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MITTAG LEFFLER FUNCTIONS ASSOCIATED WITH FUNCTIONS THAT MAP OPEN UNIT DISC ONTO A SECTOR OF THE RIGHT-HALF PLANE

AFIS SALIU*, KANWAL JABEEN, SEMIU OLADIPUPO OLADEJO, OLAIDE YETUNDE SAKA-BALOGUN

Abstract. In this present work, we inaugurated subclasses of analytic functions which are associated with generalized Mittag Leffler Functions. Inclusion implications and integral preserving properties under the Bernardi integral operator are investigated. Some consequences of these findings are also illustrated.

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1. Introduction

Ma and Minda [11] gave a comprehensive and unified treatment of the classes S^* and C of starlike and convex functions, respectively. They considered the class Φ of analytic univalent functions $\phi(z)$ with $\operatorname{Re}\phi(z) > 0$ and for which $\phi(U)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0)$ such that $\phi'(0) > 0$. They initiated the following classes of functions that generalized and unified many known subclasses of S:

$$S^*(\phi) = \left\{ f \in \mathcal{A} \colon \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$$

and

$$C(\phi) = \left\{ f \in \mathcal{A} \colon \frac{(zf'(z))'}{f'(z)} \prec \phi(z) \right\},$$

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*Corresponding author.

where $\mathcal{A}$ is the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in an open unit disc U.

In particular, for $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\eta}$, $\eta \in (0,1]$, we have the classes of strongly starlike (denote by $\widetilde{S^*(\eta)}$) and strongly convex (denoted by $\widetilde{C(\eta)}$) functions, respectively. These classes have analytic characterization:

$$\widetilde{S^*(\eta)} = \left\{ f \in A \colon \left| \arg \frac{zf'(z)}{f(z)} \right| \leq \frac{\pi}{2}\eta \right\} \tag{2}$$

and

$$\widetilde{C(\eta)} = \left\{ f \in A \colon \left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| \leq \frac{\pi}{2}\eta \right\} \tag{3}$$

were first introduced by Brannan [4] and Stankiewicz in [21]. Later, Mocanu [14] showed that $\widetilde{C(\eta)} \subset \widetilde{S^*(\beta)}$, where the relation between η and β were given by transcendental equation. This result were further improved by Nunokawa in [16] by using the Nunokawa's Lemma. For recent work on analytic functions associated with strongly starlike functions, see [2, 5, 6, 10, 15, 17].

Among one of the most applicable special functions is the Mitag-Leffler functions (MLF) given by

$$\mathbb{E}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha, z \in \mathbb{C}, \operatorname{Re}\alpha > 0. \tag{4}$$

This function was introduced and studied by Mittag-Leffler [12, 13] in relation with his method of summation of some divergent series. In recent years, interest in MLF is considerably increasing in engineering and sciences due to its wide applications in many applied problems, such as fluid flow, probability and statistical distribution theory. As a result, this functions has witnessed many generalizations and modifications, ranges from two-parametric MLF to three-parametric MLF, which was further extended by Srivastava and Tomovski [20] and recently generalized by Jain et al [9].

Corresponding to (4), Elhaddad et al [7] initiated the Mittag-Leffler linear operator $\mathrm{E}^{m,\lambda}_{\alpha,\beta} f \colon A \longleftrightarrow A$ given by

$$\mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)(1+(n-1)\lambda)^m}{\Gamma(\alpha(n-1)+\beta)} a_n z^n \ (z \in U), \tag{5}$$

where $m \in \mathbb{N} \cup \{0\}$, $\lambda \geq 0, \alpha, \beta \in \mathbb{C}, \operatorname{Re}\alpha > 0, \operatorname{Re}, \beta > 0$. For different choices of α, β and m, the operator reduces to Al-Oboudi operator [1], Salagean operator [18] and the operator studied in [19]. It is easy to see that the identity relation associated with (5) is given by

$$\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z) = (1-\lambda)\mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z) + \lambda z (\mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z))'. \tag{6}$$

Motivated with the work presented in [7] by Elhaddad et al and the above mentioned work linked to the classes $\widetilde{C(\alpha)}$ and $\widetilde{S^*(\alpha)}$, we introduce and study the following classes of analytic functions:

Definition 1.1. Let $\rho > 0, \gamma \geq 0$ and suppose that

$$\mathcal{L}_{\alpha,\beta}^{m,\lambda}(\gamma,\rho;f)(z) = (1-\gamma)\left(\frac{\mathrm{E}_{\alpha,\beta}^{m,\lambda}f(z)}{z}\right)^{\rho} + \gamma\left(\frac{\mathrm{E}_{\alpha,\beta}^{m+1,\lambda}f(z)}{z}\right)\left(\frac{\mathrm{E}_{\alpha,\beta}^{m,\lambda}f(z)}{z}\right)^{\rho-1}. \tag{7}$$

Then $f \in \mathcal{H}_{\alpha,\beta,\rho}^{m,\lambda,\gamma}(\eta)$ if

$$\left|\arg\left(\mathcal{L}_{\alpha,\beta}^{m,\lambda}(\gamma,\rho;f)(z)\right)\right| \leq \frac{\pi}{2}\eta. \quad \eta \in (0,1]. \tag{8}$$

In particular, we have the following classes of functions:

(i) For $m = \alpha = 0, \lambda = 1$, we have

$$\mathcal{H}_{\rho}^{\gamma}(\eta) = \left\{f \in \mathcal{A}\colon \left|\arg\left(\left(\frac{f(z)}{z}\right)^{\rho}\left[1+\gamma\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right)\right| \leq \frac{\pi}{2}\eta\right\}$$

(ii) For $m = \alpha = 0, \lambda = 1, \gamma = 1$, we have

$$\mathcal{H}_{\rho}(\eta) = \left\{f \in \mathcal{A}\colon \left|\arg\left(\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\rho}\right)\right| \leq \frac{\pi}{2}\eta\right\}$$

(iii) For $m = \alpha = 0, \lambda = 1, \rho = 1$, we have

$$\mathcal{H}(\eta) = \left\{f \in \mathcal{A}\colon \left|\arg\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z)\right)\right| \leq \frac{\pi}{2}\eta\right\}$$

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in ST_{\alpha,\beta}^{m,\lambda}(\eta)$ if and only if $\mathrm{E}_{\alpha,\beta}^{m,\lambda}f(z) \in \widetilde{S^*(\eta)}$. Similarly, $f \in CT_{\alpha,\beta}^{m,\lambda}(\eta)$ if and only if $\mathrm{E}_{\alpha,\beta}^{m,\lambda}f(z) \in \widetilde{C(\eta)}$.

For $m = \alpha = 0, \lambda = 1, \gamma = 0$, Definition 1.2 reduce to the classes of strongly starlike and strongly convex functions of order η introduce by Brannan [4] and Stankiewicz in [21], respectively.

We are set to present the main theorems of this work. As such, we need the following Nunokawa's Lemma.

Lemma 1.3. [16] *Let a function $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$ be analytic in U and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\eta\,(|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| < \frac{\pi}{2}\eta\,(0 < \eta \leq 1),$$

then

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = \frac{\pi}{2}\eta\right)$$

,

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = -\frac{\pi}{2}\eta\right),$$

and $(p(z_0))^{\frac{1}{\eta}} = \pm ia\,(a > 0)$.

2. Main results

Theorem 2.1. *Let* $f \in \mathcal{H}^{m,\lambda,\gamma}_{\alpha,\beta,\rho}(\eta^{\lambda,\gamma}_{1_\rho}(\eta_2))$, *where*

$$\eta^{\lambda,\gamma}_{1_\rho}(\eta_2) = \eta_2 + \frac{2}{\pi}\arctan\left\{\frac{\gamma\lambda\eta_2 \sin\left(\frac{\pi}{2}(1-\eta_2)\right)}{\rho(1-\eta_2)^{\frac{1}{2}(1-\eta_2)}(1+\eta_2)^{\frac{1}{2}(1+\eta_2)} + \gamma\lambda\eta_2\cos\left(\frac{\pi}{2}(1-\eta_2)\right)}\right\}.$$

Then

$$\left|\arg\left(\frac{\mathrm{E}^{m,\lambda}_{\alpha,\beta}f(z)}{z}\right)^{\rho}\right| \leq \frac{\pi}{2}\eta_2, \quad \eta_2 \in (0,1].$$

Proof. Let

$$\left(\frac{\mathrm{E}^{m,\lambda}_{\alpha,\beta}f(z)}{z}\right)^{\rho} = p(z), \tag{9}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Logarithmic differentiation of (9) together with the relation (6) give

$$\frac{\mathrm{E}^{m+1,\lambda}_{\alpha,\beta}f(z)}{\mathrm{E}^{m,\lambda}_{\alpha,\beta}f(z)} = \frac{\lambda}{\rho}\frac{zp'(z)}{p(z)} + 1. \tag{10}$$

In view of (7), (9) and (10), we arrive at

$$\arg\left(\mathcal{L}^{m,\lambda}_{\alpha,\beta}(\gamma,\rho;f)(z)\right) = \arg p(z) + \arg\left(1 + \frac{\gamma\lambda}{\rho}\frac{zp'(z)/p(z)}{p(z)}\right). \tag{11}$$

If there exists $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\eta_2\,(|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| < \frac{\pi}{2}\eta_2,$$

then by Lemma 1.3,

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\eta \quad \text{and} \quad (p(z_0))^{\frac{1}{\eta}} = \pm ia\,(a > 0).$$

Now, suppose $\arg(p(z_0)) = \frac{\pi}{2}\eta_2$. Then

$$\begin{aligned}&\arg\left(\mathcal{L}^{m,\lambda}_{\alpha,\beta}(\gamma,\rho;f)(z)\right)\\ =&\frac{\pi}{2}\eta_2 + \arg\left(1 + \frac{\gamma\lambda}{\rho}\frac{ik\eta_2}{a^{\eta_2}e^{i\frac{\pi}{2}\eta_2}}\right)\end{aligned}$$

$$=\frac{\pi}{2}\eta_2+\arctan\left(\frac{\frac{\gamma\lambda}{2\rho}\frac{k\eta_2}{a^{\eta_2}}\sin\left(\frac{\pi}{2}(1-\eta_2)\right)}{1+\frac{\gamma\lambda}{2\rho}\frac{k\eta_2}{a^{\eta_2}}\cos\left(\frac{\pi}{2}(1-\eta_2)\right)}\right),\quad k\geq\frac{1}{2}\left(a+\frac{1}{a}\right).$$

Since $\frac{k\eta_2}{a^{\eta_2}}\geq\frac{\eta_2}{2}\left(a^{1-\eta_2}+a^{-1-\eta_2}\right)$ $a>0$, then the function $h(a)=\frac{1}{2}\left(a^{1-\eta_2}+a^{-1-\eta_2}\right)$ assumes its minimum value at $a=\left(\frac{1+\eta_2}{1-\eta_2}\right)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}&\arg\left(\mathcal{L}^{m,\lambda}_{\alpha,\beta}(\gamma,\rho;f)(z)\right)\\ &\geq\frac{\pi}{2}\eta_2+\arctan\left(\frac{\frac{\gamma\lambda\eta_2}{2\rho}\left[\left(\frac{1+\eta_2}{1-\eta_2}\right)^{\frac{1-\eta_2}{2}}+\left(\frac{1-\eta_2}{1+\eta_2}\right)^{\frac{1+\eta_2}{2}}\right]\sin\left(\frac{\pi}{2}(1-\eta_2)\right)}{1+\frac{\gamma\lambda\eta_2}{2\rho}\left[\left(\frac{1+\eta_2}{1-\eta_2}\right)^{\frac{1-\eta_2}{2}}+\left(\frac{1-\eta_2}{1+\eta_2}\right)^{\frac{1+\eta_2}{2}}\right]\cos\left(\frac{\pi}{2}(1-\eta_2)\right)}\right).\end{aligned}$$

This contradicts our assumption of the theorem. In a similar manner, we also obtain a contradiction for
$\arg p(z_0)=-\frac{\pi}{2}\eta_2$. That is

$$\begin{aligned}&\arg\left(\mathcal{L}^{m,\lambda}_{\alpha,\beta}(\gamma,\rho;f)(z)\right)\\ &\geq-\frac{\pi}{2}\eta_2-\arctan\left(\frac{\frac{\gamma\lambda\eta_2}{2\rho}\left[\left(\frac{1+\eta_2}{1-\eta_2}\right)^{\frac{1-\eta_2}{2}}+\left(\frac{1-\eta_2}{1+\eta_2}\right)^{\frac{1+\eta_2}{2}}\right]\sin\left(\frac{\pi}{2}(1-\eta_2)\right)}{1+\frac{\gamma\lambda\eta_2}{2\rho}\left[\left(\frac{1+\eta_2}{1-\eta_2}\right)^{\frac{1-\eta_2}{2}}+\left(\frac{1-\eta_2}{1+\eta_2}\right)^{\frac{1+\eta_2}{2}}\right]\cos\left(\frac{\pi}{2}(1-\eta_2)\right)}\right).\end{aligned}$$

Thus the function $p(z)$ has to satisfy $|\arg p(z)|<\frac{\pi}{2}\eta_2$ for all $z\in U$ This concludes that

$$\left|\arg\left(\frac{\mathrm{E}^{m,\lambda}_{\alpha,\beta}f(z)}{z}\right)^{\rho}\right|\leq\frac{\pi}{2}\eta_2.$$

□

Remark 2.1. Let $f\in\mathcal{A}$. Then by specifying the parameters, we have the following implications.

(*i*) $f\in\mathcal{H}^{\gamma}_{\rho}(\eta^{1,\gamma}_{1_\rho}(\eta_2))\Longrightarrow\left|\arg\left(\frac{f(z)}{z}\right)^{\rho}\right|\leq\frac{\pi}{2}\eta_2$.

(*ii*) $f\in\mathcal{H}_{\rho}(\eta^{1,1}_{1_\rho}(\eta_2))\Longrightarrow\left|\arg\left(\frac{f(z)}{z}\right)^{\rho}\right|\leq\frac{\pi}{2}\eta_2$.

(*iii*) $f\in\mathcal{H}(\eta^{1,1}_{1_1}(\eta_2))\Longrightarrow\left|\arg\left(\frac{f(z)}{z}\right)\right|\leq\frac{\pi}{2}\eta_2$.

Theorem 2.2. *$ST^{m+1,\lambda}_{\alpha,\beta}(\eta)\subset ST^{m,\lambda}_{\alpha,\beta}(\eta)$ for all $m\in\mathbb{N}$.*

Proof. Let $f \in ST^{m+1,\lambda}_{\alpha,\beta}(\eta)$ and set

$$\frac{z(\mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z))'}{\mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z)} = p(z), \tag{12}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Therefore, from the identity (6), we obtain

$$\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z) = (\lambda p(z) + (1-\lambda))\, \mathrm{E}^{m,\lambda}_{\alpha,\beta} f(z),$$

and by logarithmic differentiation, we arrive at

$$\frac{z(\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z))'}{\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z)} = p(z) + \frac{\lambda z p'(z)}{(1-\lambda) + \lambda p(z)}. \tag{13}$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\eta \,(|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| < \frac{\pi}{2}\eta,$$

then by Lemma 1.3,

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta \quad \text{and} \quad (p(z_0))^{\frac{1}{\eta}} = \pm ia \,(a > 0).$$

Now, if $\arg(p(z_0)) = \frac{\pi}{2}\eta$. Then

$$\begin{aligned}
&\arg\left(\frac{z_0(\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z_0))'}{\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z_0)}\right) \\
&= \arg(p(z_0)) + \arg\left(\frac{\lambda z_0 p'(z_0)/p(z_0)}{(1-\lambda) + \lambda p(z_0)}\right) \\
&= \frac{\pi}{2}\eta + \arg\left(1 + \frac{ik\lambda\eta}{(1-\lambda) + \lambda e^{\frac{i\pi}{2}\eta}}\right) \\
&= \frac{\pi}{2}\eta + \arctan\left[\frac{k\lambda\eta\left((1-\lambda) + \lambda a^\eta \cos\left(\frac{\pi}{2}\eta\right)\right)}{(1-\lambda)^2 + \lambda a^\eta\left(2\cos\left(\frac{\pi}{2}\eta\right) + k\lambda\eta \sin\left(\frac{\pi}{2}\eta\right)\right)}\right] \\
&\geq \frac{\pi}{2}\eta \quad \left(k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)\right).
\end{aligned}$$

This contradicts the fact that $f \in ST^{m+1,\lambda}_{\alpha,\beta}(\eta)$. Thus, the function $p(z)$ needs to satisfy $|\arg(p(z))| < \frac{pi}{2}\eta \; (z \in U)$. Hence, $f \in ST^{m,\lambda}_{\alpha,\beta}(\eta)$. Similarly, if $\arg(p(z_0)) = \frac{\pi}{2}\eta$, then we also obtain a contradiction. Hence, we obtain the required result.

Applying Theorem 2.2, we have the following corollary. □

Corollary 2.3. *$CT^{m+1,\lambda}_{\alpha,\beta}(\eta) \subset CT^{m,\lambda}_{\alpha,\beta}(\eta)$ for all $m \in \mathbb{N}$.*

Proof.

$$f \in CT^{m+1,\lambda}_{\alpha,\beta}(\eta) \Longleftrightarrow \mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z) \in \widetilde{C(\eta)} \Longleftrightarrow z(\mathrm{E}^{m+1,\lambda}_{\alpha,\beta} f(z))' \in \widetilde{S^*(\eta)}$$

$$\Longleftrightarrow \mathrm{E}_{\alpha,\beta}^{m+1,\lambda} zf'(z) \in \widetilde{S^*(\eta)} \Longleftrightarrow zf' \in ST_{\alpha,\beta}^{m+1,\lambda}(\eta) \Longrightarrow zf' \in ST_{\alpha,\beta}^{m,\lambda}(\eta)$$
$$\Longleftrightarrow \mathrm{E}_{\alpha,\beta}^{m,\lambda} zf'(z) \in \widetilde{S^*(\eta)} \Longleftrightarrow z(\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z))' \in \widetilde{S^*(\eta)} \Longleftrightarrow \mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z) \in \widetilde{C(\eta)}$$
$$\Longleftrightarrow f \in CT_{\alpha,\beta}^{m,\lambda}(\eta)$$

□

In the next results, we present the integral preserving property associated with the classes $ST_{\alpha,\beta}^{m,\lambda}(\eta)$ and $CT_{\alpha,\beta}^{m,\lambda}(\eta)$.

Consider the Bernardi integral operator

$$\mathcal{B}_b f(z) = \frac{b+1}{z^b} \int_0^z t^{b-1} f(t)dt, \quad b \in \mathbb{N},\ f \in \mathcal{A}. \tag{14}$$

This operator was first introduced and studied by Bernardi in [3].

Theorem 2.4. *Let $f \in \mathcal{A}$. If $f \in ST_{\alpha,\beta}^{m,\lambda}(\eta)$, then $\mathcal{B}_b f(z) \in ST_{\alpha,\beta}^{m,\lambda}(\eta)$.*

Proof. Let $f \in ST_{\alpha,\beta}^{m,\lambda}(\eta)$ and set

$$\frac{z(\mathrm{E}_{\alpha,\beta}^{m,\lambda} \mathcal{B}_b f(z))'}{\mathrm{E}_{\alpha,\beta}^{m,\lambda} \mathcal{B}_b f(z)} = p(z), \tag{15}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Differentiating (14) and applying the operator $\mathrm{E}_{\alpha,\beta}^{m,\lambda}$, we obtain

$$z(\mathrm{E}_{\alpha,\beta}^{m,\lambda} \mathcal{B}_b f(z))' + b\,\mathrm{E}_{\alpha,\beta}^{m,\lambda} \mathcal{B}_b f(z) = (b+1)\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z).$$

That is

$$(p(z)+b)\mathrm{E}_{\alpha,\beta}^{m,\lambda} \mathcal{B}_b f(z) = (b+1)\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z).$$

By logarithmic differentiation, we arrive

$$\frac{z(\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z))'}{\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z)} = p(z)\left(1 + \frac{zp'/p(z)}{b+p(z)}\right). \tag{16}$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\eta\,(|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| < \frac{\pi}{2}\eta.$$

If $\arg p(z_0) = -\frac{\pi}{2}\eta$, then from Lemma 1.3,

$$\begin{aligned}
&\arg\left(\frac{z_0(\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z_0))'}{\mathrm{E}_{\alpha,\beta}^{m,\lambda} f(z_0)}\right)\\
&= \arg p(z_0) + \arg\left(1 + \frac{z_0 p'(z_0)/p(z_0)}{b+p(z_0)}\right)\\
&= -\frac{\pi}{2}\eta + \arg\left(1 + \frac{ik\eta}{b + a^\eta e^{\frac{i\pi}{2}\eta}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\pi}{2}\eta + \arctan\left(\frac{k\eta(b + a^{\eta}\cos\left(\frac{\pi}{2}\eta\right))}{b^2 + 2a^{\eta}b\cos\left(\frac{\pi}{2}\eta\right) + a^{2\eta} + k\eta a^{\eta}\sin\left(\frac{\pi}{2}\eta\right)}\right)\\
&< -\frac{\pi}{2}\eta \quad \left(k \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1\right).
\end{aligned}
$$

This contradicts the definition of $f \in ST^{m,\lambda}_{\alpha,\beta}(\eta)$. Similarly, if $\arg p(z_0) = \frac{\pi}{2}\eta$, we also arrive at a contradiction. Thus, $|\arg p(z)| < \frac{\pi}{2}\eta$ for all $z \in U$. Hence, $\mathcal{B}_b f(z) \in ST^{m,\lambda}_{\alpha,\beta}(\eta)$. □

Using Theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $f \in \mathcal{A}$. If $f \in CT^{m,\lambda}_{\alpha,\beta}(\eta)$, then $\mathcal{B}_b f(z) \in CT^{m,\lambda}_{\alpha,\beta}(\eta)$.*

Proof.

$$
\begin{aligned}
&f \in CT^{m,\lambda}_{\alpha,\beta}(\eta) \Longleftrightarrow zf' \in ST^{m,\lambda}_{\alpha,\beta}(\eta) \Longrightarrow \mathcal{B}_b zf'(z) \in ST^{m,\lambda}_{\alpha,\beta}(\eta)\\
&\Longleftrightarrow z\left(\mathcal{B}_b f(z)\right)' \in ST^{m,\lambda}_{\alpha,\beta}(\eta) \Longleftrightarrow \mathcal{B}_b f(z) \in CT^{m,\lambda}_{\alpha,\beta}(\eta).
\end{aligned}
$$

□

3. Conclusion

Using the generalized Mittag-Leffler linear operator initiated in [7], we introduced the classes $\mathcal{H}^{m,\lambda,\gamma}_{\alpha,\beta,\rho}(\eta)$, $ST^{m,\lambda}_{\alpha,\beta}(\eta)$ and $CT^{m,\lambda}_{\alpha,\beta}(\eta)$. The investigations were in twofold. Firstly, we studied and proved the inclusion implications related with the classes. Also, it was shown that the classes $ST^{m,\lambda}_{\alpha,\beta}(\eta)$ and $CT^{m,\lambda}_{\alpha,\beta}(\eta)$ are invariant under the Bernardi integral operator.

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Saliu Afis received M.Sc. from University of Ilorin, Nigeria, and Ph.D. at COMSATS University Islamabad, Pakistan. He started his academic career at Gombe State University, Nigeria and currently working at the University of the Gambia, The Gambia. His research interests include Geometric function theory, and Mathematical Analysis.

Department of Mathematics, Gombe State University P.M.B 127, Tudun Wada, Gombe, Gombe State, Nigeria.
e-mail: `saliugsu@gmail.com`

Kanwal Jabeen received M.Sc. from COMSATS University Islamabad, Pakistan, and she is currently a Ph.D. student at COMSATS University Islamabad. She has been working as a senior Mathematics teacher at Bright Future Academy, Qatar. Her research interests include Geometric function theory and Mathematical Analysis.

Department of Mathematics, COMSATS University Islamabad, Park Road, Tarlai Kalan, Islamabad 45550, Pakistan.
e-mail: `kanwaljabeen575@gmail.com`

Semiu Oladipupo Oladejo received M.Sc. from University of Ilorin, Ilorin, Nigeria and Ph.D. from University of Bradford, United Kingdom. Since 2016 he has been at Gombe State

University. His research interests include Mathematical Analysis, Quantum Information and Computing, Data Science, Artificial Intelligence and Applied Statistics.

Department of Mathematics, Gombe State University P.M.B 127, Tudun Wada, Gombe, Gombe State, Nigeria.
e-mail: abdsemiu@yahoo.com

Olaide Yetunde Saka-Balogun received M.Sc. from University of Ibadan, Nigeria, and Ph.D. from University of Ilorin. She is currently working at Afe Babalola University, Nigeria. Her research interests include Complex Analysis (Geometric function theory).

Department of Mathematical and Physical Sciences, Afe Babalola University, Ado-Ekiti, Nigeria.
e-mail: balogunld@abuad.edu.ng