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REGULARITY IN RIGHT DUO SEMINEARRINGS

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Abstract. The reason behind to investigate axiom systems with fewer axioms into investigate what types of results still hold, and what results become more general. Seminearrings obtained by the generalisation of nearrings and semirings. Clearly, seminearrings are common abstraction of semirings and nearrings. The aim of this work is to carry out an extensive study on algebraic structure of seminearrings and the major objective is to further enhance the theory of seminearrings in order to study the special structures of seminearrings, this work addresses some special structures of seminearrings such as right duo seminearrings. The right ideal of a seminearring need not be a left ideal. We focused on those seminear-rings which demonstrate this property. A seminearring S is right duo if every right ideal is two sided. Here we have concentrated on the seminearring which are right duo and regular. Main aim of this paper is to deal with properties of regularity in right duo seminearring. We have given some results on right duo seminearring. Followed by that, we have derived some theorems on the relation between the properties of seminearring such as regularity, semi simplicity and intra-regularity in right duo seminearring. We also illustrate this concept with suitable examples.

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1. Introduction

Regularity plays important role in seminearrings. For the non empty set S, semigroups $(S,+)$ and $(S,\cdot)$ which are connected by right distributive law forms a triple $(S,+,\cdot)$ known as seminearring. The left duo seminearring is characterised and its properties were studied in [10]. The motivation of this paper is to understand regularity in right duo seminearring. This paper specifically investigates relations among the regularity, semi simplicity, quasi regularity and intra-regularity in right duo seminearring. We obtain various characterizations of regularity in right duo seminearring which are not included in [7]. We prove

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that in the class of right duo seminearring, the notion of regularity coincides with the notions of quasi regularity and of being a union of groups. Further in the case of duo seminearring, we show that regularity is equivalent to semi simplicity and intra-regularity also, thus extending the results obtained by Szasz [5, 7].

2. Preliminaries

This section collects all of the seminearring theory terminologies that we use in our work. A subset I (non empty) of a seminearring $(S, +, .)$ is said to be a left (respectively right) ideal of S if (i) $p + q \in I$ for all $p, q \in I$, (ii) $a \cdot p \in I$ $(p \cdot a \in I)$ for all $p \in I$ and $a \in R$. I is said to be an ideal of S when I is both the left and right ideal of S [1, 2]. Consider a from S is said to distributive if for every $p, q \in S$, $a(p + q) = ap + aq$. If all the elements of S is distributive then S is said to be distributive. S is distributively generated, if it contains D which is multiplicative subsemigroup of distributive elements that generating $(S, +)$ such that $Sa = \{sa : s \in S\}$. Clearly we know that $aS(Sa)$ is a right (left) ideal of S [3]. If a generates the ideal $aS(Sa)$ then the ideal $aS(Sa)$ is called right(left) principal ideal. For a subset X of S, $< X >$ is denote the ideal generated by X [4]. In a seminearring S consider the subset X and S^1 is another seminearring, we observe that (i) $S^1X = X \cup SX$ $(XS^1 = X \cup XS)$ is the left(right) ideal generated by X (ii) $S^1XS^1 = X \cup SX \cup XS \cup SXS$ is the ideal generated by X. An ideal P of S is called as prime ideal, if ideals X, Y of S, satisfies that $XY \subseteq P \Rightarrow$ either $X \subseteq P$ *or* $Y \subseteq P$. An ideal P of S is said to be a quasi prime ideal, if for any ideal X of $S, X^2 \subseteq P \Rightarrow X \subseteq P$. An ideal P of S is irreducible if for ideals X, Y of $S, P = X \cap Y \Rightarrow$ either $P = X$ *or* $P = Y$. S is regular if for all a in S, there is some b in S such that a equals aba [6]. If $pxq = pq$, $\forall p, q \in S$ then the element a is said to be mid-unit. A right (left) regular seminearring S is satisfies the condition $a \in a^2S(Sa^2)$ for each $a \in S$. A seminearring S is called as intra-regular if $a \in Sa^2S$ for $a \in S$. A seminearring S is said to be quasi regular if $a \in aSaS \cap SaSa$. Suppose that $a \in S$ and $a^2 = a$ then it is called an idempotent. Suppose each $a \in S$ is idempotent then seminearring S is called as band. A seminearring S is semilattice if S is a commutative band. Suppose every left (right) ideal of S is two sided then the seminearring S is said to be left (right) duo and S is aid to be duo if every one sided ideal of S is two sided [8]. If S contains no proper ideals then the seminearring S is called simple. An ideal I of seminearring S is said to be a irreducible ideal if $A \cap B = P$ that implies either $A = P$ or $B = P$. An ideal I of a seminearring S is completely irreducible ideal if for ideal X_α of S $P = \cap A_\alpha$ that implies $P = A_\alpha$. $\sqrt{A} = \{x \in S/x^n \in A$, for some integer $n \geq 1$ depending on x$\}$. A seminearring S is semi-simple if every ideal of S is idempotent. In this entire paper S denotes the right seminearring also with an absorbing zero.

Example 2.1. Let $S = \{0, m, n, p, q\}$. In S the semigroup operations '+' and '$\cdot$' defined as follows,

+	0	m	n	p	q
0	0	m	n	p	q
m	m	m	m	m	m
n	n	m	n	n	n
p	p	m	n	p	p
q	q	m	n	p	q

TABLE 1. The semigroup operation '+' in S.

$\cdot$	0	m	n	p	q
0	0	0	0	0	0
m	0	m	m	m	m
n	0	m	n	n	n
p	0	m	n	p	p
q	0	m	n	p	q

TABLE 2. The semigroup operation '$\cdot$' in S.

then $(S,+,\cdot)$ is a seminearring where $m\cdot x = x\cdot m = m$ for $x = m,n,p,q$ and $n\cdot p = p\cdot n = n$. The seminearring $(S,+,\cdot)$ has no identity and is a regular right duo due to its commutative. So $n^2 = n$. Hence n is regular, therefore S is regular.

Example 2.2. For the seminearring $S_1 = \{0, m_1, n_1, p_1, q_1\}$. In S the semigroup operations " + " and " $\cdot$ " defined as follows,

$+$	0	m_1	n_1	p_1	q_1
0	0	m_1	n_1	p_1	q_1
m_1	m_1	m_1	n_1	q_1	q_1
n_1	n_1	n_1	n_1	q_1	q_1
p_1	p_1	q_1	q_1	p_1	q_1
q_1	q_1	q_1	q_1	q_1	q_1

TABLE 3. The semigroup operation '+' in S_1.

$\cdot$	0	m_1	n_1	p_1	q_1
0	0	0	0	0	0
m_1	0	m_1	m_1	m_1	m_1
n_1	0	m_1	n_1	n_1	n_1
p_1	0	m_1	n_1	n_1	n_1
q_1	0	m_1	q_1	q_1	q_1

TABLE 4. The semigroup operation '$\cdot$' in S_1.

In S the elements n, p, q are mid-units. For $x = 0, m_1, n_1, p_1, q_1$ then $(q_1p_1)n_1 = n_1q_1p_1 = n_1q_1{p_1}^3q_1 = {p_1}^3(n_1q_1)q_1 = {p_1}^3q_1n_1q_1 = p_1n_1q_1 = n_1(p_1q_1)$. Therefore ${p_1}^2 = p_1$. Hence p_1 is regular. So S is regular.

3. Main results

Proposition 3.1. *S is a seminearring with right duo iff $Sa \subseteq aS^1$ for each $a \in S$.*

Proof. Let us consider seminearring S which is right duo, by the definition $as \in S \implies sa \in S$, $\forall s \in S$. We have to show that $Sa \subseteq aS^1$, we know that $Sa = \{sa : s \in S\}$ and $aS^1 = \{a \cup as : s \in S\} \implies Sa \subseteq aS^1$. Since S is right duo. Conversely assume that $Sa \subseteq aS^1$ and have to show that S is right duo. By the definition $Sa = \{sa : s \in S\}$ and $aS^1 = \{a \cup as : s \in S\} \implies sa = as$ (since $Sa \subseteq aS^1$). Therefore S is right duo. □

Proposition 3.2. *S is a seminearring which is right duo iff $(a) = (a)_R$ for each a in S.*

Let e be an element in duo seminearring S which is idempotent, then we have $Se = eS$.

Proposition 3.3. *In a regular seminearring S, $P \cap Q = PQ$ for every P (right ideal) and every Q (left ideal) of S.*

Proof. Let S be a regular seminearring, let $a \in P \cap Q \implies$ there is $x \in S$ such that $axa = a$. Since L is a left ideal, $xa \in P$. Therefore $a = a(xa) \in PQ$. This shows $PQ \supset P \cap Q$ and already it is clear that $PQ \subset P \cap Q$. Hence $PQ = P \cap Q$. For the converse part, let a in S, then a generates the right ideal $\{ax/x \in S\} \cup a$. By the hypothesis, $(a) = (a) \cap S = (a)S = aS$. $\implies$ we have $a \in aS$. Similarly $a \in Sa$, Hence $a \in aS \cap Sa = aS^2a$, and there is an element x such that $a = axa$. Now, suppose that a given regular seminearring S is commutative, any ideal P in $Q \implies$ is idempotent, i.e. $P^2 = P$. Conversely, suppose that every ideal in a commutative seminearring Q is idempotent. If P and Q are ideals in S, then we have

$$P \cap Q = (P \cap Q)^2 = (P \cap Q)(P \cap Q) \subset PQ$$

Alternatively,

$$P \cap Q \supset PQ$$

Hence

$$P \cap Q = PQ$$

□

Result 1. [9](Theorem 4.3) The conditions as follows for the seminearring S is

(i) a union of rings
(ii) both left and right regular
(iii) both left regular and regular
(iv) both right regular and regular

are equivalent.

Result 2. A seminearring S is a semilattice of rings if and only if it is a regular duo seminearring.

Proof. Assume S is semilattice of rings. S is commutative band. Implies that $ab = ba, \forall a, b \in S$ and $a^2 = a, \forall a \in S$.

$$ab = a^2b = a(ab) = a(ba) = aba$$

Since S is commutative which implies every one sided ideal is two sided. Therefore S is regular duo. Conversely assume S is regular duo that implies $\forall a \in S$, $\exists\ b \in S \ni a = aba$ and every one sided ideal is two sided which implies that $ab = ba, \forall a, b \in S$ and also $a^2 = a$. Hence S is a semilattice. □

Lemma 3.4. *Any intersection of prime ideals of S is quasi prime ideal of S.*

Proof. Let $\{P_i\}_{i=1}^{k}$ be a prime ideals. Let us consider $\cap\{P_i\}_{i=1}^{k} = P$. Now we have to show that an ideal P is a quasi prime. Suppose ideal P is not a quasi prime then there exist an ideal X satisfies $X^2 \subseteq P$ but $X \nsubseteq P$. Implies that $\exists\ x \in X$, $x \notin \{P_i\}_{i=1}^{k}$ and $x^2 \notin \{P_i\}_{i=1}^{k} \implies X^2 \nsubseteq P$. Which is a contradiction. Therefore, the ideal P is a quasi prime. □

Lemma 3.5. *Every ideal X of S is the intersection of completely irreducible ideals of S that contain X.*

Proof. We split this proof as two different cases
Case 1: If $X = S$ then the result is obvious.
Case 2: Suppose X is proper ideal then $\exists$ an element $a \in X$. Take union of all ideals of S not containing a is an ideal containing X and not containing a and let it be Y. Now claim that Y is completely irreducible. Suppose $Y = \cap M_\alpha$, where M_α are ideals of S then there is atleast one $\alpha \ni a \notin M_\alpha$. Hence $M_\alpha \subseteq Y$. But $Y \subseteq M_\alpha$, so that $Y = M_\alpha$. Obviously $X \subseteq \cap T_i$ where T_i is a completely irreducible ideal of containing X. If $X \subset \cap T_i$, that implies we can find a completely irreducible ideal containing $\cap T_i$, which is a contradiction. Hence the result. □

Result 3. Every completely irreducible ideal of S is irreducible ideal of S.

Corollary 3.6. *Every ideal X of S is the intersection of irreducible ideals of S that contain X.*

Proof. Proof is obviously by lemma 3.5. □

Corollary 3.7. *A right duo seminearring S is a quasi prime iff $A = \sqrt{A}$, $\forall$ ideal A of S.*

Proposition 3.8. *S is regular iff $a \in (Sa)^n$ for every $a \in S$ and any integer $n \geq 2$.*

Proof. For S (seminearring) which is regular $\implies \forall\ a$ in S, there is $a \in aSa \subseteq Sa(aSa) \subseteq SaSa = (Sa)^2$, so that the result $a \in (Sa)^n$ holds for $n = 2$. Let the result hold for $n = m$, then

$$a \in (Sa)^m = (Sa)(Sa)...(Sa)\ (m\ times) \subseteq (Sa)(Sa)...(SaSa)$$

as

$$a \in (aSa)$$

(i.e) $a \in (Sa)^{m+1}$, proving the required condition. Conversely, let the condition $a \in (Sa)^n$ be true for any integer $n \geq 2$, then taking $n = 2$, we get $a \in (Sa)^2 = SaSa \subseteq aSa$ (by Proposition 3.1). Hence a is regular. This completes the proof. □

Proposition 3.9. *The following conditions are equivalent for the seminearring S is*

(i) *a regular.*
(ii) *a quasi-regular.*
(iii) *a union of rings.*

Proof. $(i) \Rightarrow (ii)$.

For $a \in S, a = axa$ (by regularity) $= axaxa \in aSaS \cap SaSa$. So a is a quasi-regular.

$(ii) \Rightarrow (i)$

For $a \in S$, $a = axay$ (by quasi regularity) $= ax(za)$, where $z \in S^1$ (by Proposition 3.7). Thus a is a regular.

$(ii) \Rightarrow (iii)$

Since, $(ii) \Rightarrow (i)$, S is a right regular ($\because S$ is a regular) [1]

Hence, S is a union of rings.

$(iii) \Rightarrow (i)$ is obvious. □

Proposition 3.10. *The following conditions are equivalent on seminearring S is*

(i) *a intra-regular.*
(ii) *a semi-simple.*

Proof. Since any intra-regular seminearring is semisimple, it is enough to prove that $(ii) \Rightarrow (i)$, Let S be semisimple, then for any $a \in S$

$$(a) = (a)^2 = (a)_R(a) = S^1S^1aS^1a \text{ (by Proposition 3.2).}$$

Hence

$$a = xayaz, where\ for\ some\ x, y, z \in S^1 = x(ua)az$$

where

$$u \in S^1$$

Thus, $a \in Sa^2S$, proving that a is intra-regular. As applications of result 1, to right duo seminearring, we obtained the following. □

Proposition 3.11. *Any ideal A of a seminearring S (regular right duo) is a regular right duo subseminearring of S. Further A is a union of rings.*

Proof. Obviously A is regular. Let $a, b \in A$. then

$$ab = (axa)b$$

(by regularity of S).

$$xa(ay)$$

where

$$y \in S^1$$

(by Proposition 3.2).

$$ap$$

where

$$p = xya \in SA \subseteq A$$

Thus A is right duo (by Proposition 3.1). Since S is a regular, it is a union of rings (by Proposition 3.8) and so it is right regular (by result 1). As $a \in A$, then

$$a = a^2x$$

(by right regularity of s)

$$a(ax)$$

$$a^2x(ax) \in a^2(SAS) \subseteq a^2A$$

Hence A is right regular and so A is a union of rings (By result 1). □

Proposition 3.12. *The centre C of a seminearring S (regular right duo) is a semilattice of rings.*

Proof. Clearly C is a commutative subseminearring of S. If $c \in C$, then

$$c = xc^2$$

(since S is left regular by [1])

$$= (xc)c = (xc)(xc^2)$$

$$c^2(cx^2) = c^2(cd)$$

where

$$d = x^2$$

If $b \in S$, then $(dc)b = bdc = bdc^3d = c^3(bd)d = c^3dbd = cbd = b(cd)$, Showing that $cd \in C$. Hence $c = c^2(cd) \in c^2C = cCc$. So that C is regular and C is a semilattice of rings now follows by Result 2. We now give characterizations of regularity in the case of duo seminearring. □

Proposition 3.13. *Let S be a duo seminearring.The following conditions are equivalent*

(i) *S is regular.*
(ii) *S is quasi regular.*
(iii) *S is semilattice.*

(iv) *S is intra-regular.*
(v) *S is semi-simple.*

Proof. $(i) \Leftrightarrow (ii)$ follows by Proposition 3.7.
$(i) \Leftrightarrow (iii)$ follows by Result 2.
$(iv) \Leftrightarrow (v)$ follows by Proposition 3.9
$(i) \Leftrightarrow (v)$ is obvious.
$(v) \Leftrightarrow (i)$.
Let $a \in S$. Then as in the proof of Proposition 3.9. We can show that $a = xayaz$, where $x, y, z \in S^1 \implies (au)y(va)$, for some $u, v \in S^1$ (by Proposition 3.1) Thus a is regular, completing the proof. □

Theorem 3.14. *The conditions on a duo seminearring S is*

(i) *a regular.*
(ii) *a quasi prime in S.*
(iii) *all ideal of an ideal in S is quasi prime.*
(iv) *all irreducible ideal in S is prime.*

are equivalent.

Proof. $(i) \Rightarrow (ii)$.
Let A be an ideal in S. In view of Corollary 3.6, we need only to prove that $A = \sqrt{A}$, and since $A \subseteq \sqrt{A}$, it is enough to prove the other inclusion. Let $x \in \sqrt{A}$, so that $x^n \in A$ for some integers $n \geq 1$. Since S is a regular,
$x = xyx$, where $y \in S$
$= (xy)^n x$
$= zx^n yx$, where $z \in S^1$ (S being right duo)
Thus, $x \in A$ and so since $\sqrt{A} \subseteq A$. This proves (ii).
$(ii) \Rightarrow (iii)$
If $x \in S$, then $A = xS^1x$ is an ideal in S (since S is duo) and by hypothesis, A is a quasi-prime. Hence $A = \sqrt{A}$ (by Corollary 3.6). Since $x^2 \in A$, we have $x \in \sqrt{A} = xS^1x$, so that x is regular.
$(i) \Rightarrow (iii)$
Let A be an ideal in S and I is an ideal in A, for any $x \in I$, we have $x = xyx$ (by regularity of S)

$$xs = xyxs \in ISIS \subseteq ISAS \subseteq IA \subseteq I,$$

Hence every $s \in S$ and so $IS \subseteq I$. Thus I is a right ideal and an ideal in S. Since $(i) \Rightarrow (ii)$, so that I is quasi prime of S.
$(iii) \Rightarrow (i)$
Let I be the ideal in S. then an element $x \in I$, $A = \{x^2\} \cup Ix^2 \cup x^2I \cup Ix^2I$ is an ideal of I and so by hypothesis, is a quasi prime of S. Hence $A = \sqrt{A}$ (by Corollary 3.6). But $x^2 \in A$, so that $x \in \sqrt{A} = A$.
It follows that

(i) $x = x^2$

(ii) $x \in Ix^2$
(iii) $x \in x^2 I$
(iv) $x \in Ix^2 I$

Thus either x is idempotent, left regular, right regular or intra-regular. In any case, in view of ([1]. Proposition 1.4), it is dual and Proposition 3.12, it follows that x is regular.

$(i) \Rightarrow (iv)$

Let Q be an irreducible ideal of S. Suppose Q is not prime and also not completely prime, then $\exists\ a, b \in S \ni a, b \notin Q$, but $ab \in Q$. Therefore

$$Q \subseteq (Q \cup aS^1) \cap (Q \cup bS^1) = (Q \cup S^1 a)(Q \cup S^1 b) \subseteq Q$$

(by Proposition 3.3).

$$Q = (Q \cup aS^1) \cap (Q \cup bS^1)$$

which is a contradiction. Hence $(i) = (iv)$.

$(iv) = (ii)$.

By our hypothesis and Corollary 3.5, (every ideal A in S is the intersection of every prime ideals containing it) $\implies$ A is quasi prime (by Lemma 3.4).

$\square$

Conclusion

In this paper we discussed regularity properties in right duo seminearing. We concentrated on seminear-rings that exhibit the property of right duo. We inferred the relationship between the regularity, quasi regularity and intra-regularity in right duo seminearring. Coincidence between the notion of regularity and notions of quasi regularity in the class of right duo seminearrings are interpreted. Also we have concluded some results based on the right duo seminearring and regular seminearring. Moreover we showed that regularity is equivalent to semisimplicity.

Conflicts of interest : The authors declare no conflict of interest.

References

1. J. Ahsan, *Seminear-rings characterized by their $\mathcal{S}$-ideals,I*, Proceedings of the Japan Academy, Series A, Mathematical Sciences **71** (1995), 101-103. doi: 10.3792/ pjaa. 71.101
2. J. Ahsan, *Seminear-rings characterized by their $\mathcal{S}$-ideals,I*, Proceedings of the Japan Academy, Series A, Mathematical Sciences **71** (1995), 111-113. doi: 10.3792/ pjaa. 71.111
3. M. Shabir and I. Ahmed, *Weakly regular seminearrings*, International Electronic Journal of Algebra **2** (2007), 114-126. http://www. ieja.net/files/papers/volume-2/ Volume-1–2007/10-V2-2007.pdf
4. S. Pianskool and P. Khachorncharoenkul, *Simple* Γ-*Seminearrings*, Journal of Mathematics research **1** (2009), 124. doi: 10.5539/jmr.v1n1p0.
5. G. Szasz and S. Lajos, *Generalized regularity in semigroups*, Publicationes Mathematicae Debrecen, 1975.

6. J. Von Neumann, *On regular rings*, Proceedings of the National Academy of Sciences **22** (1936), 707-713. doi: https://doi.org/10.1073/pnas.22.12.707.
7. S. Lajos and G. Szasz, *On characterizations of certain classes of semigroups*, Publicationes Mathematicae Debrecen **25** (1978), 225-227.
8. R. Balakrishnan and R. Perumal, *Left duo seminear-rings*, Scientia Magna **8** (2012), 115-120. doi: 10.1.1.455.6099
9. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, American Mathematical Society **1** (1967).
10. G. Manikandan and R. Perumal, *A note on left duo seminearrings*, Palestine Journal of Mathematics **11** (2022), 24-28. https://pjm.ppu.edu/paper/ 1002-note-left-duo-seminearrings.

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