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BI-INTERIOR SYSTEMS AND VARIOUS COMPLETENESS†

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ABSTRACT. We investigate the relationships between right (resp. left) interior systems and right (resp. left) interior operators on complete generalized residuated lattices. We show that the set induced by a right (resp. left) interior operator is right (resp. left) join complete.

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1. Introduction

Ko and Kim [4] introduced the notions of right (resp. left) interior systems in a sense as the right (resp. left) lower bound on generalized residuated lattices. In this paper, we examine the relationships between right (resp. left) interior systems and right (resp. left) interior operators (see Section 3). Finally, we show that the set $M = \{A \in L^X \mid I(A) = A\}$, where I is a right (resp. left) interior operator, is a right (resp. left) join complete (see Section 4).

2. Preliminaries

In this section, we present some preliminary concepts and properties.

Definition 2.1. [2, 5, 6, 7, 8] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following three conditions:
(GR1) $(L, \vee, \wedge, \top, \bot)$ is bounded where $\top$ is the upper bound and $\bot$ is the universal lower bound,
(GR2) $(L, \odot, \top)$ is a monoid where $\top$ is the identity,
(GR3) it satisfies a residuation; *i.e.*, $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \Rightarrow c$.

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In this paper, we always assume that $(L,\wedge,\vee,\odot,\to,\Rightarrow,\top,\bot)$ is a complete generalized residuated lattice.

Lemma 2.2. [1, 3, 7] *Let $x,y,z\in L$. Let $\{x_i\}_{i\in\Gamma},\{y_i\}_{i\in\Gamma}\subseteq L$. Then the following hold.*
(1) *If $y\le z$, then $x\odot y\le x\odot z$, $x\to y\le x\to z$, $z\to x\le y\to x$, $x\Rightarrow y\le x\Rightarrow z$ and $z\Rightarrow x\le y\Rightarrow x$.*
(2)

$$\begin{array}{ll}
x\to\left(\bigwedge_{i\in\Gamma}y_i\right) & =\bigwedge_{i\in\Gamma}\left(x\to y_i\right),\left(\bigvee_{i\in\Gamma}x_i\right)\to y=\bigwedge_{i\in\Gamma}(x_i\to y),\\
\left(\bigvee_{i\in\Gamma}x_i\right)\to\left(\bigvee_{i\in\Gamma}y_i\right) & \ge\bigwedge_{i\in\Gamma}\left(x_i\to y_i\right),\\
\left(\bigwedge_{i\in\Gamma}x_i\right)\to\left(\bigwedge_{i\in\Gamma}y_i\right) & \ge\bigwedge_{i\in\Gamma}\left(x_i\to y_i\right),\\
x\Rightarrow\left(\bigwedge_{i\in\Gamma}y_i\right) & =\bigwedge_{i\in\Gamma}\left(x\Rightarrow y_i\right),\left(\bigvee_{i\in\Gamma}x_i\right)\Rightarrow y=\bigwedge_{i\in\Gamma}(x_i\Rightarrow y),\\
\left(\bigvee_{i\in\Gamma}x_i\right)\Rightarrow\left(\bigvee_{i\in\Gamma}y_i\right) & \ge\bigwedge_{i\in\Gamma}\left(x_i\Rightarrow y_i\right),\\
\left(\bigwedge_{i\in\Gamma}x_i\right)\Rightarrow\left(\bigwedge_{i\in\Gamma}y_i\right) & \ge\bigwedge_{i\in\Gamma}\left(x_i\Rightarrow y_i\right).
\end{array}$$

(3) *$(x\odot y)\to z=x\to(y\to z)$ and $(x\odot y)\Rightarrow z=y\Rightarrow(x\Rightarrow z)$.*
(4) *$x\to(y\Rightarrow z)=y\Rightarrow(x\to z)$ and $x\Rightarrow(y\to z)=y\to(x\Rightarrow z)$.*
(5) *$x\odot(x\Rightarrow y)\le y$ and $(x\to y)\odot x\le y$. Moreover, $x\le(x\Rightarrow y)\to y$ and $x\le(x\to y)\Rightarrow y$.*
(6) *$(x\Rightarrow y)\odot z\le x\Rightarrow(y\odot z)$ and $y\odot(x\to z)\le x\to(y\odot z)$.*
(7) *$(x\Rightarrow y)\odot(y\Rightarrow z)\le x\Rightarrow z$ and $(y\to z)\odot(x\to y)\le x\to z$.*
(8) *$(x\Rightarrow z)\le(y\odot x)\Rightarrow(y\odot z)$ and $(x\to z)\le(x\odot y)\to(z\odot y)$.*
(9) *$x\to y\le(y\to z)\Rightarrow(x\to z)$ and $(x\Rightarrow y)\le(y\Rightarrow z)\to(x\Rightarrow z)$.*
(10) *$y\to z\le(x\to y)\to(x\to z)$ and $(y\Rightarrow z)\le(x\Rightarrow y)\Rightarrow(x\Rightarrow z)$.*
(11) *$x\to y=\top$ if and only if $x\le y$. Similarly, $x\Rightarrow y=\top$ if and only if $x\le y$.*

Definition 2.3. [4, 5] Let X be a set. A map $e^r_X: X\times X\to L$ is called an *r-partial order* (or *right-partial order*) if it satisfies the following three conditions:
(O1) $e^r_X(x,x)=\top$ for all $x\in X$,
(O2) If $e^r_X(x,y)=e^r_X(y,x)=\top$ where $x,y\in X$, then $x=y$,
(R) $e^r_X(x,y)\odot e^r_X(y,z)\le e^r_X(x,z)$ for all $x,y,z\in X$.

A map $e^l_X: X\times X\to L$ is called an *l-partial order* (or *left partial order*) if it satisfies the following three conditions:
(O1) $e^l_X(x,x)=\top$ for all $x\in X$,
(O2) If $e^l_X(x,y)=e^l_X(y,x)=\top$ where $x,y\in X$, then $x=y$,
(L) $e^l_X(y,z)\odot e^l_X(x,y)\le e^l_X(x,z)$ for all $x,y,z\in X$.

The pair (X,e^r_X) is called an *r-partially ordered set* (or *right partially ordered set*).

The pair $\left(X,e^l_X\right)$ is called an *l-partially ordered set* (or *left partially ordered set*).

The triple $\left(X,e^r_X,e^l_X\right)$ is called a *bi-partially ordered set*.

Using Lemma 2.2(7), one can have the following.

Lemma 2.4. [4, 5] *Let $G \subseteq L^X$. Define $e^r_G : G\times G \to L$ and $e^l_G : G\times G \to L$ by $e^r_G(A,B) = \bigwedge_{x\in X}[A(x) \Rightarrow B(x)]$ and $e^l_G(A,B) = \bigwedge_{x\in X}[A(x) \to B(x)]$. Then $\left(G, e^r_G, e^l_G\right)$ is a bi-partially ordered set.*

3. Bi-interior operators and bi-interior systems

In this section, we investigate the relationship between right (resp. left) interior systems and right (resp. left) interior operators.

Definition 3.1. A map $I^r : L^X \to L^X$ is called an *r-interior operator* (or *right interior operator*) on X if it satisfies the following three conditions:
(I1) $I^r(A) \le A$ for all $A \in L^X$,
(I2) $I^r(A) \le I^r(I^r(A))$ for all $A \in L^X$,
(IR) $e^r_{L^X}(A,B) \le e^r_{L^X}(I^r(A), I^r(B))$ for all $A, B \in L^X$.

A map $I^l : L^X \to L^X$ is called an *l-interior operator* (or *left interior operator*) on X if it satisfies the following three conditions:
(I1) $I^l(A) \le A$ for all $A \in L^X$,
(I2) $I^l(A) \le I^l\left(I^l(A)\right)$ for all $A \in L^X$,
(IL) $e^l_{L^X}(A,B) \le e^l_{L^X}\left(I^l(A), I^l(B)\right)$ for all $A, B \in L^X$.

The triple $\left(X, I^r, I^l\right)$ is called a *bi-interior space*.

Definition 3.2. Let $H^r, H^l \subseteq L^X$.
(1) A family H^r is called an *r-interior system* (or *right interior system*) on X if (a) $A \odot k \in H^r$ for all $A \in H^r$ and $k \in L$, and (b) $\bigvee_{i\in\Gamma} A_i \in H^r$ for all $\{A_i\}_{i\in\Gamma} \subseteq H^r$.
(2) A family H^l is called an *l-interior system* (or *left interior system*) on X if (a) $k\odot A \in G^l$ for all $k \in L$ and $A \in H^l$, and (b) $\bigvee_{i\in\Gamma} A_i \in H^l$ for all $\{A_i\}_{i\in\Gamma} \subseteq H^l$.

The triple $\left(X, H^r, H^l\right)$ is called a *bi-interior system*.

Remark 3.1. (1) Definitions 3.1-3.2 are consistent with those defined by Ko and Kim [4, 5].

Let $k \in L$. Let $A \in L^X$. Define four maps $A\odot k$, $k\odot A$, $k \to A$, $k \Rightarrow A : X \to L$ by $(A \odot k)(x) = A(x) \odot k$, $(k \odot A)(x) = k \odot A(x)$, $(k \to A)(x) = k \to A(x)$, $(k \Rightarrow A)(x) = k \Rightarrow A(x)$.

Lemma 3.3. *Let $k \in L$ and let $A, B \in L^X$. Then the following hold.*
(1) $k \le e^r_{L^X}(A, A\odot k)$ and $k \le e^l_{L^X}(A, k\odot A)$.
(2) Let $I^r : L^X \to L^X$ be an r-interior operator on X. If $A \le B$, then $I^r(A) \le I^r(B)$.
(3) Let $I^l : L^X \to L^X$ be an l-interior operator on X. If $A \le B$, then $I^l(A) \le I^l(B)$.

Proof. (1) Note that

$$e^r_{L^X}(A, A\odot k) = \bigwedge_{x\in X}[A(x) \Rightarrow (A(x)\odot k)] \ge k$$

and

$$e^l_{L^X}(A, k\odot A) = \bigwedge_{x\in X}[A(x) \to (k\odot A(x))] \ge k.$$

(2) Let $A \leq B$ where $A, B \in L^X$. By (IR),

$$\top = e^r_{L^X}(A, B) \leq e^r_{L^X}(I^r(A), I^r(B)) = \bigwedge_{x\in X} [I^r(A)(x) \Rightarrow I^r(B)(x)].$$

By Lemma 2.2(11), $I^r(A) \leq I^r(B)$.
(3) It can be similarly proved as in (2).

□

Theorem 3.4. (1) *Let $I^r : L^X \to L^X$ be an r-interior operator on X. Then the set $H^r_{I^r} = \{A \in L^X \mid I^r(A) = A\}$ is an r-interior system on X.*
(2) *Let $I^l : L^X \to L^X$ be an l-interior operator on X. Then the set $H^l_{I^l} = \{A \in L^X \mid I^l(A) = A\}$ is an l-interior system on X.*

Proof. (1) Let $A \in H^r_{I^r}$ and $k \in L$. Then

$$\begin{aligned} k &\leq e^r_{L^X}(A, A \odot k) \quad \text{(by Lemma 3.3(1))} \\ &\leq e^r_{L^X}(I^r(A), I^r(A \odot k)) \quad (\because I^r \text{ is an } r\text{-interior operator}) \\ &= e^r_{L^X}(A, I^r(A \odot k)) \quad (\because A \in H^r_{I^r}). \end{aligned}$$

By residuation, $A \odot k \leq I^r(A \odot k)$. On the other hand, $I^r(A \odot k) \leq A \odot k$ by (I1). Hence $A \odot k = I^r(A \odot k)$, and so $A \odot k \in H^r_{I^r}$.

Let $\{A_i\}_{i\in\Gamma} \subseteq H^r_{I^r}$. Then

$$\begin{aligned} I^r\left(\bigvee_{i\in\Gamma} A_i\right) &\geq \bigvee_{i\in\Gamma} I^r(A_i) \quad \text{(by Lemma 3.3(2))} \\ &= \bigvee_{i\in\Gamma} A_i \quad (\because A_i \in H^r_{I^r}). \end{aligned}$$

On the other hand, $I^r\left(\bigvee_{i\in\Gamma} A_i\right) \leq \bigvee_{i\in\Gamma} A_i$ by (I1). Hence $I^r\left(\bigvee_{i\in\Gamma} A_i\right) = \bigvee_{i\in\Gamma} A_i$ and $\bigvee_{i\in\Gamma} A_i \in H^r_{I^r}$.

Therefore $H^r_{I^r}$ is an r-interior system on X.
(2) It can be similarly proved as in (1). □

Lemma 3.5. *Let $A, B \in L^X$. Then the following hold.*
(1) $A \odot e^r_{L^X}(A, B) \leq B$.
(2) $e^l_{L^X}(A, B) \odot A \leq B$.

Proof. (1) By residuation, one see that

$$\begin{aligned} A \odot e^r_{L^X}(A, B) \leq B \;\; &\text{iff} \;\; A(x) \odot e^r_{L^X}(A, B) \leq B(x) \text{ for all } x \in X \\ &\text{iff} \;\; e^r_{L^X}(A, B) \leq A(x) \Rightarrow B(x) \text{ for all } x \in X. \end{aligned}$$

(2) It can be similarly done as in (1). □

Theorem 3.6. (1) *Let H^r be an r-interior system on X. Define $I^r_{H^r} : L^X \to L^X$ by $I^r_{H^r}(A) = \bigvee\{D \in H^r | D \leq A\}$. Then $I^r_{H^r}$ is an r-interior operator such that*

$$I^r_{H^r}(A) = \bigvee_{D\in H^r} [D \odot e^r_{L^X}(D, A)] \text{ for all } A \in L^X \text{ and } H^r_{I^r_{H^r}} = H^r.$$

(2) *Let H^l be an l-interior system on X. Define $I^l_{H^l} : L^X \to L^X$ by $I^l_{H^l}(A) = \bigvee\{D \in H^l | D \leq A\}$. Then $I^l_{H^l}$ is an l-interior operator such that*

$$I^l_{H^l}(A) = \bigvee_{D\in H^l} \left[e^l_{L^X}(D, A) \odot D\right] \text{ for all } A \in L^X \text{ and } H^l_{I^l_{H^l}} = H^l.$$

(3) *Let* $I^r : L^X \to L^X$ *be an* r*-interior operator on* X*. Then* $I^r_{H^r_{I^r}} = I^r$*.*
(4) *Let* $I^l : L^X \to L^X$ *be an* l*-interior operator on* X*. Then* $I^l_{H^l_{I^l}} = I^l$*.*

Proof. (1) *Claim 1:* $I^r_{H^r}(A) = \bigvee_{D\in H^r}\left[D \odot e^r_{L^X}(D, A)\right]$ *for all* $A \in L^X$*.*

Let $I(A) = \bigvee_{D\in H^r}\left[D \odot e^r_{L^X}(D, A)\right]$ where $A \in L^X$. Note that for all $D \in H^r$, we have $D \odot e^r_{L^X}(D, A) \in H^r$ and $D \odot e^r_{L^X}(D, A) \to A$ by Lemma 3.5(1). Thus $I(A) \le I^r_{H^r}(A)$. On the other hand,

$$\begin{aligned} I(A) &= \bigvee_{D\in H^r}\left[D \odot e^r_{L^X}(D, A)\right] \\ &\ge I^r_{H^r}(A) \odot e_{L^X}(I^r_{H^r}(A), A) \quad (\because I^r_{H^r}(A) \in H^r) \\ &= I^r_{H^r}(A) \odot \top = I^r_{H^r}(A). \end{aligned}$$

Hence Claim 1 is proved.

Claim 2: Let $A \in L^X$ *and* $D \in H^r$*. Then* $D \le A$ *if and only if* $D \le I^r_{H^r}(A)$*.*
$\Rightarrow$) Assume $D \le A$. Then $I^r_{H^r}(A) = \bigvee\{E \in H^r | E \le A\} \ge D$.
$\Leftarrow$) Assume $D \le I^r_{H^r}(A)$. Then $D \le I^r_{H^r}(A) = \bigvee\{E \in H^r | E \le A\} \le A$.
(C1) By definition, $I^r_{H^r}(A) \le A$ for all $A \in L^X$.
(C2) For all $A \in L^X$,

$$\begin{aligned} I^r_{H^r}(I^r_{H^r}(A)) &= \bigvee\{D \in H^r | D \le I^r_{H^r}(A)\} \\ &= \bigvee\{D \in H^r | D \le A\} \quad \text{(by Claim 2)} \\ &= I^r_{H^r}(A). \end{aligned}$$

Claim 3: For all $k \in L$ *and* $A \in L^X$*,* $I^r_{H^r}(A) \odot k \le I^r_{H^r}(A \odot k)$*.*

Note that

$$\begin{aligned} I^r_{H^r}(A) \odot k &= \bigvee\{D \in H^r | D \le A\} \odot k \\ &= \bigvee\{D \odot k | D \le A, D \in H^r\} \\ &\le \bigvee\{D \odot k | D \odot k \le A \odot k, D \odot k \in H^r\} \\ &\le I^r_{H^r}(A \odot k). \end{aligned}$$

Claim 4: If $A \le B$ *where* $A, B \in L^X$*, then* $I^r_{H^r}(A) \le I^r_{H^r}(B)$*.*

Assume $A \le B$. Since $\{D \in H^r | D \le A\} \subseteq \{D \in H^r | D \le B\}$, we have $I^r_{H^r}(A) \le I^r_{H^r}(B)$.
(CR) Let $A, B \in L^X$. Since

$$\begin{aligned} I^r_{H^r}(A) \odot e^r_{L^X}(A, B) &\ge I^r_{H^r}\left(A \odot e^r_{L^X}(A, B)\right) \quad \text{(by Claim 3)} \\ &\le I^r_{H^r}(B) \quad \text{(by Lemma 3.5(1) and Claim 4)}, \end{aligned}$$

we have by residuation that $e^r_{L^X}(A, B) \le e^r_{L^X}(I^r_{H^r}(A), I^r_{H^r}(B))$.

Therefore $I^r_{H^r}$ is an r-interior operator.

Let $A \in H^r_{I^r_{H^r}}$. Then $A = I^r_{H^r}(A) = \bigvee\{D \in H^r \mid D \le A\}$, and so $A \in H^r$. On the other hand, let $A \in H^r$. Then $I^r_{H^r}(A) = \bigvee\{D \in H^r \mid D \le A\} = A$, and so $A \in H^r_{I^r_{H^r}}$.
(3) *Claim 5: Let* $A \in L^X$ *and let* $D \in H^r_{I^r}$*. Then* $D \le A$ *if and only if* $D \le I^r(A)$*.*
$\Rightarrow$) Assume $D \le A$. By Lemma 3.3(2), $I^r(D) \le I^r(A)$. Since $D \in H^r_{I^r}$, we have $I^r(D) = D$. Hence $D \le I^r(A)$.

$\Leftarrow$) Assume $D \leq I^r(A)$. Since $I^r(A) \leq A$, we have $D \leq A$.

Let $A \in L^X$. Since $I^r_{H^r_{I^r}}(A) = \bigvee\{D \in H^r_{I^r} \mid D \leq A\}$, $I^r(A) \in H^r_{I^r}$ and $I^r(A) \leq A$, we have $I^r(A) \leq I^r_{H^r_{I^r}}(A)$. On the other hand, note that

$$\begin{aligned} I^r_{H^r_{I^r}}(A) &= \bigvee\{D \in H^r_{I^r} \mid D \leq A\} \\ &= \bigvee\{D \in H^r_{I^r} \mid D \leq I^r(A)\} \quad \text{(by Claim 5)} \\ &\geq I^r(A). \end{aligned}$$

(2) and (4) can be similarly proved as in (1) and (3), respectively. □

By Theorem 3.6, we have the following.

Corollary 3.7. *Let (X, e^r_X, e^l_X) be a bi-partially ordered set.*
(1) *There is a one to one correspondence between the set of all r-interior operators on X and the set of all r-interior systems on X.*
(2) *There is a one to one correspondence between the set of all l-interior operators on X and the set of all l-interior systems on X.*

Definition 3.8. Let (X, e^r_X, e^l_X) be a bi-partially ordered set. Define four maps $\Downarrow_r, \Downarrow_l, \Uparrow_r, \Uparrow_l \colon L^X \to L^X$ by

$$\Downarrow_r A(x) = \bigwedge_{y\in X}\left[e^r_X(y,x) \Rightarrow A(y)\right], \qquad \Downarrow_l A(x) = \bigwedge_{y\in X}\left[e^l_X(y,x) \to A(y)\right],$$
$$\Uparrow_r A(x) = \bigwedge_{y\in X}\left[e^r_X(x,y) \to A(y)\right], \qquad \Uparrow_l A(x) = \bigwedge_{y\in X}\left[e^l_X(x,y) \Rightarrow A(y)\right]$$

where $A \in L^X$.

Definition 3.9. Let (X, e^r_X, e^l_X) be a bi-partially ordered set. Define four maps $I^r_1, I^l_1, I^r_2, I^l_2 : L^X \to L^X$ by

$$\begin{aligned} I^r_1(A)(x) &= \bigvee_{y\in X}\left[e^r_X(x,y) \odot \bigwedge_{z\in X}\left[e^r_X(z,y) \Rightarrow A(z)\right]\right], \\ I^l_1(A)(x) &= \bigvee_{y\in X}\left[\bigwedge_{z\in X}\left[e^l_X(z,y) \to A(z)\right] \odot e^l_X(x,y)\right], \\ I^r_2(A)(x) &= \bigvee_{y\in X}\left[e^l_X(y,x) \odot \bigwedge_{z\in X}\left[e^l_X(y,z) \Rightarrow A(z)\right]\right], \\ I^l_2(A)(x) &= \bigvee_{y\in X}\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x)\right] \end{aligned}$$

where $A \in L^X$.

Theorem 3.10. *Let (X, e^r_X, e^l_X) be a bi-partially ordered set. Let $A, B \in L^X$. Then the following hold.*
(1) *$\Downarrow_r$ and I^r_1 are r-interior operators. Moreover, $I^r_1 \leq \Downarrow_r$.*
(2) *$\Uparrow_l$ and I^r_2 are r-interior operators. Moreover, $I^r_2 \leq \Uparrow_l$.*
(3) *$\Downarrow_l$ and I^l_1 are l-interior operators. Moreover, $I^l_1 \leq \Downarrow_l$.*
(4) *$\Uparrow_r$ and I^l_2 are l-interior operators. Moreover, $I^l_2 \leq \Uparrow_r$.*

Proof. (1) We show that $\Downarrow_r$ is an r-interior operator.
(I1) Let $A \in L^X$. Then

$$\Downarrow_r A(x) = \bigwedge_{y\in X}\left[e^r_X(y,x) \Rightarrow A(y)\right] \leq e^r_X(x,x) \Rightarrow A(x) = \top \Rightarrow A(x) = A(x).$$

(I2) Let $A \in L^X$. Then

$$\begin{aligned}
\Downarrow_r (\Downarrow_r A)(x) &= \bigwedge_{y\in X} [e^r_X(y,x) \Rightarrow \Downarrow_r A(y)] \\
&= \bigwedge_{y\in X} \left[e^r_X(y,x) \Rightarrow \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)]\right] \\
&= \bigwedge_{y\in X} \bigwedge_{z\in X} [e^r_X(y,x) \Rightarrow [e^r_X(z,y) \Rightarrow A(z)]] \text{ (by Lemma 2.2(2))} \\
&= \bigwedge_{y\in X} \bigwedge_{z\in X} [[e^r_X(z,y) \odot e^r_X(y,x)] \Rightarrow A(z)] \text{ (by Lemma 2.2(3))} \\
&= \bigwedge_{z\in X} \left[\bigvee_{y\in X} [e^r_X(z,y) \odot e^r_X(y,x)] \Rightarrow A(z)\right] \text{ (by Lemma 2.2(2))} \\
&= \bigwedge_{z\in X} [e^r_X(z,x) \Rightarrow A(z)] \\
&= \Downarrow_r A(x).
\end{aligned}$$

(IR) Let $A, B \in L^X$. Then

$$\begin{aligned}
& e^r_{L^X}(\Downarrow_r A, \Downarrow_r B) = \bigwedge_{x\in X} [\Downarrow_r A(x) \Rightarrow \Downarrow_r B(x)] \\
&= \bigwedge_{x\in X} \left[\bigwedge_{y\in X} [e^r_X(y,x) \Rightarrow A(y)] \Rightarrow \bigwedge_{y\in X} [e^r_X(y,x) \Rightarrow B(y)]\right] \\
&\geq \bigwedge_{x\in X} \bigwedge_{y\in X} \left[[e^r_X(y,x) \Rightarrow A(y)] \Rightarrow \bigwedge_{y\in X} [e^r_X(y,x) \Rightarrow B(y)]\right] \\
&\geq \bigwedge_{x\in X} \bigwedge_{y\in X} [A(y) \Rightarrow B(y)] \text{ (by Lemma 2.2(10))} \\
&= e^r_{L^X}(A, B).
\end{aligned}$$

Hence $\Downarrow_r$ is an r-interior operator.

We show that I^r_1 is an r-interior operator.

(I1) Let $A \in L^X$. Then

$$\begin{aligned}
I^r_1(A)(x) &= \bigvee_{y\in X} \left[e^r_X(x,y) \odot \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)]\right] \\
&\leq \bigvee_{y\in X} [e^r_X(x,y) \odot [e^r_X(x,y) \Rightarrow A(x)]] \\
&\leq A(x) \quad \text{(by Lemma 2.2(5))}.
\end{aligned}$$

(I2) Let $A \in L^X$.

Claim 1: $\bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)] = \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow I^r_1(A)(z)]$.

Since $I^r_1(A)(z) \leq A(z)$ by (I1), we have by Lemma 2.2(1) that

$$\bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow I^r_1(A)(z)] \leq \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)].$$

On the other hand, note that

$$\begin{aligned}
& \bigwedge_{w\in X} [e^r_X(w,y) \Rightarrow I^r_1(A)(w)] \\
&= \bigwedge_{w\in X} \left[e^r_X(w,y) \Rightarrow \bigvee_{p\in X} \left[e^r_X(w,p) \odot \bigwedge_{z\in X} [e^r_X(z,p) \Rightarrow A(z)]\right]\right] \\
&\geq \bigwedge_{w\in X} \left[e^r_X(w,y) \Rightarrow \left[e^r_X(w,y) \odot \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)]\right]\right] \\
&\geq \bigwedge_{w\in X} \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)] \text{ (by residuation)} \\
&= \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)].
\end{aligned}$$

Hence Claim 1 is proved.

Finally, we have

$$\begin{aligned}
I^r_1(I^r_1(A))(x) &= \bigvee_{y\in X} \left[e^r_X(x,y) \odot \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow I^r_1(A)(z)]\right] \\
&= \bigvee_{y\in X} \left[e^r_X(x,y) \odot \bigwedge_{z\in X} [e^r_X(z,y) \Rightarrow A(z)]\right] \quad \text{(by Claim 1)} \\
&= I^r_1(A)(x).
\end{aligned}$$

Hence $I^r_1(I^r_1(A)) = I^r_1(A)$ for all $A \in L^X$.

(IR) Let $A, B \in L^X$. By Lemma 2.2(8) and (10), we have

$$\begin{aligned}
&e^r_{L^X}(I^r_1(A), I^r_1(B)) = \bigwedge_{x\in X}[I^r_1(A)(x) \Rightarrow I^r_1(B)(x)]\\
&= \bigwedge_{x\in X}\Big[\bigvee_{y\in X}[e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)]]\\
&\qquad\qquad \Rightarrow \bigvee_{y\in X}\big[e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow B(z)]\big]\Big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\big[\big[e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)]\big]\\
&\qquad\qquad \Rightarrow \big[e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow B(z)]\big]\big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\big[\bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)]\Rightarrow \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow B(z)]\big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\bigwedge_{z\in X}[[e^r_X(z,y)\Rightarrow A(z)]\Rightarrow [e^r_X(z,y)\Rightarrow B(z)]]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\bigwedge_{z\in X}[A(z)\Rightarrow B(z)]\\
&= e^r_{L^X}(A,B).
\end{aligned}$$

Hence I^r_1 is an r-interior operator.

We show $I^r_1 \leq \Downarrow_r$. Let $A \in L^X$. Since

$$\begin{aligned}
&e^r_X(w,x)\odot e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)]\\
&\leq e^r_X(w,x)\odot e^r_X(x,y)\odot [e^r_X(w,y)\Rightarrow A(w)]\\
&\leq e^r_X(w,y)\odot [e^r_X(w,y)\Rightarrow A(w)] \quad \text{(by (R))}\\
&\leq A(w) \quad \text{(by Lemma 2.2(5))},
\end{aligned}$$

we have by residuation that

$$e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)] \leq e^r_X(w,x)\Rightarrow A(w) \quad \text{for all } y,w\in X,$$

which implies that

$$\bigvee_{y\in X}\left[e^r_X(x,y)\odot \bigwedge_{z\in X}[e^r_X(z,y)\Rightarrow A(z)]\right] \leq \bigwedge_{w\in X}[e^r_X(w,x)\Rightarrow A(w)].$$

Hence $I^r_1(A) \leq \Downarrow_r A$ for all $A \in L^X$.

(4) We show that $\Uparrow_r$ is an l-interior operator.

(I1) Let $A \in L^X$. Then

$$\Uparrow_r A(x) \ = \bigwedge_{y\in X}[e^r_X(x,y)\to A(y)] \leq e^r_X(x,x)\to A(x) = A(x).$$

(I2) Let $A \in L^X$. Then

$$\begin{aligned}
&\Uparrow_r(\Uparrow_r A)(x) = \bigwedge_{y\in X}[e^r_X(x,y)\to \Uparrow_r A(y)]\\
&= \bigwedge_{y\in X}\big[e^r_X(x,y)\to \bigwedge_{z\in X}[e^r_X(y,z)\to A(z)]\big]\\
&= \bigwedge_{y\in X}\bigwedge_{z\in X}[e^r_X(x,y)\to [e^r_X(y,z)\to A(z)]] \quad \text{(by Lemma 2.2(2))}\\
&= \bigwedge_{y\in X}\bigwedge_{z\in X}[[e^r_X(x,y)\odot e^r_X(y,z)]\to A(z)] \quad \text{(by Lemma 2.2(3))}\\
&= \bigwedge_{z\in X}\Big[\bigvee_{y\in X}[e^r_X(x,y)\odot e^r_X(y,z)]\to A(z)\Big] \quad \text{(by Lemma 2.2(2))}\\
&= \bigwedge_{z\in X}[e^r_X(x,z)\to A(z)]\\
&= \Uparrow_r A(x).
\end{aligned}$$

(IL) Let $A, B \in L^X$. Then

$$\begin{aligned}
&e^l_{L^X}\,(\Uparrow_r A, \Uparrow_r B) = \bigwedge_{x\in X}[\Uparrow_r A(x) \to \Uparrow_r B(x)]\\
&= \bigwedge_{x\in X}\Big[\bigwedge_{y\in X}[e^r_X(x,y)\to A(y)] \to \bigwedge_{y\in X}[e^r_X(x,y)\to B(y)]\Big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\Big[[e^r_X(x,y)\to A(y)] \to \bigwedge_{y\in X}[e^r_X(x,y)\to B(y)]\Big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}[A(y)\to B(y)] \quad \text{(by Lemma 2.2(10))}\\
&= e^l_{L^X}(A,B).
\end{aligned}$$

Hence $\Uparrow_r$ is an l-interior operator.

We show that I^l_2 is an l-interior operator.

(I1) Let $A \in L^X$. Then

$$\begin{aligned}
I^l_2(A)(x) &= \bigvee_{y\in X}\big[\bigwedge_{z\in X}[e^r_X(y,z)\to A(z)]\odot e^r_X(y,x)\big]\\
&\leq \bigvee_{y\in X}[[e^r_X(y,x)\to A(x)]\odot e^r_X(y,x)]\\
&\leq A(x) \quad \text{(by Lemma 2.2(5))}.
\end{aligned}$$

(I2) Let $A \in L^X$.

Claim 4: $\bigwedge_{z\in X}[e^r_X(y,z)\to A(z)] = \bigwedge_{z\in X}\big[e^r_X(y,z)\to I^l_2(A)(z)\big]$.

Since $I^l_2(A)(z) \leq A(z)$ by (I1), we have by Lemma 2.2(1) that

$$\bigwedge_{z\in X}\big[e^r_X(y,z)\to I^l_2(A)(z)\big] \leq \bigwedge_{z\in X}[e^r_X(y,z)\to A(z)].$$

On the other hand, note that

$$\begin{aligned}
&\bigwedge_{w\in X}\big[e^r_X(y,w)\to I^l_2(A)(w)\big]\\
&= \bigwedge_{w\in X}\Big[e^r_X(y,w)\to \bigvee_{p\in X}\big[\bigwedge_{z\in X}[e^r_X(p,z)\to A(z)]\odot e^r_X(p,w)\big]\Big]\\
&\geq \bigwedge_{w\in X}\big[e^r_X(y,w)\to \big[\bigwedge_{z\in X}[e^r_X(y,z)\to A(z)]\odot e^r_X(y,w)\big]\big]\\
&\geq \bigwedge_{w\in X}\bigwedge_{z\in X}[e^r_X(y,z)\to A(z)] \ \text{(by residuation)}\\
&= \bigwedge_{z\in X}[e^r_X(y,z)\to A(z)].
\end{aligned}$$

Hence Claim 4 is proved.

Finally, we have

$$\begin{aligned}
I^l_2(I^l_2(A))(x) &= \bigvee_{y\in X}\big[\bigwedge_{z\in X}\big[e^r_X(y,z)\to I^l_2(A)(z)\big]\odot e^r_X(y,x)\big]\\
&= \bigvee_{y\in X}\big[\bigwedge_{z\in X}[e^r_X(y,z)\to A(z)]\odot e^r_X(y,x)\big] \ \text{(by Claim 4)}\\
&= I^l_2(A)(x).
\end{aligned}$$

Hence $I^l_2(I^l_2(A)) = I^l_2(A)$ for all $A \in L^X$.

(IR) Let $A, B \in L^X$. By Lemma 2.2(8) and (10), we have

$$\begin{aligned}
&e^l_{L^X}\left(I^l_2(A), I^l_2(B)\right) = \bigwedge_{x\in X}\left[I^l_2(A)(x) \to I^l_2(B)(x)\right]\\
&= \bigwedge_{x\in X}\Big[\bigvee_{y\in X}\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x)\right]\\
&\qquad\qquad \to \bigvee_{y\in X}\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to B(z)\right] \odot e^r_X(y,x)\right]\Big]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\left[\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x)\right]\right.\\
&\qquad\qquad \left.\to \left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to B(z)\right] \odot e^r_X(y,x)\right]\right]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \to \bigwedge_{z\in X}\left[e^r_X(y,z) \to B(z)\right]\right]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\bigwedge_{z\in X}\left[\left[e^r_X(y,z) \to A(z)\right] \to \left[e^r_X(y,z) \to B(z)\right]\right]\\
&\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\bigwedge_{z\in X}\left[A(z) \to B(z)\right] \text{ (by Lemma 2.2(10))}\\
&= e^l_{L^X}(A, B).
\end{aligned}$$

Hence I^l_2 is an l-interior operator.

We show $I^l_2 \leq \Uparrow_r$. Let $A \in L^X$. Since

$$\begin{aligned}
&\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x) \odot e^r_X(x,w)\\
&\leq \bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,w) \quad \text{(by (R))}\\
&\leq \left[e^r_X(y,w) \to A(w)\right] \odot e^r_X(y,w)\\
&\leq A(w) \quad \text{(by Lemma 2.2(5))},
\end{aligned}$$

we have by residuation that

$$\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x) \leq e^r_X(x,w) \to A(w) \quad \text{for all } y, w \in X,$$

which implies that

$$\bigvee_{y\in X}\left[\bigwedge_{z\in X}\left[e^r_X(y,z) \to A(z)\right] \odot e^r_X(y,x)\right] \leq \bigwedge_{w\in X}\left[e^r_X(x,w) \to A(w)\right].$$

Hence $I^l_2(A) \leq \Uparrow_r A$ for all $A \in L^X$.

(2) and (3) can be similarly proved. □

4. Various completeness

In this section, we demonstrate that the set $M = \{A \in L^X \mid I(A) = A\}$, where I is a right (resp. left) interior operator, is a right (resp. left) join complete.

Definition 4.1. [5] Let (X, e^r_X) be an r-partially ordered set. Let $A \in L^X$.

(1) A point x_0 is called an *r-join* (or *right-join*) of A, denoted by $x_0 = \sqcup_r A$, if it satisfies

(RJ1) $A(x) \leq e^r_X(x, x_0)$ for all $x \in X$,

(RJ2) $\bigwedge_{x\in X}\left[A(x) \Rightarrow e^r_X(x,y)\right] \leq e^r_X(x_0, y)$ for all $y \in X$.

(2) A point x_1 is called an *r-meet* (or *right-meet*) of A, denoted by $x_1 = \sqcap_r A$, if it satisfies

(RM1) $A(x) \leq e^r_X(x_1, x)$ for all $x \in X$, (RM2) $\bigwedge_{x\in X}[A(x)\to e^r_X(y,x)]\le e^r_X(y,x_1)$ for all $y\in X$.

Let (X,e^l_X) be an l-partially ordered set. Let $A\in L^X$.

(3) A point x_0 is called an *l-join* (or *left-join*) of A, denoted by $x_0=\sqcup_l A$, if it satisfies

(LJ1) $A(x)\le e^l_X(x,x_0)$ for all $x\in X$,

(LJ2) $\bigwedge_{x\in X}\left[A(x)\to e^l_X(x,y)\right]\le e^l_X(x_0,y)$ for all $y\in X$.

(4) A point x_1 is called an *l-meet* (or *left-meet*) of A, denoted by $x_1=\sqcap_l A$, if it satisfies

(LM1) $A(x)\le e^l_X(x_1,x)$ for all $x\in X$,

(LM2) $\bigwedge_{x\in X}\left[A(x)\Rightarrow e^l_X(y,x)\right]\le e^l_X(y,x_1)$ for all $y\in X$.

(5) An r-partially ordered set (X,e^r_X) is *r-join complete* (resp. *r-meet complete*) if there exists $\sqcup_r A$ (resp. $\sqcap_r A$) for all $A\in L^X$.

(6) An r-partially ordered set (X,e^r_X) is *r-complete* if is r-join complete and r-meet complete.

(7) An l-partially ordered set (X,e^l_X) is *l-join complete*(resp. *l-meet complete*) if there exists $\sqcup_l A$ (resp. $\sqcap_l A$) for all $A\in L^X$.

(8) An l-partially ordered set (X,e^l_X) is *l-complete* if it is l-join complete and l-meet complete.

Lemma 4.2. [5] *Let (X,e^r_X,e^l_X) be a bi-partially ordered set. Let $x_0,x_1\in X$. Let $A\in L^X$. Then the following hold.*

(1) $x_0=\sqcup_r A$ if and only if $\bigwedge_{x\in X}[A(x)\Rightarrow e^r_X(x,y)]=e^r_X(x_0,y)$ for all $y\in X$.

(2) $x_1=\sqcap_r A$ if and only if $\bigwedge_{x\in X}[A(x)\to e^r_X(y,x)]=e^r_X(y,x_1)$ for all $y\in X$.

(3) $x_0=\sqcup_l A$ if and only if $\bigwedge_{x\in X}\left[A(x)\to e^l_X(x,y)\right]=e^l_X(x_0,y)$ for all $y\in X$.

(4) $x_1=\sqcap_l A$ if and only if $\bigwedge_{x\in X}\left[A(x)\Rightarrow e^l_X(y,x)\right]=e^l_X(y,x_1)$ for all $y\in X$.

(5) $\sqcup_r A$, $\sqcap_r A$, $\sqcup_l A$ and $\sqcap_l A$ are unique if each exists.

Theorem 4.3. (1) *Let $I^r:L^X\to L^X$ be an r-interior operator. Let $H^r_{I^r}=\{A\in L^X\mid I^r(A)=A\}$. Then $\left(H^r_{I^r},e^r_{H^r_{I^r}}\right)$ is r-join complete where*

$$\sqcup_r\Psi=\bigvee_{A\in H^r_{I^r}}[A\odot\Psi(A)]\quad \textit{for all } \Psi\in L^{L^X}.$$

(2) *Let $I^l:L^X\to L^X$ be an l-interior operator. Let $H^l_{I^l}=\{A\in L^X\mid I^l(A)=A\}$. Then $\left(H^l_{I^l},e^l_{H^l_{I^l}}\right)$ is l-join complete where*

$$\sqcup_l\Psi=\bigvee_{A\in H^l_{I^l}}[\Psi(A)\odot A]\quad \textit{for all } \Psi\in L^{L^X}.$$

Proof. (1) Let $\Psi : H^r_{I^r} \to L$ be a map. Note that for all $B \in L^X$,

$$\begin{aligned}
&\bigwedge_{A\in H^r_{I^r}} \left[\Psi(A) \Rightarrow e^r_{A\in H^r_{I^r}}(A,B)\right] = \bigwedge_{A\in H^r_{I^r}} \left[\Psi(A) \Rightarrow \bigwedge_{x\in X} [A(x) \Rightarrow B(x)]\right] \\
&= \bigwedge_{A\in H^r_{I^r}} \bigwedge_{x\in X} [\Psi(A) \Rightarrow [A(x) \Rightarrow B(x)]] \quad \text{(by Lemma 2.2(2))} \\
&= \bigwedge_{A\in H^r_{I^r}} \bigwedge_{x\in X} [[A(x) \odot \Psi(A)] \Rightarrow B(x)] \quad \text{(by Lemma 2.2(3))} \\
&= \bigwedge_{x\in X} \left[\bigvee_{A\in H^r_{I^r}} [A(x) \odot \Psi(A)] \Rightarrow B(x)\right] \quad \text{(by Lemma 2.2(2))} \\
&= e^r_{A\in H^r_{I^r}} \left(\bigvee_{A\in H^r_{I^r}} [A \odot \Phi(A)], B\right).
\end{aligned}$$

By Lemma 4.2, $\sqcup_r \Psi = \bigvee_{A\in H^r_{I^r}} [A \odot \Phi(A)]$.
(2) Let $\Psi : H^l_{I^l} \to L$ be a map. Note that for all $B \in H^l_{I^l}$,

$$\begin{aligned}
&\bigwedge_{A\in H^l_{I^l}} \left[\Psi(A) \to e^l_{H^l_{I^l}}(A,B)\right] = \bigwedge_{A\in H^l_{I^l}} \left[\Psi(A) \to \bigwedge_{x\in X} [A(x) \to B(x)]\right] \\
&= \bigwedge_{A\in H^l_{I^l}} \bigwedge_{x\in X} [\Psi(A) \to [A(x) \to B(x)]] \quad \text{(by Lemma 2.2(2))} \\
&= \bigwedge_{A\in H^l_{I^l}} \bigwedge_{x\in X} [[\Psi(A) \odot A(x)] \to B(x)] \quad \text{(by Lemma 2.2(3))} \\
&= \bigwedge_{x\in X} \left[\bigvee_{A\in H^l_{I^l}} [\Psi(A) \odot A(x)] \to B(x)\right] \quad \text{(by Lemma 2.2(2))} \\
&= e^l_{H^l_{I^l}} \left(\bigvee_{A\in H^l_{I^l}} [\Phi(A) \odot A], B\right).
\end{aligned}$$

By Lemma 4.2, $\sqcup_l \Psi = \bigvee_{A\in H^l_{I^l}} [\Phi(A) \odot A]$.

□

By Theorems 3.10 and 4.3, we have the following.

Corollary 4.4. (1) *The pair* $\left(H^r_{I^r}, e^r_{H^r_{I^r}}\right)$ *is r-join complete where* $I^r = \Downarrow_r$ *or* $\Uparrow_l$.
(2) *The pair* $\left(H^l_{I^l}, e^l_{H^l_{I^l}}\right)$ *is l-join complete where* $I^l = \Downarrow_l$ *or* $\Uparrow_r$.

Lemma 4.5. (1) *Let* $\{k_i\}_{i\in\Gamma} \subseteq L$ *and* $\{A_i\}_{i\in\Gamma} \subseteq H^r_{I^r}$ *where* $I^r = \Downarrow_r$ *or* $\Uparrow_l$. *Then* $\bigwedge_{i\in\Gamma} [k_i \to A_i] \in H^r_{I^r}$.
(2) *Let* $\{k_i\}_{i\in\Gamma} \subseteq L$ *and* $\{A_i\}_{i\in\Gamma} \subseteq H^l_{I^l}$ *where* $I^l = \Downarrow_l$ *or* $\Uparrow_r$. *Then* $\bigwedge_{i\in\Gamma} [k_i \Rightarrow A_i] \in H^l_{I^l}$.

Proof. (1) *Case 1:* $I^r = \Downarrow_r$. Note that

$$\begin{aligned}
&\Downarrow_r \left(\bigwedge_{i\in\Gamma} [k_i \to A_i]\right)(x) = \bigwedge_{y\in X} \left[e^r_X(y,x) \Rightarrow \bigwedge_{i\in\Gamma} [k_i \to A_i](y)\right] \\
&= \bigwedge_{y\in X} \bigwedge_{i\in\Gamma} [e^r_X(y,x) \Rightarrow [k_i \to A_i(y)]] \ \text{(by Lemma 2.2(2))} \\
&= \bigwedge_{y\in X} \bigwedge_{i\in\Gamma} [k_i \to [e^r_X(y,x) \Rightarrow A_i(y)]] \ \text{(by Lemma 2.2(4))} \\
&= \bigwedge_{i\in\Gamma} \left[k_i \to \bigwedge_{y\in X} [e^r_X(y,x) \Rightarrow A_i(y)]\right] \ \text{(by Lemma 2.2(2))} \\
&= \bigwedge_{i\in\Gamma} [k_i \to \Downarrow_r A_i(x)] \\
&= \bigwedge_{i\in\Gamma} [k_i \to A_i(x)] \ (\because A_i \in H^r_{I^r}) \\
&= \left(\bigwedge_{i\in\Gamma} [k_i \to A_i]\right)(x).
\end{aligned}$$

Hence $\bigwedge_{i\in\Gamma} [k_i \to A_i] \in H^r_{\Downarrow_r}$.

Case 2: $I^r = \Uparrow_l$. Note that

$$\begin{aligned}
&\Uparrow_l \left(\bigwedge_{i\in\Gamma}[k_i \to A_i]\right)(x) = \bigwedge_{y\in X}\left[e^l_X(x,y) \Rightarrow \bigwedge_{i\in\Gamma}[k_i \to A_i](y)\right]\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}\left[e^l_X(x,y) \Rightarrow [k_i \to A_i(y)]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}\left[k_i \to \left[e^l_X(x,y) \Rightarrow A_i(y)\right]\right] \text{ (by Lemma 2.2(4))}\\
&= \bigwedge_{i\in\Gamma}\left[k_i \to \bigwedge_{y\in X}\left[e^l_X(x,y) \Rightarrow A_i(y)\right]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{i\in\Gamma}[k_i \to \Uparrow_l A_i(x)]\\
&= \bigwedge_{i\in\Gamma}[k_i \to A_i(x)] \ (\because A_i \in H^r_{I^r})\\
&= \left(\bigwedge_{i\in\Gamma}[k_i \to A_i]\right)(x).
\end{aligned}$$

Hence $\bigwedge_{i\in\Gamma}[k_i \to A_i] \in H^r_{\Uparrow_l}$.

(2) *Case 1:* $I^l = \Downarrow_l$. Note that

$$\begin{aligned}
&\Downarrow_l \left(\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i]\right)(x) = \bigwedge_{y\in X}\left[e^l_X(y,x) \to \bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i](y)\right]\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}\left[e^l_X(y,x) \to [k_i \Rightarrow A_i(y)]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}\left[k_i \Rightarrow \left[e^l_X(y,x) \to A_i(y)\right]\right] \text{ (by Lemma 2.2(4))}\\
&= \bigwedge_{i\in\Gamma}\left[k_i \Rightarrow \bigwedge_{y\in X}\left[e^l_X(y,x) \to A_i(y)\right]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{i\in\Gamma}[k_i \Rightarrow \Downarrow_l A_i(x)]\\
&= \bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i(x)] \ (\because A_i \in H^l_{I^l})\\
&= \left(\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i]\right)(x).
\end{aligned}$$

Hence $\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i] \in H^l_{\Downarrow_l}$.

Case 2: $I^l = \Uparrow_r$. Note that

$$\begin{aligned}
&\Uparrow_r \left(\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i]\right)(x) = \bigwedge_{y\in X}\left[e^r_X(x,y) \to \bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i](y)\right]\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}[e^r_X(x,y) \to [k_i \Rightarrow A_i(y)]] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{y\in X}\bigwedge_{i\in\Gamma}[k_i \Rightarrow [e^r_X(x,y) \to A_i(y)]] \text{ (by Lemma 2.2(4))}\\
&= \bigwedge_{i\in\Gamma}\left[k_i \Rightarrow \bigwedge_{y\in X}[e^r_X(x,y) \to A_i(y)]\right] \text{ (by Lemma 2.2(2))}\\
&= \bigwedge_{i\in\Gamma}[k_i \Rightarrow \Uparrow_r A_i(x)]\\
&= \bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i(x)] \ (\because A_i \in H^l_{I^l})\\
&= \left(\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i]\right)(x).
\end{aligned}$$

Hence $\bigwedge_{i\in\Gamma}[k_i \Rightarrow A_i] \in H^l_{\Uparrow_r}$. □

Theorem 4.6. (1) *The pair* $\left(H^r_{I^r}, e^r_{H^r_{I^r}}\right)$ *is r-complete where* $I^r = \Downarrow_r$ *or* $\Uparrow_l$.
(2) *The pair* $\left(H^l_{I^l}, e^l_{H^l_{I^l}}\right)$ *is l-complete where* $I^l = \Downarrow_l$ *or* $\Uparrow_r$.

Proof. (1) By Theorems 3.10 and 4.3, $\left(H^r_{I^r}, e^r_{H^r_{I^r}}\right)$ is r-join complete. It is enough to show that $\left(H^r_{I^r}, e^r_{H^r_{I^r}}\right)$ is r-meet complete.

Let $\Psi : H^r_{I^r} \to L$ be a map. Note that for all $B \in H^r_{I^r}$,

$$\begin{aligned}
&\bigwedge_{A\in H^r_{I^r}}\Big[\Psi(A)\to e^r_{H^r_{I^r}}(B,A)\Big]=\bigwedge_{A\in H^r_{I^r}}\big[\Psi(A)\to\bigwedge_{x\in X}[B(x)\Rightarrow A(x)]\big]\\
&=\bigwedge_{A\in H^r_{I^r}}\bigwedge_{x\in X}[\Psi(A)\to[B(x)\Rightarrow A(x)]]\quad\text{(by Lemma 2.2(2))}\\
&=\bigwedge_{A\in H^r_{I^r}}\bigwedge_{x\in X}[B(x)\Rightarrow[\Psi(A)\to A(x)]]\quad\text{(by Lemma 2.2(4))}\\
&=\bigwedge_{x\in X}\Big[B(x)\Rightarrow\bigwedge_{A\in H^r_{I^r}}[\Psi(A)\to A(x)]\Big]\quad\text{(by Lemma 2.2(2))}\\
&=e^r_{H^r_{I^r}}\Big(B,\bigwedge_{A\in H^r_{I^r}}[\Psi(A)\to A]\Big).
\end{aligned}$$

By Lemmas 4.2 and 4.5, $\sqcap_r\Psi=\bigwedge_{A\in H^r_{I^r}}[\Phi(A)\to A]$. Hence $\left(H^r_{I^r},e^r_{H^r_{I^r}}\right)$ is r-meet complete.

(2) By Theorems 3.10 and 4.3, $\left(H^l_{I^l},e^l_{H^l_{I^l}}\right)$ is l-join complete. It is enough to show that $\left(H^l_{I^l},e^l_{H^l_{I^l}}\right)$ is l-meet complete.

Let $\Psi:H^l_{I^l}\to L$ be a map. For all $B\in H^l_{I^l}$,

$$\begin{aligned}
&\bigwedge_{A\in H^l_{I^l}}\Big[\Psi(A)\Rightarrow e^l_{H^l_{I^l}}(B,A)\Big]=\bigwedge_{A\in H^l_{I^l}}\big[\Psi(A)\Rightarrow\bigwedge_{x\in X}[B(x)\to A(x)]\big]\\
&=\bigwedge_{A\in H^l_{I^l}}\bigwedge_{x\in X}[\Psi(A)\Rightarrow[B(x)\to A(x)]]\quad\text{(by Lemma 2.2(2))}\\
&=\bigwedge_{A\in H^l_{I^l}}\bigwedge_{x\in X}[B(x)\to[\Psi(A)\Rightarrow A(x)]]\quad\text{(by Lemma 2.2(4))}\\
&=\bigwedge_{x\in X}\Big[B(x)\to\bigwedge_{A\in H^l_{I^l}}[\Psi(A)\Rightarrow A(x)]\Big]\quad\text{(by Lemma 2.2(2))}\\
&=e^l_{H^l_{I^l}}\Big(B,\bigwedge_{A\in H^l_{I^l}}[\Psi(A)\Rightarrow A]\Big).
\end{aligned}$$

By Lemmas 4.2 and 4.5, $\sqcap_l\Psi=\bigwedge_{A\in H^l_{I^l}}[\Psi(A)\Rightarrow A]$. Hence $\left(H^l_{I^l},e^l_{H^l_{I^l}}\right)$ is l-meet complete. □

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References

1. R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.
2. R. Bělohlávek, *Lattices of fixed points of Galois connections*, Math. Logic Quart. **47** (2001), 111-116.
3. G. Georgescu and A. Popescue, *Non-commutative Galois connections*, Soft Computing **7** (2003), 458-467.
4. J.M. Ko and Y.C. Kim, *Bi-closure systems and bi-closure operators on generalized residuated lattices*, Journal of Intelligent and Fuzzy Systems **36** (2019), 2631-2643.
5. J.M. Ko and Y.C. Kim, *Various operations and right completeness in generalized residuated lattices*, Journal of Intelligent and Fuzzy Systems **40** (2021), 149-164.

6. S.P. Tiwari, I. Perfilieva and A.P. Singh, *Generalized residuated lattices based F-transform*, Iranian Journal of Fuzzy Systems **15** (2018), 165-182.
7. E. Turunen, *Mathematics Behind Fuzzy Logic*, A Springer-Verlag Co., 1999.
8. C.Y. Wang and B.Q. Hu, *Fuzzy rough sets based on general residuated lattices*, Information Sciences **248** (2013), 31-49.

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