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APPROXIMATION OF LIPSCHITZ CLASS BY DEFERRED-GENERALIZED NÖRLUND $(D^{\gamma}_{\beta}.N_{pq})$ PRODUCT SUMMABILITY MEANS

JITENDRA KUMAR KUSHWAHA, LAXMI RATHOUR*, LAKSHMI NARAYAN MISHRA, KRISHNA KUMAR

Abstract. In this paper, we have determined the degree of approximation of function belonging of Lipschitz class by using Deferred-Generalized Nörlund $(D^{\gamma}_{\beta}.N_{pq})$ means of Fourier series and conjugate series of Fourier series, where $\{p_n\}$ and $\{q_n\}$ is a non-increasing sequence. So that results of DEĞER and BAYINDIR [23] become special cases of our results.

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1. Introduction

Many researchers like Leindler [7], Rhoades [21], Agnew [22], Qureshi and Neha, [20], Khatri and Mishra [4], Mishra et al. ([9], [11], [12]) Mishra and Mishra [10], Deepmala et al. [3], Nigam [18], Nigam and Sharma [19], Lal ([6], [8]), Khan [5], Chandra [1] have studied the degree of approximation of functions belonging to various Lipschitz classes by using summability methods of Fourier series and conjugate series of Fourier series. Working in similar direction Mishra et al. ([14], [15], [16], [17]) have determined the degree of approximation of conjugate of function belonging to Lipschitz class by Cesáro-Nörlund product means of conjugate series of Fourier series . Later on DEG̃ER and BAYINDIR [23] established the trigonometric approximation of functions belonging to Lipschitz class by Deferred-Nörlund $(D^{b}_{a}.N_{p})$ product means . Now, we are using more general method to determine the degree of approximation of functions belonging

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to Lipschitz class which reduces to the results of DEĞER and BAYINDIR [23] as particular cases.

2. Definitions

Definition 2.1. Let a function f is 2π periodic and Lebesgue integrable in $[0, 2\pi]$ with n^{th} partial sums $s_n(f)$. Then,

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \qquad n \in N \tag{2.1}$$

with $s_0(f;x) = \frac{1}{2}a_0$.

The conjugate series of Fourier series (1.1) is given by

$$\tilde{S}_n[f] = \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx). \tag{2.2}$$

A function $f \in Lip\alpha$, if

$$f(x+t) - f(x) = O(|t^{\alpha}|) \text{ for } 0 < \alpha \leqslant 1, \ t > 0.$$

The L_∞-norm of function $f : R \to R$ is defined by

$$\|f\|_\infty = sup|f(x)|, x \in R.$$

The L^r-norm of function is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r}, 1 \leq r < \infty.$$

The degree of approximation of function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\|.\|_\infty$ is defined by Mac.Fadden [13].

$$\|t_n - f\|_\infty = sup\{|t_n(x) - f(x)|, x \in R\}$$

and the degree of approximation of function $E_n(f)$ of a function $f \in L^r$ is defined by

$$E_n(f) = min\|t_n(x) - f(x)\|_r.$$

We use following notations through out the paper

$$\phi_x(t) = f(x+t) - 2f(x) + f(x-t)$$

$$\phi_x(t) = f(x+t) - f(x-t)$$

and

$$\bar{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \phi_x(t)\cot(t/2)dt.$$

If $f \in L^r$, then $\tilde{f}$ exists for almost all x.

Let $\{p_n\}$ and $\{q_n\}$ be a non-negative sequence of real line. The transformation given by

$$t_n^N = \frac{1}{R_n}\sum_{\nu=0}^{n} p_{n-\nu}q_\nu s_\nu(f;x)$$

is called the generalized Nörlund mean (N, p_n, q_n) of the sequence $s_n(f;x)$ where

$$R_n = \sum_{k=0}^{n} p_{n-k}q_k, \qquad \forall n \geq 0 \qquad and \qquad R_n \to \infty, \qquad as \qquad n \to \infty.$$

In the special case in which the generalized Nörlund mean N_{pq} reduces to the familiar C^α mean with a N_{pq} mean define $(C^1.N_{pq})$ summability. Thus the $(C^1.N_{pq})$ and $(C^1.\widetilde{N_{pq}})$ means are given respectively by the transformations

$$t_n^{CN} = \frac{1}{n+1}\sum_{k=0}^{n} R_k^{-1}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu s_\nu\left(f;x\right), \tag{2.3}$$

and

$$\tilde{t}_n^{CN} = \frac{1}{n+1}\sum_{k=0}^{n} R_k^{-1}\sum_{\nu=0}^{k} p_{k-\nu}q_\nu \tilde{s}_\nu\left(f;x\right) \tag{2.4}$$

The Fourier series of f is said to be $(C^1.N_{pq})$ summable to $s(x)$ if
$t_n^{CN} \to s(x) \qquad as \qquad n \to \infty.$

We can easily seen that $(C^1.N_{pq})$ method is regular.

In this paper, we have determined the degree of approximation for the functions belonging to the $Lip\alpha, (0 < \alpha \leq 1)$ class using Deferred-Generalized Nörlund $(D^{\gamma}_{\beta}.Npq)$ means of Fourier series and conjugate series of Fourier series, where $\{p_n\}$ and $\{q_n\}$ is a non-increasing sequence.

3. Definition of Deferred Cesáro mean and its product with Generalized Nörlund means in Lipschitz class

The Deferred Cesàro means is defined as following.
Let $\beta = (a_n)$ and $\gamma = (b_n)$ be sequence of non-negative integers with conditions

$$a_n < b_n; \qquad n = 1, 2, 3, \ldots \tag{3.1}$$

$$\lim_{n\to\infty} b_n = +\infty \tag{3.2}$$

The Deferred Cesàro means, D^{γ}_{β} determined by β and γ

$$D^{\gamma}_{\beta} = \frac{s_{a_{n+1}} + s_{a_{n+2}} + \ldots\ldots\cdots + s_{b_n}}{(b_n - a_n)} = \frac{1}{(b_n - a_n)}\sum_{k=a_{n+1}}^{b_n} s_k$$

Where (s_k) is a sequence of real or complex numbers.

Taking into deferred Cesàro means Deferred generalized Nörlund ($D^{\gamma}_{\beta}.N_{pq}$) means which the product of D^{γ}_{β} means with a N_{pq} mean are defined by transformation

$$t_n^{D^{\gamma}_{\beta}.N_{pq}}(f;x) = t_n^{D^{\gamma}_{\beta}.N_{pq}} = \frac{1}{\gamma-\beta}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k} p_{\nu}q_{k-\nu}s_{\nu}(f;x)\right).$$

and similarly; ($D^{\gamma}_{\beta}.N_{pq}$) means of conjugate Fourier series are given transformation

$$\tilde{t}_n^{D^{\gamma}_{\beta}.N_{pq}}(f;x) = \tilde{t}_n^{D^{\gamma}_{\beta}.N_{pq}} = \frac{1}{\gamma-\beta}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k} p_{\nu}q_{k-\nu}\tilde{s}_{\nu}(f;x)\right).$$

This results related to trigonometric approximation of function belonging to Lipschitz class by the $D^{\gamma}_{\beta}.N_{pq}$ means of it's Fourier series. The second result states the degree of approximation to conjugates of function belonging to Lipschitz class by the $D^{\gamma}_{\beta}.N_{pq}$ means of conjugate series of Fourier series.

4. Lemmas

We prove following lemmas for the proof of main theorems.

Lemma 4.1. *If R_n is positive and $R_n^{-1} \geq R_{n+1}^{-1}$ for every $n \geq 0$. Then for $0 \leq \beta < \gamma \leq \infty$; $0 < t \leq \pi$. and for any n, we have*

$$\left|\sum_{k=\beta}^{\gamma} R_k^{-1} e^{i(n-k)t}\right| = \begin{cases} O(t^{-1}); & \beta \\ O(t^{-1}R_{\beta}^{-1}); & \beta \geq \tau \end{cases}$$

Where $\tau = [t^{-1}]$ denotes the integer part of $1/t$.

Lemma 4.2. *Let $\{p_n\}$ and $\{q_n\}$ be a non-negative sequence and $\{\alpha_n\} \in \pounds^{+}$ with $n\alpha_n = O(1)$, For $\frac{\pi}{\gamma-\beta} < t \leq \pi$.*

We have

$$(i).\quad \left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma} R_k^{-1}\sum_{\nu=0}^{k} p_{\nu}q_{k-\nu}\frac{\sin{(k-\nu+1/2)t}}{\sin(t/2)}\right| = O\left(\frac{\tau^2}{\gamma-\beta}+\tau\right)$$

$$(ii).\quad \left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma} R_k^{-1}\sum_{\nu=0}^{k} p_{\nu}q_{k-\nu}\frac{\cos{(k-\nu+1/2)t}}{\sin(t/2)}\right| = O\left(\frac{\tau^2}{\gamma-\beta}+\tau\right)$$

Proof (i) -

$$\left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma} R_k^{-1}\sum_{\nu=0}^{k} p_{\nu}q_{k-\nu}\frac{\sin{(k-\nu+1/2)t}}{\sin(t/2)}\right|$$

$$\leq \left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\tau} R_k^{-1}\sum_{\nu=0}^{k} p_\nu q_{k-\nu}\frac{\sin{(k-\nu+1/2)t}}{\sin(t/2)}\right|$$
$$+\left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\tau+1}^{\gamma} R_k^{-1}\sum_{\nu=0}^{k} p_\nu q_{k-\nu}\frac{\sin{(k-\nu+1/2)t}}{\sin(t/2)}\right|$$
$$= M_1 + M_2$$

By Jordan's inequality we can write $(\sin(t/2))^{-1} \geq \frac{\pi}{t}$; for $\frac{\pi}{\gamma-\beta} < t \leq \pi$.

Therefore,

$$\begin{aligned} M_1 &\leq \frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\tau} R_k^{-1}\sum_{\nu=0}^{k} p_\nu q_{k-\nu}\frac{1}{|\sin(t/2)|} \\ &= \frac{1}{2t(\gamma-\beta)}\sum_{k=\beta+1}^{\tau} R_k^{-1}\sum_{\nu=0}^{k} p_\nu q_{k-\nu} \\ &= \frac{1}{2t(\gamma-\beta)}\sum_{k=\beta+1}^{\tau} R_k^{-1}\sum_{\nu=0}^{k} R_k \\ &= \frac{1}{2t(\gamma-\beta)}\sum_{k=\beta+1}^{\tau} 1 \\ &= \frac{1}{2t(\gamma-\beta)}\tau \end{aligned}$$

$$M_1 = O\left(\frac{\tau}{(\gamma-\beta)}\tau\right) = O\left(\frac{\tau^2}{(\gamma-\beta)}\right) \tag{4.1}$$

$$\begin{aligned} M_2 &= \left|\frac{1}{2t(\gamma-\beta)}\sum_{k=\tau+1}^{\gamma} R_k^{-1}\sum_{\nu=0}^{k} p_\nu q_{k-\nu}\sin{(k-\nu+1/2)\,t}\right| \\ &= \left(\frac{1}{2t(\gamma-\beta)}\right)\left|\sum_{\nu=0}^{\gamma} p_\nu q_{k-\nu}\sum_{k=\tau+1}^{\gamma} R_k^{-1}\sin{(k-\nu+1/2)\,t}\right| \end{aligned}$$

Let us divide in to two part of the last sum. Thus we have

$$M_2 \leq \frac{1}{2t(\gamma-\beta)}\left|\sum_{\nu=0}^{\tau+1} p_\nu q_{k-\nu}\sum_{k=\tau+1}^{\gamma} R_k^{-1}\sin{(k-\nu+1/2)\,t}\right|$$

$$+\frac{1}{2t(\gamma-\beta)}\left|\sum_{\nu=\tau+1}^{\gamma} p_\nu q_{k-\nu}\sum_{k=\tau+1}^{\gamma} R_k^{-1}\sin{(k-\nu+1/2)\,t}\right|$$

$$= \frac{1}{2t(\gamma-\beta)}\left(M_{21}+M_{22}\right)$$

First of all let us estimate M_{21}. By elementary calculation , we have

$$\begin{aligned} M_{21} &\leq \sum_{\nu=0}^{\tau+1} p_\nu q_{k-\nu}\left|\sum_{k=\tau+1}^{\gamma} R_k^{-1} e^{i(k-\nu)t} e^{it/2}\right| \\ &= \sum_{\nu=0}^{\tau+1} p_\nu q_{k-\nu}\left|\sum_{k=\tau+1}^{\gamma} R_k^{-1} e^{i(k-\nu)t}\right| \end{aligned}$$

Therefore by lemma (4.1) we get

$$M_{21} \leq \sum_{\nu=0}^{\tau+1} p_\nu q_{k-\nu} O\left(\tau R_{\tau+1}^{-1}\right) = R_{\tau+1} O(\tau R_{\tau+1}^{-1}) = O(\tau) \tag{4.2}$$

Now, let us consider the second sum. Taking into account of Abel transformation. We obtain

$$\begin{aligned} &\sum_{k=\nu}^{\gamma} R_k^{-1} \sin(k-\nu+1/2)t \\ &= \sum_{k=\nu}^{\gamma-1}\left(\Delta R_k^{-1}\right)\sum_{m=0}^{k}\sin(k-\nu+1/2)t + R_\gamma^{-1}\sum_{m=0}^{\gamma}\sin(k-m+1/2)t \\ &- R_\nu^{-1}\sum_{m=0}^{\nu-1}\sin(k-m+1/2)t \end{aligned}$$

Where $\Delta R_k^{-1} = R_k^{-1} - R_{k+1}^{-1}$,
by using $\sum_{k=\lambda}^{\mu} e^{(-ikt)} = O(t)$

For $k \geq 0$, and then $\{R_n\}$ is non-decreasing sequence, we have

$$\begin{aligned} &\left|\sum_{k=\nu}^{\gamma} R_k^{-1} \sin(k-\nu+1/2)t\right| \\ &\leq \sum_{k=\nu}^{\gamma-1}\left|\Delta R_k^{-1}\right|\left|\sum_{m=0}^{k}\sin(k-\nu+1/2)t\right| + R_\gamma^{-1}\left|\sum_{m=0}^{\gamma}\sin(k-m+1/2)t\right| \\ &+ R_\nu^{-1}\left|\sum_{m=0}^{\nu-1}\sin(k-m+1/2)t\right| \end{aligned}$$

$$= O(t)\left(\sum_{k=\nu}^{\gamma-1}\left|\left(\Delta R_k^{-1}\right)\right| + R_\gamma^{-1} + R_\nu^{-1}\right) = O(t)\left(R_\gamma^{-1} + R_\nu^{-1}\right). \tag{4.3}$$

Owing to (4.2) and (4.3) we get

$$\begin{aligned} M_2 &= O\left(\frac{\tau}{\gamma-\beta}\right)\left(O(\tau)+\sum_{\nu=\tau+1}^{\gamma} p_\nu q_{k-\nu} O(\tau)\left(R_\gamma^{-1}+R_\nu^{-1}\right)\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}\right)\left(1+\sum_{\nu=\tau+1}^{\gamma} p_\nu q_{k-\nu}\left(R_\gamma^{-1}+R_\nu^{-1}\right)\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}\right)\left(1+R_\gamma^{-1}\sum_{\nu=\tau+1}^{\gamma} p_\nu q_{k-\nu}+\sum_{\nu=\tau+1}^{\gamma}\frac{p_\nu q_{k-\nu}}{R_\nu}\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}\right)\left(1+1+\sum_{\nu=\tau+1}^{\gamma}\alpha_\nu\right) \end{aligned}$$

By considering with $n\alpha = O(1)$ and $\tau = [t^{-1}]$ we write

$$\begin{aligned} M_2 &= O\left(\frac{\tau^2}{\gamma-\beta}\right)\left(2+\sum_{\nu=\tau+1}^{\gamma}\alpha_\nu\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}\right)\left(2+O\left(\gamma-\tau\right)\alpha_{\tau+1}\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}\right)+O\left(\frac{\tau^2}{\gamma-\beta}\left(\gamma-\tau\right)\alpha_{\tau+1}\right) \\ &= O\left(\frac{\tau^2}{\gamma-\beta}+\tau\right) \end{aligned}$$

Lemma 4.3. *The following statements are satisfied-*

$$(i).\left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k}p_\nu q_{k-\nu}\frac{\sin\left(k-\nu+1/2\right)t}{\sin(t/2)}\right)\right| = O\left([1/t]\right)$$

$$(ii).\left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k}p_\nu q_{k-\nu}\frac{\cos\left(k-\nu+1/2\right)t}{\sin(t/2)}\right)\right| = O\left([1/t]\right)$$

for $0 < t \leq \pi/(n+1)$.

Proof:- Since $\sin(t/2) \geq t/\pi$. for $0 < t \leq \pi/(n+1)$. (Jordan's Inequality). We have-

$$\left|\frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k}p_\nu q_{k-\nu}\frac{\sin\left(k-\nu+1/2\right)t}{\sin(t/2)}\right)\right|$$

$$\begin{aligned}
&\leq \frac{1}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k}p_\nu q_{k-\nu}\frac{1}{|\sin(t/2)|}\right)\\
&= \frac{\pi/t}{2\pi(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}\left(R_k^{-1}\sum_{\nu=0}^{k}p_\nu q_{k-\nu}\right)\\
&= \frac{1}{2t(\gamma-\beta)}\left(\sum_{k=\beta+1}^{\gamma}1\right)=\frac{1}{2t(\gamma-\beta)}(\gamma-\beta)\\
&= \frac{1}{2t}=O(\tau).
\end{aligned}$$

The proof of the case (ii) runs along the same as that of (i).

5. Main Theorems

Theorem 5.1. *Let (p_n) and (q_n) be a non-negative sequence and $\alpha_n \in \pounds_0^+$ with $n\alpha_n = O(1)$. If $f \in Lip\alpha$, then the degree of approximation by the $D_\beta^\gamma.N_{pq}$ means of Fourier series is given by*

$$\|t_n^{D_\beta^\gamma.N_{pq}}-f(x)\|_\infty=\sup_{0\leq x\leq 2\pi}\left|t_n^{D_\beta^\gamma.N_{pq}}-f(x)\right|=\begin{cases}O((\gamma-\beta)^{-\alpha}); & 0<\alpha<1.\\ O(\frac{log(\gamma-\beta)}{(\gamma-\beta)}); & \alpha=1\end{cases}$$

$$f(x)=\frac{1}{(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}R_k^{-1}\sum_{\nu=0}^{k}p_{k-\nu}q_\nu f(x)$$

Proof:- since

$$f(x)=\frac{1}{(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}R_k^{-1}\sum_{\nu=0}^{k}p_{k-\nu}q_\nu f(x)$$

We have

$$t_n^{D_\beta^\gamma.N_{pq}}(f;x)-f(x)=\frac{1}{(\gamma-\beta)}\sum_{k=\beta+1}^{\gamma}R_k^{-1}\sum_{\nu=0}^{k}p_{k-\nu}q_\nu(s_\nu(f;x)-f(x))$$

Where $s_\nu(f;x)$ is the partial sum of Fourier series of f.
On the other hand , we know that

$$s_n^{D_\beta^\gamma.N_{pq}}(f;x)-f(x)=\frac{1}{2\pi}\int_0^\pi \phi_x(t)\frac{\sin(\nu+1/2)t}{\sin(t/2)}dt$$

where $\phi_x(t)=f(x+t)-f(x-t)$.

Therefore we write

$$\begin{aligned} & t_n^{D_\beta^\gamma.N_{pq}}(f;x) - f(x) \\ = & \frac{1}{(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} R_k^{-1} \sum_{\nu=0}^{k} p_{k-\nu} q_\nu \left(\frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\sin(\nu+1/2)t}{\sin(t/2)} dt \right) \quad (1) \\ = & \int_0^\pi \phi_x(t) \frac{1}{2\pi(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} R_k^{-1} \sum_{\nu=0}^{k} p_{k-\nu} q_\nu \frac{\sin(\nu+1/2)t}{\sin(t/2)} dt \\ = & \int_0^\pi \phi_x(t) \frac{1}{2\pi(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} R_k^{-1} \sum_{\nu=0}^{k} p_\nu q_{k-\nu} \frac{\sin(k-\nu+1/2)t}{\sin(t/2)} dt \end{aligned}$$

Let us divide into two parts the integral. Thus we have

$$t_n^{D_\beta^\gamma.N_{pq}}(f;x) - f(x)$$

$$= \left[\int_0^{\pi/(\gamma-\beta)} + \int_{(\pi/(\gamma-\beta))}^{\pi} \right] \phi_x(t) \frac{1}{2\pi(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} R_k^{-1} \sum_{\nu=0}^{k} p_\nu q_{k-\nu} \frac{\sin(k-\nu+1/2)t}{\sin(t/2)} dt$$

$$I_1 + I_2$$

Firstly let consider I_1

$$|I_1| \le \int_0^{\pi/(\gamma-\beta)} |\phi_x(t)| \left| \frac{1}{2\pi(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} \left(R_k^{-1} \sum_{\nu=0}^{k} p_\nu q_{k-\nu} \right) \frac{\sin(k-\nu+1/2)t}{\sin(t/2)} \right| dt$$

Since $f(x) \in Lip\alpha$, we know that $\phi_x(t) \in Lip\alpha$. Therefore, from lemma 4.3 -(i) we have

$$\begin{aligned} |I_1| &= \int_0^{\pi/(\gamma-\beta)} |\phi_x(t)| \left| \frac{\pi/t}{2\pi(\gamma-\beta)} \sum_{k=\beta+1}^{\gamma} 1 \right| dt \\ &= O\left(\int_0^{\pi/(\gamma-\beta)} \frac{\pi/t}{2\pi(\gamma-\beta)}.(\gamma-\beta)\, |t^\alpha|\, dt \right) \\ &= O\left(\int_0^{\pi/(\gamma-\beta)} |t^\alpha| \frac{1}{t} dt \right) = O((\gamma-\beta)^{-\alpha}) \qquad (5.1) \end{aligned}$$

Now let us consider I_2. By using Lemma 4.2-(i) and $\phi_x(t) \in Lip\alpha$ we obtain

$$\begin{aligned} |I_2| &= O\left(\int_{\pi/(\gamma-\beta)}^{\pi} t^\alpha \left(\frac{\tau^2}{\gamma-\beta} + \tau \right) dt \right) \\ &= O\left(\int_{\pi/(\gamma-\beta)}^{\pi} t^\alpha \left(\frac{\tau^2}{\gamma-\beta} \right) dt \right) + O\left(\int_{\pi/(\gamma-\beta)}^{\pi} t^\alpha.\tau dt \right) \end{aligned}$$

$$= \quad O(I_2^1) + O(I_2^2)$$

Hence we get

$$I_2^1 = \frac{1}{(\gamma-\beta)} \int_{\pi/(\gamma-\beta)}^{\pi} t^{\alpha-2} dt = \begin{cases} O\left((\gamma-\beta)^{-\alpha}\right); & 0 < \alpha < 1. \\ O\left(\frac{log(\gamma-\beta)}{(\gamma-\beta)}\right); & \alpha = 1 \end{cases}$$

and

$$I_2^2 = \int_{\pi/(\gamma-\beta)}^{\pi} t^{\alpha-1} dt = O((\gamma-\beta)^{-\alpha}).$$

Taking into account $1/(b_n - a_n) \leq \left(\frac{log(b_n-a_n)}{(b_n-a_n)}\right)$ for sufficiently large values of n and combining the last results, we obtain

$$|I_2| = \begin{cases} O\left((\gamma-\beta)^{-\alpha}\right); & 0 < \alpha < 1. \\ O\left(\frac{log(\gamma-\beta)}{(\gamma-\beta)}\right); & \alpha = 1 \end{cases} \tag{5.2}$$

According to (5.1)and (5.2), we have

$$\left|t_n^{D_\beta^\gamma.N_{pq}} - f\right| = |I_1 + I_2| = \begin{cases} O\left((\gamma-\beta)^{-\alpha}\right); & 0 < \alpha < 1. \\ O\left(\frac{log(\gamma-\beta)}{(\gamma-\beta)}\right); & \alpha = 1 \end{cases}$$

Therefore,

$$\|t_n^{D_\beta^\gamma.N_{pq}} - f(x)\|_\infty = \sup_{0\leq x\leq 2\pi} \left|t_n^{D_\beta^\gamma.N_{pq}} - f(x)\right| = \begin{cases} O\left((\gamma-\beta)^{-\alpha}\right); & 0 < \alpha < 1. \\ O\left(\frac{log(\gamma-\beta)}{(\gamma-\beta)}\right); & \alpha = 1 \end{cases}$$

Next theorem is related to approximation of conjugate of functions belonging to Lipschitz class by the generalized Deferred-Nörlund mean of conjugate series of Fourier series.

Theorem 5.2. *Let $\{p_n\}$ and $\{q_n\}$ be a non-negative sequence and $\{\alpha\} \in \pounds_0^+$ with $n\alpha = O(1)$. If $f \in Lip\alpha$ with $0 < \alpha \leq 1$.Then the degree of approximation of the conjugate function $\tilde{f}$ by the$\left(D_\beta^\gamma.N_{pq}\right)$ means of the conjugate series of Fourier series is given by*

$$\|\tilde{t}_n^{D_\beta^\gamma.N_{pq}} - \tilde{f}(x)\|_\infty = \begin{cases} O\left((\gamma-\beta)^{-\alpha}\right); & 0 < \alpha < 1. \\ O\left(\frac{log(\gamma-\beta)}{(\gamma-\beta)}\right); & \alpha = 1 \end{cases}$$

Proof:- We know that

$$\tilde{S}_n(f;x) - \tilde{f} = \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\cos(n+1/2)t}{\sin(t/2)} dt$$

where $\phi_x(t) = f(x+t) - f(x-t)$.

Therefore the proof is done similar to the theorem (5.1) taking into account Lemma (4.2)-(ii) and Lemma (4.3)-(ii).

Remark 5.1. If we consider $\{q_n\} = 1$ in these results then, the results of DEG̃ER and BAYINDIR [23] become the particular cases of our results.

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Dr. Jitendra Kumar Kushwaha received Ph.D. from Banaras Hindu University. Varanasi, India. He is currently working as an Assistant Professor at Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur.

Department of Mathematics and Statistics, Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, India.
e-mail: k.jitendrakumar@yahoo.com

Laxmi Rathour born in Anuppur, India. She received her Master's degree from Department of Mathematics, Indira Gandhi National Tribal University, Amarkantak, M.P., India. She has research interests in the areas of pure and applied mathematics specially Approximation theory, Nonlinear analysis and optimization, Fixed Point Theory and applications, Operation Research etc. In the meantime, she has published several scientific and professional papers in the country and abroad. Moreover, she serves voluntary as reviewer for Mathematical Reviews (USA) and Zentralblatt Math (Germany).

Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl-796012, Mizoram, India.
https://orcid.org/0000-0002-2659-7568
e-mail: laxmirathour817@gmail.com, rathourlaxmi562@gmail.com

Dr. Lakshmi Narayan Mishra is working as Assistant Professor in the Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India. He completed his Ph.D. programme from National Institute of Technology, Silchar, Assam, India. His research interests are in the areas of pure and applied mathematics including Special Functions, Non-linear analysis and Optimization, Fractional Integral and differential equations, Measure of non-compactness, Local and Global attractivity, Approximation Theory, Fourier approximation, Fixed Point Theory and applications in dynamic programming, q-series and q-polynomials, signal analysis and Image processing etc. He has published more than 125 research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of several international journals in frame of pure and applied Mathematics and applied economics. He has presented research papers and delivered invited talks at several international and National conferences, STTPs, Workshops and Refresher programmes in Universities in India. Citations of his research contributions can be found in many books and monographs, Ph.D. thesis, and scientific journal articles, much too numerous to be recorded here. Moreover, he serves voluntary as reviewer for Mathematical Reviews (USA) and Zentralblatt Math (Germany).

Department of Mathematics, Vellore Institute of Technology, Vellore-632014,Tamil Nadu, India.
e-mail: lakshminarayanmishra04@gmail.com

Krishna Kumar received M.Sc. from Shibli National PG College Azamgarh affiliated by VBS Purvanchal University Jaunpur UP and pursuing Ph.D. at DDU Gorakhpur University Gorakhpur UP of India. His research interests are approximation of functions by summability techniques and its applications include degree of approximations.

Department of Mathematics and Statistics, Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, India.
e-mail: kkmaths1986@gmail.com