OD + OE$ and OC?

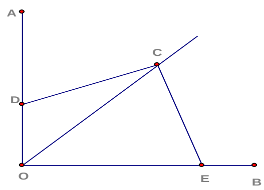

Figure 5

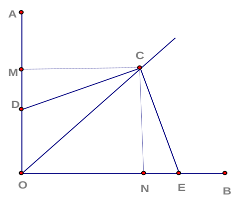

Figure 6

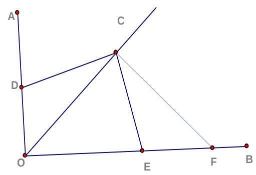

Figure 7

Teacher: Who can solve this extended problem 1?

Student: Based on pattern similarity, I can solve this extended problem 1 by analogy. As shown in figure 6, pass point C to do $CM \perp OA, CN \perp OB$, and the vertical foot are M, N respectively. It can be obtained from the known conditions that $CM = CN, \angle MCN = 90^{\circ}$. Then the quadrilateral OMCN is square.

By $\angle AOB = \angle DCE = 90^{\circ}$, we can get four points O, D, C and E are circular. So, $\angle MCD = \angle NCE$.

In ΔMCD and ΔNCE, $\begin{cases} \angle CMD = \angle CNE = 90^{\circ} \\ CM = CN \\ \angle MCD = \angle NCE \end{cases}$, then $\Delta MCD \cong \Delta NCE$ (AAS). So, $MD = NE$. Then $OD + OE = OD + ON + NE = 2ON = \sqrt{2}OC$.

Teacher: Very good, the student has identified the essential pattern of this extended problem 1 just now and solved it. Can any other students use different methods to solve this extended problem 1?

Student: As shown in figure 7, pass point C to do $CF \perp OC$, which intersects OB at point F. Then $\angle DOC = \angle EFC = 45^{\circ}$, CO = CF, $\angle DCO = \angle ECF$. So, $\Delta DCO \cong \Delta ECF$ (ASA).

Then OD = EF. So, $OD + OE = OF = \sqrt{2}OC$.

Teacher: How do you find this solution?

Student: I solved this extended problem 1 by analogizing one previous student's solution based on pattern similarity.

Teacher: Good, can you further pose new problems based on pattern similarity?

Student: If one side of $\angle DCE$ intersects the extension line of AO or BO, we can get the following variant extended problem 1.

Variant extended problem 1. As shown in figure 8, $\angle AOB = \angle DCE = 90^{\circ}$,OC is angular bisector of $\angle AOB$, one side of $\angle DCE$ intersects the extension line of AO at point D. what is quantitative relationship among OD, OE and OC?

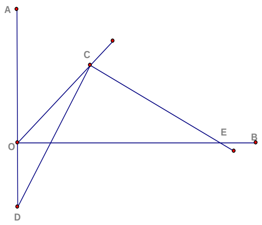

Figure 8

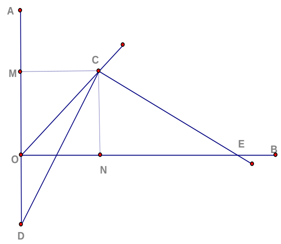

Figure 9

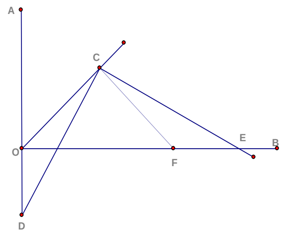

Figure 10

Teacher: Who can solve this extended problem 1?

Student: I can obtain two solutions based on pattern similarity, which are the same as that of extended problem 1. As shown in figure 9 and figure 10, I can get $OE - OD = \sqrt{2}OC$ by analogy.

Teacher: Good, who can further pose new problems based on pattern similarity?

Student: If we further change the condition of $\angle AOB = \angle DCE = 90^{\circ}$, replace it with other similar conditions (for example, $\angle AOB = 120^{\circ}, \angle DCE = 60^{\circ}$), we can get extended problem 2.

Extended problem 2. As shown in figure 11, if $\angle AOB = 120^{\circ}, \angle DCE = 60^{\circ}$, OC is angular bisector of $\angle AOB$, what is quantitative relationship among OD, OE and OC?

Teacher: Who can solve this extended problem 2?

Student: Based on pattern similarity, I can also find two solutions by analogy with extended problem 1. As shown in figure 12 and figure 13, I can get $OD + OE = OC$.

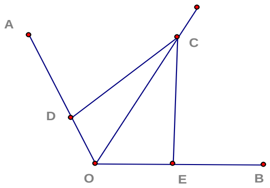

Figure 11

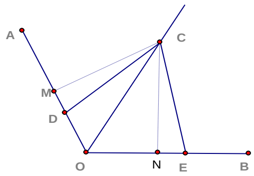

Figure 12

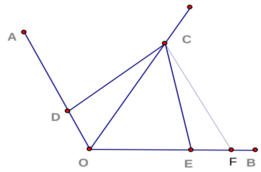

Figure 13

Teacher: Good, who can further pose new problems based on pattern similarity?

Student: By analogy with variant extended problem 1 based on pattern similarity, if one side of $\angle DCE$ intersects with the extension line of AO or BO, we can obtain the following variant extended problem 2.

Variant extended problem 2. As shown in figure 14, $\angle AOB = 120^{\circ}, \angle DCE = 60^{\circ}$, OC is angular bisector of $\angle AOB$, one side of $\angle DCE$ intersects at point E with the extension line of BO. what is quantitative relationship among OD, OE and OC?

Teacher: Who can solve this variant extended problem 2?

Student: I can obtain two solutions based on pattern similarity, which are same as that of extended problem 2. As shown in figure 15 and figure 16, I can get $OD - OE = OC$.

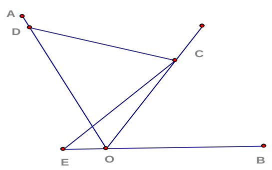

Figure 14

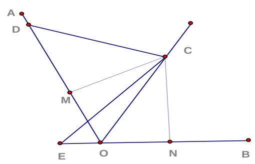

Figure 15

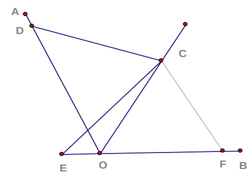

Figure 16

Teacher: Good, who can further pose new problems based on pattern similarity?

Student: If we further generalize the conditions and let $\angle AOB = 2\alpha, \angle DCE = 180^{\circ} - 2\alpha$.

based on pattern similarity, we can also get extended problem 3.

Extended problem 3. As shown in figure 17, $\angle AOB = 2\alpha, \angle DCE = 180^{\circ} - 2\alpha$, OC is angular bisector of $\angle AOB$, what is quantitative relationship among OD, OE and OC?

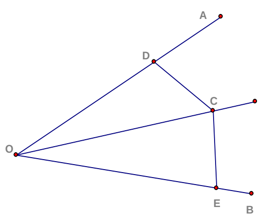

Figure 17

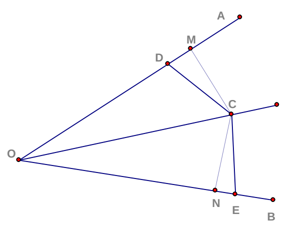

Figure 18

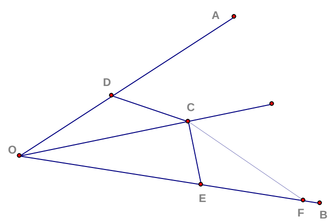

Figure 19

Teacher: Who can solve this extended problem 3?

Student: Based on pattern similarity, I can get the following solution. As shown in figure 18, pass point C to do $CM \perp OA, CN \perp OB$, and the vertical foot are M, N respectively. Then $CM = CN$. Let $\angle OCD = \beta$, then $\angle MCD = 90^{\circ} - \alpha - \beta, \angle NCE = 180^{\circ} - 2\alpha - \beta - (90^{\circ} - \alpha) = 90^{\circ} - \alpha - \beta$.

So, $\angle MCD = \angle NCE$.

In ΔMCD and ΔNCE, $\begin{cases} \angle CMD = \angle CNE = 90^{\circ} \\ CM = CN \\ \angle MCD = \angle NCE \end{cases}$, then $\Delta MCD \cong \Delta NCE$ (AAS). So, $MD = NE$.

Then $OD + OE = OD + ON + NE = 2ON = 2OC\cos\alpha$.

Teacher: Very good, can any other students use different methods to solve this extended problem 3?

Student: As shown in figure 19, take CO as one side to do $\angle FCO = 180^{\circ} - 2\alpha$, intersects with OB at point F. Then $\angle DCO = \angle ECF$, $\angle COF = \angle CFO = \angle CFE = \alpha = \angle COD$.

Then CO = CF. So, $\Delta DCO \cong \Delta ECF$ (ASA). Then OD = EF. $OD + OE = OF = 2OC\cos\alpha$.

Teacher: Good, who can further pose new problems based on pattern similarity?

Student: Similarly, if one side of $\angle DCE$ intersects with the extension line of AO or BO, we can obtain the following variant extended problem 3 by analogy.

Variant extended problem 3. As shown in figure 20, $\angle AOB = 2\alpha, \angle DCE = 180^{\circ} - 2\alpha$, OC is angular bisector of $\angle AOB$, One side of $\angle DCE$ intersects with the extension line of BO at point E. what is quantitative relationship among OD, OE and OC?

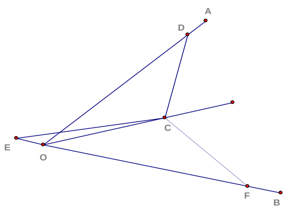

Figure 20

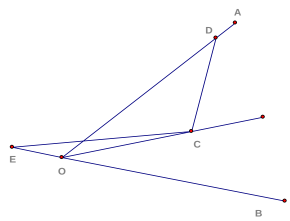

Figure 21

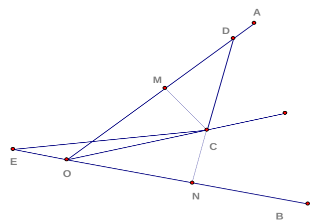

Figure 22

Teacher: Who can solve this variant extended problem 3?

Student: As shown in figure 21, pass point C to do $CM \perp OA, CN \perp OB$, and the vertical foot are M, N respectively. Then $CM = CN$. Let $\angle OCE = \beta$, then $\angle ECN = 90^{\circ} - \alpha + \beta$, $\angle MCD = 180^{\circ} - 2\alpha - (90^{\circ} - \alpha - \beta) = 90^{\circ} - \alpha + \beta$, then $\angle DCM = \angle ECN$.

In ΔDCM and ΔECN, $\begin{cases} \angle CMD = \angle CNE = 90^{\circ} \\ CM = CN \\ \angle DCM = \angle ECN \end{cases}$, then $\Delta DCM \cong \Delta ECN$ (ASA).

So, $MD = NE$. Then $OD - OE = 2ON = 2OC\cos\alpha$.

Teacher: Very good, can any other students use different methods to solve this variant extended problem 3?

Student: As shown in figure 22, take CO as one side to do $\angle FCO = 180^{\circ} - 2\alpha$, intersects with OB at point F. Then $\angle OCD = \angle FCE$, $\angle OFC = \angle OCF = \alpha = \angle EFC = \angle DOC$. Then CO = CF. So, $\Delta DCO \cong \Delta ECF$ (ASA). Then OD = EF, $OD - OE = OF = 2OC\cos\alpha$.

Ⅳ. CONCLUSION

Any meaningful mathematics problems are not isolated. Practices of inquiry based on pattern similarity are foundational for mathematical endeavor. A common trait among mathematicians is to rely on particular cases, isomorphic re-formulations, or analogous problems that simulate the original problem situations in their search for a solution (Pólya, 1954; Skemp, 1986). The literature suggests that most creative individuals tend to be attracted to complexity, which most school math curricula has very little to offer. Classroom practices and math curricula rarely use problems with an underlying mathematical structure and allow students a prolonged period of engagement and independence to work on such problems (Sriraman, 2009). Sriraman (2009) conjectures that in order for mathematical creativity to manifest in the school classroom, students should be given the opportunity to tackle non-routine problems with complexity and structure, which require not only motivation and persistence but also considerable reflection. This implies that educators should recognize the value of allowing students to reflect on previously solved problems and draw comparisons between various isomorphic problems (Sriraman, 2003, 2004). In addition, encouraging students to look for similarities in a class of problems also fosters ''mathematical'' behavior (Pólya, 1954), leading some students to discover fairly sophisticated mathematical structures and principles in a manner akin to creative mathematicians.

Teachers should be good at guiding students to look for the relationship among mathematics problems based on pattern similarity. In this research, we also explored how teacher may guide students to find the pattern hidden in mathematics problems in classroom, and guide students to use pattern similarity to constantly pose new mathematics problems and give their solutions by analogy. It is also a basic strategy and method of mathematicians' engaging in mathematics research. It is crucial to recognize the internal hidden pattern that embodies the essence of the problem. It needs some kind of mathematics intuition and talent, but it can also be taught and cultivated. As long as teachers often guide students to analyze problems based on pattern similarity, students can gradually be influenced imperceptibly, and learn mathematics inquiring based on pattern similarity. Practice has shown that students can learn to inquire mathematics based on pattern similarity. Specifically, they can pay attention to the process of constructing and discovering mathematics pattern based on their own experience, and develop the ability of generalizing and extending original problem and pose new mathematics problems based on pattern similarity. Through pattern recognition, extended problem and variant extended problem, students can gradually learn to develop the ability of recognizing, analyzing, appreciating, creating, expanding and applying mathematics pattern based on pattern similarity.

When implementing mathematical inquiry teaching based on pattern similarity, teachers take up a critical role of supporting student inquiry, "establishing a radically different set of social norms and values in the classroom as well as finding ways to invite students into the inquiry process, and support them as they engage in the process" (Siegel & Borasi, 1994, p. 210). For example, in this case, the teacher continuously guides students to pose new problems and look for different solutions based on pattern similarity. As emphasized by Pólya (1954), in looking for conspicuous patterns, mathematicians use a variety of heuristics such as (1) verifying consequences, (2) successively verifying several consequences, (3) verifying an improbable consequence, (4) inferring from analogy, (5) deepening the analogy. Without the guidance and inspiration, as well as the questioning and organization of teachers, it is difficult to carry out mathematical inquiry classrooms based on pattern similarity. When students have no ideas, teachers should constantly inspire and guide students to explore the essential characteristics (i.e., patterns) reflected in the problem. Only in this way can students find solutions to problems and pose new mathematics problems. Without a solid mathematical foundation and ample experience of inquiring, teachers are usually unable to organize and guide students' mathematical exploration based on pattern similarity. Because it is often difficult to find solutions that reflect the deep-seated essential characteristics of mathematics problems. Especially when posing mathematics problems based on pattern similarity, it requires a certain mathematician's creative way of thinking, which is often a challenge for teachers.

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