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LU-FACTORIZATION OF THE SQUARE-TYPE MATRIX OF THE STIRLING MATRIX

Ji-Hwan Jung

Abstract. Let $S_n = [S(i,j)]_{1\leq i,j\leq n}$ and $S^*_n = [S(i+j,j)]_{1\leq i,j\leq n}$ where $S(i,j)$ is the Stirling number of the second kind. Choi and Jo [*On the determinants of the square-type Stirling matrix and Bell matrix*, Int. J. Math. Math. Sci. 2021] obtained the diagonal entries of matrix U in the LU-factorization of S^*_n for calculating the determinant of S^*_n, where $L = S_n$.

In this paper, we compute the all entries of U in the LU-factorization of matrix S^*_n. This implies the identities related to Stirling numbers of both kinds.

1. Introduction

We denote the set of nonnegative integers by $\mathbb{N}_0$. Let $L = [\ell_{n,k}]_{n,k\in\mathbb{N}_0}$ be an infinity lower triangular matrix. Then we define a *square-type matrix* $L^* = [\ell^*_{n,k}]_{n,k\in\mathbb{N}_0}$ of L by $\ell^*_{n,k} = \ell_{n+k,k}$.

The Stirling number of the second kind counts the number of ways to partition an n elements set into k non-empty subsets and it is denoted by $S(n,k)$. The exponential generating function of $S(n,k)$ is

$$\sum_{n\geq k} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \tag{1}$$

for each $k \in \mathbb{N}_0$.

Now, consider the Stirling matrix S

$$S = [S(n+1,k+1)]_{n,k\in\mathbb{N}_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 7 & 6 & 1 & 0 & \\ 1 & 15 & 25 & 10 & 1 & \\ & & \vdots & & & \end{pmatrix}.$$

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Then we have the square-type matrix S^* of S as

$$S^* = [S(n+k+1,k+1)]_{n,k\in\mathbb{N}_0} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \\ 1 & 3 & 6 & 10 & 15 & \\ 1 & 7 & 25 & 65 & 140 & \cdots \\ 1 & 15 & 90 & 350 & 1050 & \\ 1 & 31 & 301 & 1701 & 6951 & \\ & & \vdots & & & \end{pmatrix}.$$

Many combinatorial counting problems can be treated systematically using the Riordan matrix introduced by Shapiro, Getu, Woan, and Woodson [6]. In this paper, we use some elements from the exponential version of the Riordan matrix.

Definition 1 ([1])**.** An *exponential Riordan matrix* is an infinite lower triangular matrix $L = [\ell_{n,k}]_{n,k\geq 0}$ whose kth column has the exponential generating function $g(t)f(t)^k/k!$ where $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. Equivalently, $\ell_{n,k} = n![t^n]g(t)f(t)^k/k!$ where $[t^n]\sum_{i\geq 0} a_i t^i = a_n$. The matrix R is denoted by $(g(t), f(t))$.

It is known [1] that if we multiply $R = (g(t), f(t))$ by a column vector $\mathbf{v} = (v_0, v_1, \ldots)^T$ corresponding to the exponential generating function $v(t) = \sum_{n\geq 0} v_n t^n/n!$, then the resulting column vector $R\mathbf{v} = (h_0, h_1, \ldots)^T$ has the exponential generating function $g(t)v(f(t)) = \sum_{n\geq 0} h_n t^n/n!$. This observation is known as the *fundamental theorem of Riordan matrix* (FTRM), and we write this as

$$(g(t), f(t))v(z) = g(t)\cdot(v\circ f)(t) = g(t)v(f(t))$$

where $\circ$ is a composition operator.

The FTRM implies that the set of exponential Riordan matrices forms a group under the matrix multiplication defined by

$$(g(t), f(t))(h(t), l(t)) = (g(t)h(f(t)), l(f(t))).$$

The identity element is $(1, t)$ the usual identity matrix, and the inverse of $(g(t), f(t))$ is

$$(g(t), f(t))^{-1} = \left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right) \tag{2}$$

where $\overline{f}(t)$ is the compositional inverse, i.e., $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

The importance of the exponential Riordan group is emphasized by the fact that well-known combinatorial sequences such as the Stirling numbers of both kinds, Lah numbers, Bessel numbers etc. can be expressed as exponential Riordan matrices. Moreover, exponential Riordan matrix methods give simple proofs of their identities, see [2, 4, 5] and references there in.

In 2021, Choi and Jo [3] obtained the diagonal entries of the matrix U in the LU-factorization of S^* for calculating the determinant of $S_n^* = [S(i+j,j)]_{1\le i,j\le n}$, where $L = S$.

In this paper, we compute the all entries of U in the LU-factorization of matrix S^* (see Theorem 2.2). This implies the identities in Corollaries 2.3 and 2.4.

2. Main results

Theorem 2.1. *Let $L = (g(t), f(t))$ be an exponential Riordan matrix and L^* be its square-type matrix. Then $[m_{n,k}]_{n,k\in\mathbb{N}_0} = L^{-1}L^*$ has a kth-column exponential generating function*

$$\sum_{n\ge 0} m_{n,k}\frac{t^n}{n!} = \left(\frac{\frac{d^k}{dt^k}\left(gf^k\right)}{k!g}\circ \bar{f}\right)(t)$$

where $\frac{d^k}{dt^k}$ is the kth derivative with respect to t.

Proof. Let $L = (g(t), f(t)) = [\ell_{n,k}]_{n,k\in\mathbb{N}_0}$. Then square-type matrix of L is $L^* = [\ell_{n+k,k}]_{n,k\in\mathbb{N}_0}$. Note that the kth column exponential generating function of L is $g(t)\frac{f^k(t)}{k!} = \sum_{n\ge k}\ell_{n,k}\frac{t^n}{n!}$. Thus we obtain

$$\frac{d^k}{dt^k}\left(g(t)\frac{f^k(t)}{k!}\right) = \frac{d^k}{dt^k}\sum_{n\ge k}\ell_{n,k}\frac{t^n}{n!} = \sum_{n\ge 0}\ell_{n+k,k}\frac{t^n}{n!}$$

which is the kth column exponential generating function of L^*. We denote it by $L_k^*(t) = \sum_{n\ge 0}\ell_{n+k,k}\frac{t^n}{n!}$.

Let $M = [m_{n,k}]_{n,k\in\mathbb{N}_0} = L^{-1}L^*$. Then

$$\begin{pmatrix} m_{0,k} \\ m_{1,k} \\ m_{2,k} \\ \vdots \end{pmatrix} = (g, f)^{-1}\begin{pmatrix} \ell_{k,k} \\ \ell_{k+1,k} \\ \ell_{k+2,k} \\ \vdots \end{pmatrix}.$$

Hence, by FTRM and (2), the kth column exponential generating function $M_k(t) = \sum_{n\ge 0} m_{n,k}\frac{t^n}{n!}$ of M is

$$\begin{aligned} M_k(t) = (g(t), f(t))^{-1}L_k^*(t) &= \left(\frac{1}{g(\overline{f}(t))}, \overline{f}(t)\right)\frac{d^k}{dt^k}\left(g(t)\frac{f^k(t)}{k!}\right) \\ &= \left(\frac{\frac{d^k}{dt^k}\left(gf^k\right)}{k!g}\circ\bar{f}\right)(t). \end{aligned}$$

□

If we differentiate both sides of (1) with respect to t and then substitute n and k with $n+1$ and $k+1$ respectively, then we obtain

$$\sum_{n\geq k} S(n+1,k+1)\frac{t^n}{n!} = e^t\frac{(e^t-1)^k}{k!}. \tag{3}$$

Thus, by Definition 1, the Stirling matrix S can be expressed as the exponential Riordan matrix

$$S = [S(n+1,k+1)]_{n,k\in\mathbb{N}_0} = \left(e^t, e^t-1\right). \tag{4}$$

Theorem 2.2. *The square-type matrix S^* of the Stirling matrix S has the LU-factorization $S^* = LU$ where*

$$L = S \quad \textit{and} \quad U = \left[\sum_{i=n}^{k}(-1)^{k-i}\frac{(i+1)^k}{(k-i)!(i-n)!}\right]_{n,k\in\mathbb{N}_0}.$$

Proof. Let $g(t) = e^t$ and $f(t) = e^t - 1$. Then, by binomial expansion,

$$\begin{aligned}\frac{d^k}{dt^k}g(t)f^k(t) &= \frac{d^k}{dt^k}e^t(e^t-1)^k = \frac{d^k}{dt^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{(i+1)t}\\ &= \sum_{i=0}^{k}(-1)^{k-i}(i+1)^k\binom{k}{i}e^{(i+1)t}.\end{aligned}$$

Since the compositional inverse of f is $\bar{f} = \ln(1+t)$,

$$\begin{aligned}\left(\frac{\frac{d^k}{dt^k}\left(gf^k\right)}{k!g}\circ\bar{f}\right)(t) &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}(i+1)^k\binom{k}{i}(1+t)^i\\ &= \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{k-i}\frac{(i+1)^k}{k!}\binom{k}{i}\binom{i}{j}t^j\\ &= \sum_{j=0}^{k}\sum_{i=j}^{k}(-1)^{k-i}\frac{(i+1)^k}{k!}\binom{k}{i}\binom{i}{j}t^j\\ &= \sum_{j=0}^{k}\left(\sum_{i=j}^{k}(-1)^{k-i}\frac{(i+1)^k}{(k-i)!(i-j)!}\right)\frac{t^j}{j!}. \qquad (5)\end{aligned}$$

We note $L = (g(t), f(t)) = S$ by (4). Then $L^* = S^*$. Thus, by Theorem 2.1, the kth-column exponential generating function of $U = L^{-1}L^*$ is (5). Hence we obtain the desired result. □

For instance, the first entries of the matrix U in Theorem 2.2 are

$$U = \left[\sum_{i=n}^{k} (-1)^{k-i} \frac{(i+1)^k}{(k-i)!(i-n)!} \right]_{n,k\in\mathbb{N}_0} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \\ 0 & 2 & 5 & 9 & 14 & \\ 0 & 0 & 9 & 37 & 97 & \cdots \\ 0 & 0 & 0 & 64 & 369 & \\ 0 & 0 & 0 & 0 & 625 & \\ & & \vdots & & & \end{pmatrix}.$$

By Theorem 2.2, $S^* = SU$ where U is the matrix in Theorem 2.2. Thus we obtain the following result.

Corollary 2.3. *For any nonnegative integers n and k, the Stirling number of the second kind satisfies the following identity:*

$$S(n+k+1, k+1) = \sum_{j=0}^{\min\{n,k\}} \sum_{i=j} (-1)^{k-i} \frac{(i+1)^k}{(k-i)!(i-n)!} S(n+1, j+1).$$

The Stirling number of the first kind counts the number of ways to permutations of n elements set into k cycles and is denoted by $s(n,k)$. We note

$$\sum_{i=0}^{n} (-1)^{i-k} S(n,i) s(i,k) = \delta_{n,k} \tag{6}$$

where $\delta_{n,k}$ is the Kronecker symbol. Since $S(n,0) = s(n,0) = 0$ for any positive integer n, $S^{-1} = [(-1)^{n-k} s(n+1,k+1)]_{n\in\mathbb{N}_0}$. By applying (4) to (2), S^{-1} can be expressed as the exponential Riordan matrix

$$S^{-1} = \left(\frac{1}{1+t}, \ln(1+t) \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 0 & \\ 1 & -3 & 1 & 0 & 0 & \cdots \\ -1 & 7 & -6 & 1 & 0 & \\ 1 & -15 & 25 & -10 & 1 & \\ & & \vdots & & & \end{pmatrix}.$$

By Theorem 2.2, $U = S^{-1} S^*$ where U is the matrix in Theorem 2.2. Thus we obtain the following result.

Corollary 2.4. *For any nonnegative integers n and k, we obtain the following identities*

(i) *if $k \geq n$,*

$$\sum_{i=0}^{n} (-1)^{n-i} s(n+1, i+1) S(i+k+1, k+1) = \sum_{i=n}^{k} (-1)^{k-i} \frac{(i+1)^k}{(k-i)!(i-n)!};$$

(ii) *otherwise,*

$$\sum_{i=0}^{n} (-1)^{n-i} s(n+1, i+1) S(i+k+1, k+1) = 0.$$

In particular, if $k = n$ of (i), then

$$\sum_{i=0}^{n}(-1)^{n-i}s(n+1,i+1)S(i+n+1,n+1) = (n+1)^k.$$

Remark 1. The entries in U part of the LU-factorization of S^* do not appear in the On-Line Encyclopedia of Integer Sequences (OEIS) [7]. Thus we may say that the identities in Corollaries 2.3 and 2.4 are new.

For instance, consider the first 4×4 entries of the matrix S^*. Then its LU-factorization is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 \\ 0 & 0 & 9 & 37 \\ 0 & 0 & 0 & 64 \end{pmatrix}.$$

Also we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 5 & 9 \\ 0 & 0 & 9 & 37 \\ 0 & 0 & 0 & 64 \end{pmatrix}.$$

3. Closing remark

If $L = S$ and $L^* = S^*$ in Theorem 2.1, $L^{-1}L^*$ is an upper triangular matrix by Theorem 2.2. However, in most cases, $L^{-1}L^*$ is not an upper triangular matrix for an exponential Riordan matrix L. Thus we have the following open problem.

Open problem 1. Characterize an exponential Riordan matrix L such that $L^{-1}L^*$ becomes an upper triangular matrix with nonzero diagonal entries.

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Ji-Hwan Jung
Chinju National University of Education
369-3, Jinyangho-ro, Jinju-si, Gyeongsangnam-do, Republic of Korea
Email address: jhjung@cue.ac.kr