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MAXIMAL STRONG PRODUCT AND BALANCED FUZZY GRAPHS†

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Abstract. The notion of maximal product of two fuzzy graphs was introduced by Radha and Arumugam in 2015 and the notion of balanced fuzzy graph was introduced by Al-Hawary in 2011. In this paper, we give a modification of the maximal product definition, which we call maximal strong product. We also introduce the relatively new notion of maximal-balanced fuzzy graphs. We give necessary and sufficient conditions for the maximal strong product of two balanced (resp., maximal-balanced) fuzzy graphs to be balanced (resp., maximal-balanced) and we prove that these two independent notions are preserved under isomorphism.

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A fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$ and a fuzzy relation μ on a fuzzy subset σ, is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 0.1. [16] A *fuzzy graph,* with V as the underlying set, is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$.$H = (\sigma', \mu')$ is a *fuzzy subgraph* of G if there exists $X \subseteq V$ such that $\sigma' : X \to [0,1]$ is a fuzzy subset and $\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ' such that $\mu^{'}(x,y) \leq \sigma^{'}(x) \wedge \sigma^{'}(y)$ for all $x, y \in X$.

Definition 0.2. [16] *Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$ are* isomorphic *if there exists a bijection $h : V_1 \to V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$* for all $x \in V_1$ *and*

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$\mu_1(x,y) = \mu_2(h(x), h(y))$ for all $(x,y) \in E_1$. We then write $G_1 \simeq G_2$ and h is called an *isomorphism.* If $G_1 = G_2$, h is called an *automorphism.*

A fuzzy graph $G : (\sigma, \mu)$ with underlying graph $G^* : (V, E)$ is said to be complete *if* $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$ and G is strong *if* $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in E$.

Definition 0.3. [17] *The* complement *of fuzzy graph $G : (\sigma, \mu)$ is defined to be the fuzzy graph $\overline{G} : (\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} = \sigma$ and*

$$\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y).$$

Graph theory has many interesting applications in system analysis, economics and operations research. Most of the time the aspects of graph problems are uncertain and so it is nice to deal with these aspects via the methods of fuzzy logic. The concept of fuzzy relation which has a widespread application in pattern recognition was introduced by Zadeh [20] in his paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were first introduced by Rosenfeld [16] in 1975. Fuzzy graph theory is finding more and more increasing number of applications in modelling real time systems where the level of information inherent in the system varies with distinct levels of precision. Fuzzy models are becoming useful because of their aim is to reduce the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.

Since the notions of degree, complement, completeness, regularity, connectedness and many others play very important rules in the crisp graph case, the idea is to find what corresponds to these notions in the case of fuzzy graphs. Several authors have studied operations on fuzzy graphs, see for example [2, 3, 4, 5, 6, 7, 8, 9, 10]. AL-Hawary [1] introduced the concept of balanced fuzzy graphs. He defined three new operations on fuzzy graphs and explored what classes of fuzzy graphs are balanced. Sense then, many authors have studied the idea of balanced on distinct kinds of fuzzy graphs, see for example [11, 12, 13, 14, 18, 19].

Our aim in this paper is to study the notions of complete, strong, balanced and maximal-balanced product fuzzy graphs. Moreover, the relatively new operation of maximal strong product on fuzzy graphs are provided and properties are deeply explored.

Definition 0.4. *The* maximal strong product *of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with underlying graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with underlying graph $G_2^* : (V_2, E_2)$, where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \circledast G_2 : (\sigma_1 \circledast \sigma_2, \mu_1 \circledast \mu_2)$ with underlying graph $G^* : (V_1 \times V_2, E)$ where $E = \{(x, y_1)(x, y_2) : x \in V_1, y_1y_2 \in E_2\} \cup \{(x_1, y)(x_2, y) : x_1x_2 \in E_1, y \in V_2\} \cup \{(x_1, y_1)(x_2, y_2) : x_1x_2 \in E_1, y_1y_2 \in E_2\}$,*

$$\begin{aligned} (\sigma_1 \circledast \sigma_2)(x,y) &= \sigma_1(x) \vee \sigma_2(y) \text{ for all } x \in V_1, y \in V_2 \\ (\mu_1 \circledast \mu_2)((x,y_1)(x,y_2)) &= \sigma_1(x) \vee \mu_2(y_1,y_2), \end{aligned}$$

$$\begin{aligned}(\mu_1 \circledast \mu_2)((x_1,y)(x_2,y)) &= \sigma_2(y)) \vee \mu_1(x_1,x_2) \text{ and}\\ (\mu_1 \circledast \mu_2)((x_1,y_1)(x_2,y_2)) &= \mu_1(x_1,x_2) \vee \mu_2(y_1,y_2).\end{aligned}$$

Next, we show that the above definition is well defined.

Lemma 0.5. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs, then $G_1 \circledast G_2$ is a fuzzy graph.*

Proof. Case 1. If $x \in V_1, y_1y_2 \in E_2$, then

$$\begin{aligned}(\mu_1 \circledast \mu_2)((x,y_1)(x,y_2)) &= \sigma_1(x) \vee \mu_2(y_1,y_2)\\ &\leq \sigma_1(x) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))\\ &= (\sigma_1(x) \vee \sigma_2(y_1)) \wedge (\sigma_1(x) \vee \sigma_2(y_2))\\ &= ((\sigma_1 \circledast \sigma_2)(x,y_1)) \wedge ((\sigma_1 \circledast \sigma_2)(x,y_2)).\end{aligned}$$

Case 2. If $x \in V_1, y_1y_2 \in E_2$, then by a similar argument to that in Case 1,

$$(\mu_1 \circledast \mu_2)((x_1,y)(x_2,y)) \leq ((\sigma_1 \circledast \sigma_2)(x_1,y)) \wedge ((\sigma_1 \circledast \sigma_2)(x_2,y)).$$

Case 3. If $x_1x_2 \in E_1, y_1y_2 \in E_2$, then

$$\begin{aligned}&(\mu_1 \circledast \mu_2)((x_1,y_1)(x_2,y_2))\\ =\ & \mu_1(x_1,x_2) \vee \mu_2(y_1,y_2)\\ \leq\ & (\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))\\ =\ & ((\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee \sigma_2(y_1)) \wedge ((\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee \sigma_2(y_2))\\ \leq\ & (\sigma_1(x_1) \vee \sigma_2(y_1)) \wedge (\sigma_1(x_2) \vee \sigma_2(y_2))\\ =\ & ((\sigma_1 \circledast \sigma_2)(x_1,y_1)) \wedge ((\sigma_1 \circledast \sigma_2)(x_2,y_2)).\end{aligned}$$

□

Lemma 0.6. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two complete fuzzy graphs, then $G_1 \circledast G_2$ is complete.*

Proof. Since G_1 *and* G_2 are complete, then $\mu_1(x_1,x_2) = \sigma_1(x_1) \wedge \sigma_1(x_2)$ for all $x_1, x_2 \in V_1$ and $\mu_2(y_1,y_2) = \sigma_2(y_1) \wedge \sigma_2(y_2)$ for all $y_1, y_2 \in V_2$. Thus

$$\begin{aligned}(\mu_1 \circledast \mu_2)((x,y_1)(x,y_2)) &= \sigma_1(x) \vee \mu_2(y_1,y_2)\\ &= \sigma_1(x) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))\\ &= (\sigma_1(x) \vee \sigma_2(y_1)) \wedge (\sigma_1(x) \vee \sigma_2(y_2))\\ &= ((\sigma_1 \circledast \sigma_2)(x,y_1)) \wedge ((\sigma_1 \circledast \sigma_2)(x,y_2)).\end{aligned}$$

A similar argument gives

$$(\mu_1 \circledast \mu_2)((x_1,y)(x_2,y)) = ((\sigma_1 \circledast \sigma_2)(x_1,y)) \wedge ((\sigma_1 \circledast \sigma_2)(x_2,y)).$$

Finally,

$$\begin{aligned}&(\mu_1 \circledast \mu_2)((x_1,y_1)(x_2,y_2))\\ =\ & \mu_1(x_1,x_2) \vee \mu_2(y_1,y_2)\\ =\ & (\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))\end{aligned}$$

$$\begin{aligned} &= ((\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee \sigma_2(y_1)) \wedge ((\sigma_1(x_1) \wedge \sigma_1(x_2)) \vee \sigma_2(y_2)) \\ &= (\sigma_1(x_1) \vee \sigma_2(y_1)) \wedge (\sigma_1(x_2) \vee \sigma_2(y_2)) \\ &= ((\sigma_1 \circledast \sigma_2)(x_1, y_1)) \wedge ((\sigma_1 \circledast \sigma_2)(x_2, y_2)). \end{aligned}$$

□

Corollary 0.7. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two strong (complete) fuzzy graphs, then $G_1 \circledast G_2$ is strong.*

Theorem 0.8. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $\overline{G_1 \circledast G_2} \simeq \bar{G}_1 \circledast \bar{G}_2$.*

Proof. Let $\overline{G_1 \circledast G_2} = (\sigma_1 \circledast \sigma_2, \overline{\mu_1 \circledast \mu_2})$. We only need to show that $\overline{(\mu_1 \circledast \mu_2)}((x, y_1)(x, y_2)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x, y_1)(x, y_2))$, $\overline{(\mu_1 \circledast \mu_2)}((x_1, y)(x_2, y)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x_1, y)(x_2, y))$ and $\overline{(\mu_1 \circledast \mu_2)}((x_1, y_1)(x_2, y_2)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x_1, y_1)(x_2, y_2))$. Since G_1 and G_2 are two complete fuzzy graphs, then by Lemma 0.6, $G_1 \circledast G_2$ is complete. Hence

$$\overline{\mu_1 \circledast \mu_2}((x, y_1)(x, y_2)) = 0.$$

On the other hand, since G_1 and G_2 are two complete fuzzy graphs, their complements are empty fuzzy graphs and the maximal strong product of two empty fuzzy graphs is empty. So $(\bar{\mu}_1 \circledast \bar{\mu}_2)((x, y_1)(x, y_2)) = 0$.
Thus $\overline{(\mu_1 \circledast \mu_2)}((x, y_1)(x, y_2)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x, y_1)(x, y_2))$. By a similar argument, $\overline{(\mu_1 \circledast \mu_2)}((x_1, y)(x_2, y)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x_1, y)(x_2, y))$ and $\overline{(\mu_1 \circledast \mu_2)}((x_1, y_1)(x_2, y_2)) = (\bar{\mu}_1 \circledast \bar{\mu}_2)((x_1, y_1)(x_2, y_2))$. □

1. Balanced product fuzzy graphs.

We start this section by recalling the definition of balanced fuzzy graph.

Definition 1.1. [1] *The* density *of a fuzzy graph is* $D(G) = \dfrac{2 \sum_{(x,y) \in E} (\mu(x, y))}{\sum_{x,y \in V} (\sigma(x) \wedge \sigma(y))}$. *$G$ is* balanced *if $D(H) \leq D(G)$ for any non-empty product fuzzy subgraphs H of G.*

We next provide a necessary and sufficient condition for the density of the maximal product of two fuzzy graphs to be equal to the density of both fuzzy graphs.

Lemma 1.2. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two complete fuzzy graphs. Then $D(G_1 \circledast G_2) \geq D(G_i)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \circledast G_2)$.*

Proof. If $D(G_1 \circledast G_2) \geq D(G_i)$ for $i = 1, 2$, *then*

$$\begin{aligned} D(G_1) &= \frac{2 \sum\limits_{x_1, x_2 \in V_1} (\mu_1(x_1, x_2))}{\sum\limits_{x_1, x_2 \in V_1} (\sigma_1(x_1) \wedge \sigma_1(x_2))} \\ &\geq \frac{2 \sum\limits_{x \in V_1, y_1, y_2 \in V_2} \sigma_1(x) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))}{\sum\limits_{x \in V_1, y_1, y_2 \in V_2} \sigma_1(x) \vee (\sigma_2(y_1) \wedge \sigma_2(y_2))} \\ &\geq \frac{2 \sum\limits_{x \in V_1, y_1 y_2 \in E_2} \sigma_1(x) \vee \mu_2(y_1, y_2)}{\sum\limits_{x \in V_1, y_1, y_2 \in V_2} (\sigma_1(x) \vee \sigma_2(y_1)) \wedge (\sigma_1(x) \vee \sigma_2(y_2))} \\ &= D(G_1 \circledast G_2). \end{aligned}$$

Other cases are similar.
The converse is trivial. □

Next, a necessary and sufficient condition for the maximal product of two balanced fuzzy graphs to be balanced is provided.

Theorem 1.3. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two balanced fuzzy graphs. Then $G_1 \circledast G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \circledast G_2)$.*

Proof. If $G_1 \circledast G_2$ is balanced, then $D(G_i) \leq D(G_1 \circledast G_2)$ for $i = 1, 2$ and by Lemma 1.2, $D(G_1) = D(G_2) = D(G_1 \circledast G_2)$.

Conversely, if $D(G_1) = D(G_2) = D(G_1 \circledast G_2)$ and H is a fuzzy subgraph of $G_1 \circledast G_2$, then there exist fuzzy subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = n_1/r_1$, then $D(H_1) = a_1/b_1 \leq n_1/r_1$ and $D(H_2) = a_2/b_2 \leq n_1/r_1$. Thus $a_1 r_1 + a_2 r_1 \leq b_1 n_1 + b_2 n_1$ and hence $D(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D(G_1 \circledast G_2)$. Therefore, $G_1 \circledast G_2$ is balanced. □

2. Maximal-balanced fuzzy graphs

In this section, we introduce the relatively new notion of maximal-balanced. We note that using this notion, we get better results than using balanced one.

Definition 2.1. *The* maximal-density *of a fuzzy graph G is* $CD(G) = 2\frac{\sum_{(x,y) \in E} \mu(x,y))}{\sum_{x,y \in V} \sigma(x) \vee \sigma(y)}$. *$G$ is* maximal-balanced *if $MD(H) \leq MD(G)$ for all fuzzy non-empty subgraphs H of G.*

Theorem 2.2. *Let G be a fuzzy graph. Then $MDG) \leq 2$ if and only if and only if G is complete.*

Proof. Let G be a complete fuzzy graph. Then $MD(G) = \frac{2\sum\limits_{x,y\in V}\sigma(x)\wedge\sigma(y)}{\sum\limits_{x,y\in V}\sigma(x)\vee\sigma(y)} \leq 2$. Conversely, suppose G is not complete with maximal-density less than or equals to 2. Then $MD(G) = \frac{2\sum\limits_{(x,y)\in E}\mu(x,y))}{\sum\limits_{x,y\in V}\sigma(x)\vee\sigma(y)} \leq 2$. So $\sum_{(x,y)\in E}\mu(x,y)) \geq \sum_{x,y\in V}\sigma(x)\vee\sigma(y)$. Since G is not complete, $\mu(x,y) < \sigma(x)\wedge\sigma(y)$ for some $x,y\in V$. That means $\mu(\acute{x},\acute{y}) > \sigma(\acute{x})\vee\sigma(\acute{y})$ for some $\acute{x},\acute{y}\in V-\{x,y\}$, a contradiction. □

Theorem 2.3. *Any complete fuzzy graph is maximal-balanced.*

Proof. Let G be a complete fuzzy graph. Then by Theorem 2.2, $MD(G) \leq 2$. If H is a non-empty fuzzy subgraph of G, then we have two cases: □

Case I If H has less edges than G, then $\sum_{(x,y)\in E(H)}\mu(x,y)) \leq \sum_{(x,y)\in E}\mu(x,y))$ and $\sum_{x,y\in V(H)}\sigma(x)\vee\sigma(y) = \sum_{x,y\in V}\sigma(x)\vee\sigma(y)$. Thus

$$\begin{aligned} MD(H) &= \frac{2\sum\limits_{(x,y)\in E(H)}(\mu(x,y))}{\sum\limits_{x,y\in V(H)}(\sigma(x)\vee\sigma(y))} = \frac{2\sum\limits_{(x,y)\in E(H)}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\vee\sigma(y))} \\ &\leq \frac{2\sum\limits_{(x,y)\in E}(\mu(x,y))}{\sum\limits_{x,y\in V}(\sigma(x)\vee\sigma(y))} \leq 2 \leq MD(G). \end{aligned}$$

Case II If H has vertices lass than G, then it is clear that H is a complete fuzzy graph. We conclude that $MD(H) = MD(G)$.

Thus G is maximal-balanced product fuzzy graph.

The converse of preceding result need not be true.

Example 2.4. Consider the fuzzy graph G such that $\sigma(x_1) = 0.1$, $\sigma(x_2) = 0.2 = \sigma(x_3)$, $\mu(x_1,x_2) = 0.01 = \mu(x_1,x_3)$ and $\mu(x_2,x_3) = 0.02$. Then G is a maximal-balanced fuzzy graph that is not complete.

Theorem 2.5. *Every self-complementary fuzzy graph has maximal-density less than or equal to* 1.

Proof. Let G be self-complementary fuzzy graph. Then

$$MD(G) = \frac{2\sum\limits_{(x,y)\in E}\mu(x,y)}{\sum\limits_{x,y\in V}(\sigma(x)\vee\sigma(y))} = \frac{2\frac{1}{2}\sum\limits_{x,y\in V}(\sigma(x)\wedge\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\vee\sigma(y))} = \frac{\sum\limits_{x,y\in V}(\sigma(x)\wedge\sigma(y))}{\sum\limits_{x,y\in V}(\sigma(x)\vee\sigma(y))} \leq 1$$

□

The converse of the above result need not be true.

Example 2.6. Consider the fuzzy graph G such that $\sigma(x_1) = 0.1$, $\sigma(x_2) = 0.2$, $\sigma(x_3) = 0.4$, $\mu(x_1,x_2) = 0.02$, $\mu(x_1,x_3) = 0$ and $\mu(x_2,x_3) = 0.05$. Then $MD(G) \leq 1$, but G is not self-complementary.

Theorem 2.7. *Let $G : (\sigma, \mu)$ be a fuzzy graph such that $\mu(x, y) = \frac{1}{2}(\sigma(x)\sigma(y))$ for all $x, y \in V$. Then $MD(G) \leq 1$.*

Proof. By Lemma 1.2, G is self-complementary and by Theorem 2.5, $MD(G_1) \leq 1$. □

Lemma 2.8. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two complete fuzzy graphs. Then $MD(G_i) \leq MD(G_1 \circledast G_2)$ for $i = 1, 2$ if and only if $MD(G_1) = MD(G_2) = MD(G_1 \circledast G_2)$.*

Proof. If $D(G_i) \leq D(G_1 \circledast G_2)$ for $i = 1, 2$, then since G_1 and G_2 are complete fuzzy graphs, by Theorem 2.7,

$$MD(G_1), MD(G_2) \leq 2.$$

By the preceding Corollary, $G_1 \circledast G_2$ is strong and hence by Theorem 2.7, $MD(G_1 \circledast G_2) < 2$. Thus $MD(G_i) \geq MD(G_1 \circledast G_2)$ for $i = 1, 2$ and so $MD(G_1) = MD(G_2) = MD(G_1 \circledast G_2)$. □

The converse is trivial.

Theorem 2.9. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two maximal-balanced fuzzy graphs. Then $G_1 \circledast G_2$ is maximal-balanced if and only if $MD(G_1) = MD(G_2) = MD(G_1 \circledast G_2)$.*

Proof. If $G_1 \circledast G_2$ is maximal-balanced, then $MD(G_i) \leq MD(G_1 \circledast G_2)$ for $i = 1, 2$ and by Lemma 2.8, $MD(G_1) = MD(G_2) = MD(G_1 \circledast G_2)$. Conversely, If $MD(G_1) = MD(G_2) = MD(G_1 \circledast G_2)$ and H is a fuzzy subgraph of $G_1 \circledast G_2$, then there exist fuzzy subgraph H_1 of G_1 and H_2 of G_2 such that $H \simeq H_1 \circledast H_2$. As G_1 and G_2 are maximal-balanced and say $MD(G_1) = MD(G_2) = \frac{n_1}{r_1}$, then $MD(H_1) = \frac{a_1}{b_1} \leq \frac{n_1}{r_1}$ and $MD(H_2) = \frac{a_2}{b_2} \leq \frac{n_1}{r_1}$. Thus $a_1 r_1 + a_2 r_1 \leq b_1 n_1 + b_2 n_1$ and hence $MD(H) \leq \frac{a_1 + a_2}{b_1 + b_2} \leq \frac{n_1}{r_1} = MD(G_1 \circledast G_2)$. Therefore $G_1 \circledast G_2$ is maximal-balanced. □

Theorem 2.10. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be isomorphic fuzzy graphs. If one of them is maximal-balanced, then the other is maximal-balanced.*

Proof. Suppose G_2 is maximal-balanced and let $h : V_1 \to V_2$ be a bijection such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$. Thus $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ and $\sum_{x,y \in E_1} \mu_1(x, y) = \sum_{x,y \in E_2} \mu_2(x, y)$. If $H_1 = (\acute{\sigma}_1, \acute{\mu}_1)$ is a fuzzy subgraph of G_1 with underlying set W, then $H_2 = (\acute{\sigma}_2, \acute{\mu}_2)$ is a fuzzy subgraph of G_2 with underlying set $h(W)$ where $\acute{\sigma}_2(h(x)) = \acute{\sigma}_1(x)$ and $\acute{\mu}_2(h(x), h(y)) = \acute{\mu}_1(x, y)$ for all $x, y \in W$. Since G_2 is maximal-balanced, $MD(H_1) \leq MD(G_2)$ and so $2\frac{\sum_{x,y \in E_1} \mu_2(h(x), h(y))}{\sum_{x,y \in V(H_2)} (\acute{\sigma}_2(x) \vee \acute{\sigma}_2(y))} \leq 2\frac{\sum_{x,y \in E_1} \mu_2(x, y)}{\sum_{x,y \in V_2} (\sigma_2(x) \vee \sigma_2(y))} \leq 2\frac{\sum_{x,y \in E_1} \mu_1(x, y)}{\sum_{x,y \in V_2} (\sigma_2(x) \vee \sigma_2(y))}$. Therefore, G_1 is maximal-balanced. □

Next, we show that the notions of balanced and maximal-balanced are independent.

Example 2.11. Consider the fuzzy graph G such that $\sigma(x_1) = 0.3$, $\sigma(x_2) = 0.2$, $\sigma(x_3) = 0.1$, $\mu(x_1, x_2) = 0.06$, $\mu(x_1, x_3) = 0.03$ and $\mu(x_2, x_3) = 0.02$. Then G is maximal-balanced, but is not balanced since $D(G) = 0.275$, but if we take $H = (x_1, x_3)$, then $D(H) = 0.6$.

The fuzzy graph G with $\sigma(x_1) = 0.5$, $\sigma(x_2) = 0.7 = \sigma(x_3)$, $\mu(x_1, x_2) = 0.1 = \mu(x_1, x_3)$ and $\mu(x_2, x_3) = 0.4$. is balanced, but is not maximal-balanced since if we take $H = (x_2, x_3)$, then $MD(H) = 1.14$ while $MD(G) = 1.04$.

Theorem 2.12. *Every balanced complete fuzzy graph is maximal-balanced.*

Proof. Let G be a balanced complete fuzzy graph and H be a non-empty fuzzy subgraph of G. Then as G is balanced , $D(H) \leq D(G)$. Since G is complete, G is maximal-balanced by Theorem 2.3. □

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References

1. T. Al-Hawary, *Complete fuzzy graphs*, Inter. J. Math. Comb. **4** (2011), 26-34.
2. T. Al-Hawary, S. Al-Shalaldeh and M. Akram, *Certain matrices and energies of fuzzy graphs*, TWMS JPAM **14** (2023), 50-68.
3. M. Akram., D. Saleem and T. Al-Hawary, *Spherical fuzzy graphs with application to decision-making*, Math. Comp. Appl. **25** (2020), 1-22.
4. T. Al-Hawary, *Density results for perfectly regular and perfectly edge-regular fuzzy graphs*, J. Disc. Math. Scie. & Cryptography **2** (2022), 1-10.
5. T. Al-Hawary and B. Horani,, *On intuitionistic product fuzzy graphs*, Ital. J. Pure. Appl. Math. **38** (2017), 113-126.
6. T. Al-Hawary, *On Modular flats and pushouts of matroids*, Ital. J. Pure. Appl. Math. **43** (2020), 273-241.
7. T. Al-Hawary, *Matroid filterbase*, Indian J. math. **60** (2018), 301-310.
8. T. Al-Hawary, *β-greedoids*, Matematicki Vesnik **66** (2014), 343-350.
9. T. Al-Hawary, *Complete Hamacher fuzzy graphs*, J. Appl. Math. and Informatics **40** (2022), 1043-1052.
10. T. Al-Hawary, *Strong modular product and complete fuzzy graphs*, Ital. J. Pure Appl. Math. (2023)
11. M. Karunambigai, M. Akram, S. Sivasankar and K. Palanivel, *Balanced intuitionistic fuzzy graphs*, Appl. Math. Sci. **7** (2013), 2501-2514.
12. M. Karunambigai, S. Sivasankar and K Palanivel, *Properties of balanced intuitionistic fuzzy graphs*, Inte. J. Res. Sci. **1** (2014), 1-5.
13. H. Rashmanlou and M. Pal, *Balanced interval-valued fuzzy graphs*, J. Phys. Sci. **17** (2013), 43-57.

14. H. Rashmanlou and Y. Benjun, *Complete interval-valued fuzzy graph*, Annals Fuzzy Math. Infor. **1** (2013), 1-11.
15. K. Radha and S. Arumugam, *On maximal product of two fuzzy graphs*, Inter. J. Current Res. **1** (2015), 11508-11515.
16. A. Rosenfeld, *Fuzzy graphs*, in Zadeh. L. A, K. S. Fu, K, Tanaka and M. Shirmura (Eds), *Fuzzy sets and their applications to cognitive and processes* Academic Press, New York, 1975.
17. M. Sunitha, *Complement of fuzzy graphs*, Indian J. Pure Appl. Math. **33** (2002), 1451-1464.
18. K. Tatanassov, *Intuitionistic fuzzy sets: Theory and applications*, Studiesin fuzziness and soft computing, Heidelberg, New York, Physicaverl, 1999, 1451-1464.
19. N. Vinoth Kumar and G. Geetha Ramani, *Product intuitionistic fuzzy graph*, Inter. J. Comp. Appl. **18** (2011), 11-33.
20. L. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.

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