

NON-LINEAR PRODUCT $\mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$ ON PRIME $*$ -ALGEBRAS

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ABSTRACT. In this paper, we explore the additivity of the map $\Omega : \mathcal{A} \rightarrow \mathcal{A}$ that satisfies

$$\Omega([\mathcal{L}, \mathcal{M}]_*) = [\Omega(\mathcal{M}), \mathcal{L}]_* + [\mathcal{M}, \Omega(\mathcal{L})]_*,$$

where $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$, for all $\mathcal{L}, \mathcal{M} \in \mathcal{A}$, a prime $*$ -algebra with unit \mathcal{I} . Additionally we show that if $\Omega(\alpha\mathcal{I})$ is self-adjoint operator for $\alpha \in \{1, i\}$, then $\Omega = 0$.

1. Introduction

Let \mathcal{A} be a $*$ -algebra. For any $\mathcal{L}, \mathcal{M} \in \mathcal{A}$, the $*$ -Jordan ($*$ -Lie) product presented as $\mathcal{L} \diamond \mathcal{M} = \mathcal{L}\mathcal{M} + \mathcal{M}\mathcal{L}^*$ ($[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}^*$). These products have garnered a significant amount of attention, and allusions show a widening interest in literature [1–6, 9, 11]. Notice that a map $\Omega : \mathcal{A} \rightarrow \mathcal{A}$ is referred to as a reverse derivation if $\Omega(\mathcal{L} + \mathcal{M}) = \Omega(\mathcal{L}) + \Omega(\mathcal{M})$ and $\Omega(\mathcal{L}\mathcal{M}) = \Omega(\mathcal{M})\mathcal{L} + \mathcal{M}\Omega(\mathcal{L})$ for all $\mathcal{L}, \mathcal{M} \in \mathcal{A}$. A map Ω is additive $*$ -derivation if it is an additive derivation and $\Omega(\mathcal{L}^*) = \Omega(\mathcal{L})^*$. Define a new product $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$. In 2020, Taghavi and Razeghi [10] studied $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}^*\mathcal{M} - \mathcal{M}^*\mathcal{L}$ product on $*$ -algebras. In particular, they established that the map $\Omega : \mathcal{A} \rightarrow \mathcal{A}$ satisfying $\Omega(\mathcal{L}\mathcal{M}) = \Omega(\mathcal{L})\mathcal{M} + \mathcal{L}\Omega(\mathcal{M})$ is a $*$ -derivation. The study of non-linear preserving problems is one of the premier areas in matrix theory as well as operator theory. It was Martindale [7] who first asked the question that when are multiplicative/nonadditive maps additive? He answered his question for a multiplicative isomorphism of a ring under the existence of a family of idempotent elements of rings which satisfies some conditions. This spawned a wealth of diverse methods on different algebraic structures like operator algebras, von Neumann algebras, Banach algebras etcetera to establish a variety of interesting results concerning nonadditive maps. Possibly the most obvious approach is the method of algebraic decompositions. A wealth of fundamentally different methods to deal with nonlinear mappings can be found in [1–8, 11] and references therein.

Our main objective of this manuscript is to explore the structure of a non-linear maps on prime $*$ -algebras satisfying $\Omega([\mathcal{L}, \mathcal{M}]_*) = [\Omega(\mathcal{M}), \mathcal{L}]_* + [\mathcal{M}, \Omega(\mathcal{L})]_*$, where $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$, for all $\mathcal{L}, \mathcal{M} \in \mathcal{A}$. We systematize the proof of above theorem in two parts. Firstly, we prove the additivity of Ω by using several claims.

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Secondly, we shall provide numerous constructive facts to elaborate the assertion of our main theorem.

2. Main results

THEOREM 2.1. *Let \mathcal{A} be a prime $*$ -algebra with unit \mathcal{I} and a nontrivial projection. Then the map $\Omega : \mathcal{A} \rightarrow \mathcal{A}$ that satisfies*

$$(2.1) \quad \Omega([\mathcal{L}, \mathcal{M}]_*) = [\Omega(\mathcal{M}), \mathcal{L}]_* + [\mathcal{M}, \Omega(\mathcal{L})]_*,$$

where $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$, for all $\mathcal{L}, \mathcal{M} \in \mathcal{A}$ is additive.

Proof. Let \mathcal{P}_1 be a nontrivial projection in \mathcal{A} and $\mathcal{P}_2 = \mathcal{I}_{\mathcal{A}} - \mathcal{P}_1$. Denote $\mathcal{A}_{ij} = \mathcal{P}_i\mathcal{A}\mathcal{P}_j$, $i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^2 \mathcal{A}_{ij}$. For every $\mathcal{L} \in \mathcal{A}$, we may write $\mathcal{L} = \mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}$. From now on, if we mention \mathcal{L}_{ij} , it means that $\mathcal{L}_{ij} \in \mathcal{A}_{ij}$. To illustrate the additivity of Ω on \mathcal{A} , we take the aforementioned partition of \mathcal{A} and present several claims that prove Ω is additive on each \mathcal{A}_{ij} , $i, j = 1, 2$.

Several claims are used to prove the preceding theorem.

Claim 1. $\Omega(0) = 0$.

Take $\mathcal{L} = \mathcal{M} = 0$, then

$$\Omega(0) = \Omega([0, 0]_*) = [\Omega(0), 0]_* + [0, \Omega(0)]_* = 0.$$

Claim 2. $\Omega(i\mathcal{L}) = i\Omega(\mathcal{L}) + \mathcal{L}^*K$, where $K = \Omega(i\mathcal{I}) - i\Omega(\mathcal{I})$.

Consider

$$\Omega([-i\mathcal{L}, \mathcal{I}]_*) = \Omega([\mathcal{L}, i\mathcal{I}]_*).$$

So, we have

$$(2.2) \quad \begin{aligned} [\Omega(\mathcal{I}), (-i\mathcal{L})]_* + [\mathcal{I}, \Omega(-i\mathcal{L})]_* &= [\Omega(i\mathcal{I}), \mathcal{L}]_* + [(i\mathcal{I}), \Omega(\mathcal{L})]_* \\ i\mathcal{L}^*\Omega(\mathcal{I}) + i\mathcal{L}\Omega(\mathcal{I})^* + \Omega(-i\mathcal{L})^* &- \Omega(-i\mathcal{L}) \\ &= \mathcal{L}^*\Omega(i\mathcal{I}) - \Omega(i\mathcal{I})^*\mathcal{L} + i\Omega(\mathcal{L})^* + i\Omega(\mathcal{L}). \end{aligned}$$

Consider

$$\Omega([-i\mathcal{L}, i\mathcal{I}]_*) = \Omega([\mathcal{I}, \mathcal{L}]_*).$$

So, we have

$$(2.3) \quad \begin{aligned} [\Omega(i\mathcal{I}), (-i\mathcal{L})]_* + [(i\mathcal{I}), \Omega(-i\mathcal{L})]_* &= [\Omega(\mathcal{L}), \mathcal{I}]_* + [\mathcal{L}, \Omega(\mathcal{I})]_* \\ i\mathcal{L}^*\Omega(i\mathcal{I}) + i\mathcal{L}\Omega(i\mathcal{I})^* + i\Omega(-i\mathcal{L})^* &+ i\Omega(-i\mathcal{L}) \\ &= \Omega(\mathcal{L}) - \Omega(\mathcal{L})^* + \Omega(\mathcal{I})^*\mathcal{L} - \mathcal{L}^*\Omega(\mathcal{I}). \end{aligned}$$

Equivalently we obtain

$$(2.3) \quad \begin{aligned} -\mathcal{L}^*\Omega(i\mathcal{I}) - \mathcal{L}\Omega(i\mathcal{I})^* - \Omega(-i\mathcal{L})^* &- \Omega(-i\mathcal{L}) \\ &= i\Omega(\mathcal{L}) - i\Omega(\mathcal{L})^* + i\Omega(\mathcal{I})^*\mathcal{L} - i\mathcal{L}^*\Omega(\mathcal{I}). \end{aligned}$$

By adding equations (2.2) and (2.3), we have

$$i\Omega(\mathcal{L}) + \Omega(-i\mathcal{L}) = i\mathcal{L}^*\Omega(\mathcal{I}) - \mathcal{L}^*\Omega(i\mathcal{I}).$$

Substituting $i\mathcal{L}$ instead of \mathcal{L} in the above equation, we get

$$i\Omega(i\mathcal{L}) + \Omega(\mathcal{L}) = \mathcal{L}^*\Omega(\mathcal{I}) + i\mathcal{L}^*\Omega(i\mathcal{I})$$

$$\Omega(i\mathcal{L}) = i\Omega(\mathcal{L}) + \mathcal{L}^*(\Omega(i\mathcal{I}) - i\Omega(\mathcal{I}))$$

So $\Omega(i\mathcal{L}) = i\Omega(\mathcal{L}) + \mathcal{L}^*K$, where $K = \Omega(i\mathcal{I}) - i\Omega(\mathcal{I})$.

Claim 3. $\Omega(-\mathcal{L}) = -\Omega(\mathcal{L})$.

By considering $\Omega(i\mathcal{L}) = i\Omega(\mathcal{L}) + \mathcal{L}^*K$ and applying $i\mathcal{L}$ instead of \mathcal{L} , we have

$$\begin{aligned} \Omega(-\mathcal{L}) &= i\Omega(i\mathcal{L}) - i\mathcal{L}^*K \\ \Omega(-\mathcal{L}) &= i(i\Omega(\mathcal{L}) + \mathcal{L}^*K) - i\mathcal{L}^*K \\ \Omega(-\mathcal{L}) &= -\Omega(\mathcal{L}) + i\mathcal{L}^*K - i\mathcal{L}^*K \\ (2.4) \quad \Omega(-\mathcal{L}) &= -\Omega(\mathcal{L}). \end{aligned}$$

Claim 4. For each $\mathcal{L}_{11} \in \mathcal{A}_{11}, \mathcal{L}_{12} \in \mathcal{A}_{12}$, we have

$$\Omega(\mathcal{L}_{11} + \mathcal{L}_{12}) = \Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}).$$

Let $\mathcal{T} = \Omega(\mathcal{L}_{11} + \mathcal{L}_{12}) - \Omega(\mathcal{L}_{11}) - \Omega(\mathcal{L}_{12})$, we will prove that $\mathcal{T} = 0$. For $\mathcal{X}_{21} \in \mathcal{A}_{21}$, we can write that

$$\begin{aligned} &[\Omega(\mathcal{X}_{21}), (\mathcal{L}_{11} + \mathcal{L}_{12})]_* + [\mathcal{X}_{21}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12})]_* \\ &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{21}]_*) \\ &= \Omega([\mathcal{L}_{11}, \mathcal{X}_{21}]_*) + \Omega([\mathcal{L}_{12}, \mathcal{X}_{21}]_*) \\ &= [\Omega(\mathcal{X}_{21}), \mathcal{L}_{11}]_* + [\mathcal{X}_{21}, \Omega(\mathcal{L}_{11})]_* + [\Omega(\mathcal{X}_{21}), \mathcal{L}_{12}]_* + [\mathcal{X}_{21}, \Omega(\mathcal{L}_{12})]_* \\ &= [\mathcal{X}_{21}, (\Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}))]_* + [\Omega(\mathcal{X}_{21}), (\mathcal{L}_{11} + \mathcal{L}_{12})]_* \end{aligned}$$

So, we obtain

$$[\mathcal{X}_{21}, \mathcal{T}]_* = 0.$$

Since $\mathcal{T} = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{21} + \mathcal{T}_{22}$, we have

$$\mathcal{X}_{21}\mathcal{T}_{21}^* + \mathcal{X}_{21}\mathcal{T}_{11}^* - \mathcal{T}_{21}\mathcal{X}_{21}^* - \mathcal{T}_{11}\mathcal{X}_{21}^* = 0.$$

From the above equation and primeness of \mathcal{A} , we have $\mathcal{T}_{11} = 0$, and

$$(2.5) \quad \mathcal{X}_{21}\mathcal{T}_{21}^* - \mathcal{T}_{21}\mathcal{X}_{21}^* = 0.$$

Alternatively, by substituting $i\mathcal{X}_{21}$ for \mathcal{X}_{21} in the preceding equation, we obtain

$$(2.6) \quad -\mathcal{X}_{21}\mathcal{T}_{21}^* - \mathcal{T}_{21}\mathcal{X}_{21}^* = 0.$$

From (2.5) and (2.6), we get $\mathcal{T}_{21}\mathcal{X}_{21}^* = 0$. Since \mathcal{A} is prime, then $\mathcal{T}_{21} = 0$. It suffices to show that $\mathcal{T}_{12} = \mathcal{T}_{22} = 0$. For this purpose take $\mathcal{X}_{12} \in \mathcal{A}_{12}$, we write

$$\begin{aligned} &\Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{12}]_*, \mathcal{P}_1)_* \\ &= [\Omega(\mathcal{P}_1), [(\mathcal{L}_{11} + \mathcal{L}_{12}), \mathcal{X}_{12}]_*]_* + [\mathcal{P}_1, \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{12}]_*)]_* \\ &= [\Omega(\mathcal{P}_1), [(\mathcal{L}_{11} + \mathcal{L}_{12}), \mathcal{X}_{12}]_*]_* + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12})]_*]_* \\ &\quad + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), (\mathcal{L}_{11} + \mathcal{L}_{12})]_*]_* \\ &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*]_* + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*]_* \\ &\quad + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{11}]_*]_* + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{12}]_*]_* \\ &\quad + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12})]_*]_* \end{aligned}$$

So, we showed that

$$(2.7) \quad \begin{aligned} \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{12}]_*, \mathcal{P}_1)_* &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_* \\ &\quad + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{11}]_*)_* + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{12}]_*)_* \\ &\quad + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12})]_*)_*. \end{aligned}$$

Since $[[\mathcal{L}_{12}, \mathcal{X}_{12}]_*, \mathcal{P}_1]_* = 0$, we hvae

$$\begin{aligned} &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\mathcal{P}_1, \Omega([\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_* \\ &\quad + [\mathcal{P}_1, \Omega([\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_*]_* \\ &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\mathcal{P}_1, ([\Omega(\mathcal{X}_{12}), \mathcal{L}_{11}]_* + [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11})]_*]_* \\ &\quad + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_* + [\mathcal{P}_1, ([\mathcal{X}_{12}, \Omega(\mathcal{L}_{12})]_* + [\Omega(\mathcal{X}_{12}), \mathcal{L}_{12}]_*)_*]_* \\ &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_* + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{11}]_*)_* \\ &\quad + [\mathcal{P}_1, \Omega(\mathcal{X}_{12}), [\mathcal{L}_{12}]_*)_* + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11})]_*)_* + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{12})]_*)_*. \end{aligned}$$

Therefore,

$$(2.8) \quad \begin{aligned} \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{12}]_*, \mathcal{P}_1)_* &= [\Omega(\mathcal{P}_1), [\mathcal{L}_{11}, \mathcal{X}_{12}]_*)_* + [\Omega(\mathcal{P}_1), [\mathcal{L}_{12}, \mathcal{X}_{12}]_*)_* \\ &\quad + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{12}]_*)_* + [\mathcal{P}_1, [\Omega(\mathcal{X}_{12}), \mathcal{L}_{12}]_*)_* \\ &\quad + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11})]_*)_* + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{12})]_*)_*. \end{aligned}$$

From (2.7) and (2.8), we have

$$[\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12})]_*)_* = [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11})]_*)_* + [\mathcal{P}_1, [\mathcal{X}_{12}, \Omega(\mathcal{L}_{12})]_*)_*.$$

It follows that $[\mathcal{P}_1, [\mathcal{X}_{12}, \mathcal{T}]_*)_* = 0$, so $\mathcal{X}_{12}\mathcal{T}_{22}^* - \mathcal{T}_{22}\mathcal{X}_{12}^* = 0$. We have $\mathcal{T}_{22}\mathcal{X}_{12}^* = 0$ or $\mathcal{P}_1\mathcal{X}\mathcal{T}_{22} = 0$ for all $\mathcal{X} \in \mathcal{A}$, then we have $\mathcal{T}_{22} = 0$. Similarly, we can show that $\mathcal{T}_{12} = 0$ by applying \mathcal{P}_2 instead of \mathcal{P}_1 in above.

Claim 5. For each $\mathcal{L}_{11} \in \mathcal{A}_{11}, \mathcal{L}_{12} \in \mathcal{A}_{12}, \mathcal{L}_{21} \in \mathcal{A}_{21}$ and $\mathcal{L}_{22} \in \mathcal{A}_{22}$, we have

$$\Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}) = \Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21})$$

$$\Omega(\mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) = \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}) + \Omega(\mathcal{L}_{22}).$$

We show that $\mathcal{T} = \Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}) - \Omega(\mathcal{L}_{11}) - \Omega(\mathcal{L}_{12}) - \Omega(\mathcal{L}_{21}) = 0$. For $\mathcal{X}_{21} \in \mathcal{A}_{21}$, we have

$$\begin{aligned} [\Omega(\mathcal{X}_{21}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_* &+ [\mathcal{X}_{21}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_* \\ &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}, \mathcal{X}_{21}]_*)_* \\ &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}, \mathcal{X}_{21}]_*)_* + \Omega([\mathcal{L}_{21}, \mathcal{X}_{21}]_*)_* \\ &= \Omega([\mathcal{L}_{11}, \mathcal{X}_{21}]_*)_* + \Omega([\mathcal{L}_{12}, \mathcal{X}_{21}]_*)_* + \Omega([\mathcal{L}_{21}, \mathcal{X}_{21}]_*)_* \\ &= [\mathcal{X}_{21}, (\Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}))]_* \\ &\quad + [\Omega(\mathcal{X}_{21}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_*. \end{aligned}$$

It follows that $[\mathcal{X}_{21}, \mathcal{T}]_* = 0$. Since $\mathcal{T} = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{21} + \mathcal{T}_{22}$, we have

$$\mathcal{X}_{21}\mathcal{T}_{21}^* + \mathcal{X}_{21}\mathcal{T}_{11}^* - \mathcal{T}_{21}\mathcal{X}_{21}^* - \mathcal{T}_{11}\mathcal{X}_{21}^* = 0.$$

Therefore, $\mathcal{T}_{11} = \mathcal{T}_{21} = 0$.

From Claim 4, we obtain

$$\begin{aligned}
 [\Omega(\mathcal{X}_{12}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_* &+ [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_* \\
 &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}], \mathcal{X}_{12}]_* \\
 &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{21}], \mathcal{X}_{12}]_* \\
 &= \Omega([\mathcal{L}_{11}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{12}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{21}], \mathcal{X}_{12}]_* \\
 &= [\mathcal{X}_{12}, (\Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}))]_* \\
 &\quad + [\Omega(\mathcal{X}_{12}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21})]_*.
 \end{aligned}$$

Hence,

$$\mathcal{X}_{12}^* \mathcal{T}_{12} + \mathcal{X}_{12}^* \mathcal{T}_{11} - \mathcal{T}_{12}^* \mathcal{X}_{12} - \mathcal{T}_{11}^* \mathcal{X}_{12} = 0.$$

Then, $\mathcal{T}_{11} = \mathcal{T}_{12} = 0$. Similarly

$$\Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}) = \Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}).$$

Claim 6. For each $\mathcal{L}_{11} \in \mathcal{A}_{11}$, $\mathcal{L}_{12} \in \mathcal{A}_{12}$, $\mathcal{L}_{21} \in \mathcal{A}_{21}$ and $\mathcal{L}_{22} \in \mathcal{A}_{22}$, we have

$$\Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) = \Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}) + \Omega(\mathcal{L}_{22}).$$

We show that

$$\mathcal{T} = \Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) - \Omega(\mathcal{L}_{11}) - \Omega(\mathcal{L}_{12}) - \Omega(\mathcal{L}_{21}) - \Omega(\mathcal{L}_{22}) = 0.$$

From Claim 5, we have

$$\begin{aligned}
 &[\Omega(\mathcal{X}_{12}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22})]_* + [\mathcal{X}_{12}, \Omega(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22})]_* \\
 &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}], \mathcal{X}_{12}]_* \\
 &= \Omega([\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{22}], \mathcal{X}_{12}]_* \\
 &= \Omega([\mathcal{L}_{11}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{12}], \mathcal{X}_{12}]_* + \Omega([\mathcal{L}_{21}], \mathcal{X}_{12}]_* \\
 &\quad + \Omega([\mathcal{L}_{22}], \mathcal{X}_{12}]_* \\
 &= [\mathcal{X}_{12}, (\Omega(\mathcal{L}_{11}) + \Omega(\mathcal{L}_{12}) + \Omega(\mathcal{L}_{21}) + \Omega(\mathcal{L}_{22}))]_* \\
 &\quad + [\Omega(\mathcal{X}_{12}), (\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22})]_*.
 \end{aligned}$$

So, $[\mathcal{X}_{12}, \mathcal{T}]_* = 0$. It follows that

$$\mathcal{X}_{12} \mathcal{T}_{12}^* + \mathcal{X}_{12} \mathcal{T}_{22}^* - \mathcal{T}_{12}^* \mathcal{X}_{12} - \mathcal{T}_{22}^* \mathcal{X}_{12} = 0.$$

Then $\mathcal{T}_{22} = \mathcal{T}_{12} = 0$

Similarly, by applying \mathcal{X}_{21} instead of \mathcal{X}_{12} in above, we obtain $\mathcal{T}_{22} = \mathcal{T}_{21} = 0$.

Claim 7. For each $\mathcal{L}_{ij}, \mathcal{M}_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$, we have

$$\Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) = \Omega(\mathcal{L}_{ij}) + \Omega(\mathcal{M}_{ij}).$$

It is easy to show that

$$(\mathcal{P}_i + \mathcal{L}_{ij})(\mathcal{P}_j + \mathcal{M}_{ij}) - (\mathcal{P}_j + \mathcal{M}_{ij}^*)(\mathcal{P}_i + \mathcal{L}_{ij}^*) = \mathcal{L}_{ij} + \mathcal{M}_{ij} - \mathcal{L}_{ij}^* - \mathcal{M}_{ij}^*.$$

So, we can write

$$\begin{aligned}
& \Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) + \Omega(-\mathcal{L}_{ij}^* - \mathcal{M}_{ji}^*) \\
&= \Omega([\mathcal{P}_i + \mathcal{L}_{ij}], (\mathcal{P}_j + \mathcal{M}_{ij}^*)_*]_* \\
&= [\Omega(\mathcal{P}_j + \mathcal{M}_{ij}^*), (\mathcal{P}_i + \mathcal{L}_{ij})_*]_* + [(\mathcal{P}_j + \mathcal{M}_{ij}^*), \Omega(\mathcal{P}_i + \mathcal{L}_{ij})_*]_* \\
&= [(\Omega(\mathcal{P}_j) + \Omega(\mathcal{M}_{ij}^*)), (\mathcal{P}_i + \mathcal{L}_{ij})_*]_* \\
&\quad + [(\mathcal{P}_j + \mathcal{M}_{ij}^*), (\Omega(\mathcal{P}_i) + \Omega(\mathcal{L}_{ij}))_*]_* \\
&= [\Omega(\mathcal{M}_{ij}^*), \mathcal{P}_i]_* + [\mathcal{M}_{ij}^*, \Omega(\mathcal{P}_i)]_* + [\Omega(\mathcal{P}_j), \mathcal{L}_{ij}]_* + [\mathcal{P}_j, \Omega(\mathcal{L}_{ij})_*]_* \\
&= \Omega([\mathcal{P}_i, \mathcal{M}_{ij}^*]_*) + \Omega([\mathcal{L}_{ij}, \mathcal{P}_j]_*) \\
&= \Omega(\mathcal{M}_{ij}) - \Omega(\mathcal{M}_{ij}^*) + \Omega(\mathcal{L}_{ij}) - \Omega(\mathcal{L}_{ij}^*)
\end{aligned}$$

Therefore, we show that

$$(2.9) \quad \Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) + \Omega(-\mathcal{L}_{ij}^* - \mathcal{M}_{ji}^*) = \Omega(\mathcal{M}_{ij}) - \Omega(\mathcal{M}_{ij}^*) - \Omega(\mathcal{L}_{ij}^*) + \Omega(\mathcal{L}_{ij})$$

By an easy computation, we can write

$$(i\mathcal{P}_i + i\mathcal{L}_{ij})(\mathcal{P}_j + \mathcal{M}_{ij}) - (\mathcal{P}_j + \mathcal{M}_{ij}^*)(-i\mathcal{P}_i - i\mathcal{L}_{ij}^*) = i\mathcal{L}_{ij} + i\mathcal{M}_{ij} + i\mathcal{L}_{ij}^* + i\mathcal{M}_{ij}^*.$$

Then, we have

$$\begin{aligned}
& \Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) + \Omega(i\mathcal{L}_{ij}^* + i\mathcal{M}_{ji}^*) \\
&= \Omega([(i\mathcal{P}_i + i\mathcal{L}_{ij}), (\mathcal{P}_j + \mathcal{M}_{ij}^*)_*]_* \\
&= [\Omega(\mathcal{P}_j + \mathcal{M}_{ij}^*), (i\mathcal{P}_i + i\mathcal{L}_{ij})_*]_* + [(\mathcal{P}_j + \mathcal{M}_{ij}^*), \Omega(i\mathcal{P}_i + i\mathcal{L}_{ij})_*]_* \\
&= [(\Omega(\mathcal{P}_j) + \Omega(\mathcal{M}_{ij}^*)), (i\mathcal{P}_i + i\mathcal{L}_{ij})_*]_* \\
&\quad + [(\mathcal{P}_j + \mathcal{M}_{ij}^*), (\Omega(i\mathcal{P}_i) + \Omega(i\mathcal{L}_{ij}))_*]_* \\
&= [\Omega(\mathcal{M}_{ij}^*), i\mathcal{P}_i]_* + [\mathcal{M}_{ij}^*, \Omega(i\mathcal{P}_i)]_* + [\Omega(\mathcal{P}_j), i\mathcal{L}_{ij}]_* + [\mathcal{P}_j, \Omega(i\mathcal{L}_{ij})_*]_* \\
&= \Omega([i\mathcal{P}_i, \mathcal{M}_{ij}^*]_*) + \Omega([i\mathcal{L}_{ij}, \mathcal{P}_j]_*) \\
&= \Omega(i\mathcal{M}_{ij}) + \Omega(i\mathcal{M}_{ij}^*) + \Omega(\mathcal{L}_{ij}) + \Omega(i\mathcal{L}_{ij}^*)
\end{aligned}$$

We showed that

$$\Omega(i\mathcal{L}_{ij} + i\mathcal{M}_{ij}) + \Omega(i\mathcal{L}_{ij}^* + i\mathcal{M}_{ji}^*) = \Omega(i\mathcal{M}_{ij}) + \Omega(i\mathcal{M}_{ij}^*) + \Omega(i\mathcal{L}_{ij}^*) + \Omega(i\mathcal{L}_{ij})$$

From Claims 2, 3 and the above equation, we have

$$(2.10) \quad \Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) + \Omega(\mathcal{L}_{ij}^* + \mathcal{M}_{ji}^*) = \Omega(\mathcal{M}_{ij}) + \Omega(\mathcal{M}_{ij}^*) + \Omega(\mathcal{L}_{ij}^*) + \Omega(\mathcal{L}_{ij}).$$

By adding equations (2.10) and (2.9), we obtain

$$\Omega(\mathcal{L}_{ij} + \mathcal{M}_{ij}) = \Omega(\mathcal{L}_{ij}) + \Omega(\mathcal{M}_{ij}).$$

Claim 8. For each $\mathcal{L}_{ii}, \mathcal{M}_{ii} \in \mathcal{A}_{ii}$ such that $1 \leq i \leq 2$, we have

$$\Omega(\mathcal{L}_{ii} + \mathcal{M}_{ii}) = \Omega(\mathcal{L}_{ii}) + \Omega(\mathcal{M}_{ii}).$$

We show that $\mathcal{T} = \Omega(\mathcal{L}_{ii} + \mathcal{M}_{ii}) - \Omega(\mathcal{L}_{ii}) - \Omega(\mathcal{M}_{ii}) = 0$. We can write that

$$\begin{aligned} & [\Omega(\mathcal{P}_j), (\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* + [\mathcal{P}_j, \Omega(\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* \\ &= \Omega([\mathcal{L}_{ii} + \mathcal{M}_{ii}, \mathcal{P}_j]_*) \\ &= \Omega([\mathcal{L}_{ii}, \mathcal{P}_j]_*) + \Omega([\mathcal{M}_{ii}, \mathcal{P}_j]_*) \\ &= [\Omega(\mathcal{P}_j), \mathcal{L}_{ii}]_* + [\mathcal{P}_j, \Omega(\mathcal{L}_{ii})]_* + [\Omega(\mathcal{P}_j), \mathcal{M}_{ii}]_* \\ &\quad + [\mathcal{P}_j, \Omega(\mathcal{M}_{ii})]_* \\ &= [\Omega(\mathcal{P}_j), (\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* + [\mathcal{P}_j, (\Omega(\mathcal{L}_{ii}) + \Omega(\mathcal{M}_{ii}))]_* \end{aligned}$$

So, we have

$$[\mathcal{P}_j, \mathcal{T}]_* = 0.$$

Therefore, we obtain $\mathcal{T}_{ij} = \mathcal{T}_{ji} = \mathcal{T}_{jj} = 0$. On the other hand, for every $\mathcal{X}_{ji} \in \mathcal{A}_{ji}$, we have

$$\begin{aligned} & [\Omega(\mathcal{X}_{ji}), (\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* + [\mathcal{X}_{ij}, \Omega(\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* \\ &= \Omega([\mathcal{L}_{ii} + \mathcal{M}_{ii}, \mathcal{X}_{ji}]_*) \\ &= \Omega([\mathcal{L}_{ii}, \mathcal{X}_{ji}]_*) + \Omega([\mathcal{M}_{ii}, \mathcal{X}_{ij}]_*) \\ &= [\Omega(\mathcal{X}_{ji}), \mathcal{L}_{ii}]_* + [\mathcal{X}_{ji}, \Omega(\mathcal{L}_{ii})]_* + [\Omega(\mathcal{X}_{ji}), \mathcal{M}_{ii}]_* \\ &\quad + [\mathcal{X}_{ji}, \Omega(\mathcal{M}_{ii})]_* \\ &= [\Omega(\mathcal{X}_{ji}), (\mathcal{L}_{ii} + \mathcal{M}_{ii})]_* + [\mathcal{X}_{ji}, (\Omega(\mathcal{L}_{ii}) + \Omega(\mathcal{M}_{ii}))]_* \end{aligned}$$

So,

$$[(\Omega(\mathcal{L}_{ii} + \mathcal{M}_{ii}) - \Omega(\mathcal{L}_{ii}) - \Omega(\mathcal{M}_{ii})), \mathcal{X}_{ji}]_* = 0.$$

It follows that $[\mathcal{X}_{ji}, \mathcal{T}]_* = 0$ or $\mathcal{X}_{ji}\mathcal{T}_{ii} = 0$. By knowing that \mathcal{A} is prime, we have $\mathcal{T}_{ii} = 0$. Hence, the additivity of Ω comes from the above claims. \square

In the rest of this paper we show that $\Omega = 0$.

THEOREM 2.2. *Taking reference to the preceding theorem, if $\Omega(\alpha\mathcal{F})$ is self-adjoint operator for $\alpha \in \{1, i\}$, then $\Omega = 0$.*

Proof. Several claims are used to verify the above theorem.

Claim 9. $\Omega(i\mathcal{F}) = \Omega(\mathcal{F}) = 0$.

Consider $\Omega([i\mathcal{F}, \mathcal{F}]_*) = [\Omega(\mathcal{F}), i\mathcal{F}]_* + [\mathcal{F}, \Omega(i\mathcal{F})]_*$ that imply

$$\begin{aligned} \Omega(2i\mathcal{F}) &= -i\Omega(\mathcal{F}) - i\Omega(\mathcal{F})^* + \Omega(i\mathcal{F})^* - \Omega(i\mathcal{F}) \\ (2.11) \quad 2\Omega(i\mathcal{F}) &= -2i\Omega(\mathcal{F}) \end{aligned}$$

By taking the adjoint of above equation we have $\Omega(i\mathcal{F}) = \Omega(\mathcal{F}) = 0$.

Claim 10. Ω preserves $*$.

Since $\Omega(i\mathcal{F}) = \Omega(\mathcal{F}) = 0$, then we can write

$$\begin{aligned} \Omega([\mathcal{F}, (i\mathcal{L})]_*) &= [\Omega(i\mathcal{L}), \mathcal{F}]_* + [(i\mathcal{L}), \Omega(\mathcal{F})]_* \\ \Omega(i\mathcal{L} + i\mathcal{L}^*) &= \Omega(i\mathcal{L})^* - \Omega(i\mathcal{L}) \end{aligned}$$

Substiting $i\mathcal{L}$ instead of \mathcal{L} in the above equation, we get

$$(2.12) \quad \Omega(\mathcal{L}^* - \mathcal{L}) = \Omega(\mathcal{L}) - \Omega(\mathcal{L})^*$$

Replace \mathcal{L} by \mathcal{L}^* in (2.12), we have

$$(2.13) \quad \Omega(\mathcal{L} - \mathcal{L}^*) = \Omega(\mathcal{L}^*) - \Omega(\mathcal{L}^*)^*$$

Adding (2.12) and (2.13), we get

$$(2.14) \quad \begin{aligned} \Omega(0) &= \Omega(\mathcal{L}) - \Omega(\mathcal{L})^* + \Omega(\mathcal{L}^*) - \Omega(\mathcal{L}^*)^* \\ 0 &= \Omega(\mathcal{L} + \mathcal{L}^*) - \Omega(\mathcal{L})^* - \Omega(\mathcal{L}^*)^* \end{aligned}$$

Replace \mathcal{L} by $i\mathcal{L}$ in (2.14), we obtain

$$(2.15) \quad \begin{aligned} 0 &= \Omega(i\mathcal{L} - i\mathcal{L}^*) - \Omega(i\mathcal{L})^* - \Omega(-i\mathcal{L}^*)^* \\ 0 &= i\Omega(\mathcal{L} - \mathcal{L}^*) + i\Omega(\mathcal{L})^* + i\Omega(-\mathcal{L}^*)^* \\ 0 &= \Omega(\mathcal{L} - \mathcal{L}^*) + \Omega(\mathcal{L})^* - \Omega(\mathcal{L}^*)^* \end{aligned}$$

By adding (2.14) and (2.15), we obtain

$$\Omega(\mathcal{L}) = \Omega(\mathcal{L}^*)^*$$

Therefore, Ω preserves $*$.

Claim 11. We prove that $\Omega = 0$.

For every $\mathcal{L}, \mathcal{M} \in \mathcal{A}$, we have

$$(2.16) \quad \begin{aligned} \Omega(\mathcal{L}\mathcal{M} - \mathcal{M}^*\mathcal{L}^*) &= \Omega([\mathcal{L}, \mathcal{M}^*]_*) \\ &= [\Omega(\mathcal{M}^*), \mathcal{L}]_* + [\mathcal{M}^*, \Omega(\mathcal{L})]_* \\ &= \Omega(\mathcal{M}^*)\mathcal{L}^* - \mathcal{L}\Omega(\mathcal{M}^*)^* + \mathcal{M}^*\Omega(\mathcal{L})^* - \Omega(\mathcal{L})\mathcal{M} \\ \Omega(\mathcal{L}\mathcal{M} - \mathcal{M}^*\mathcal{L}^*) &= \Omega(\mathcal{M})^*\mathcal{L}^* - \mathcal{L}\Omega(\mathcal{M}) + \mathcal{M}^*\Omega(\mathcal{L})^* - \Omega(\mathcal{L})\mathcal{M} \end{aligned}$$

Replace \mathcal{M} by $i\mathcal{M}$ in (2.16) and using Claims 2 and 9, we obtain

$$(2.17) \quad \Omega(\mathcal{L}\mathcal{M} + \mathcal{M}^*\mathcal{L}^*) = -\Omega(\mathcal{M})^*\mathcal{L}^* - \mathcal{L}\Omega(\mathcal{M}) - \mathcal{M}^*\Omega(\mathcal{L})^* - \Omega(\mathcal{L})\mathcal{M}$$

By adding (2.16) and (2.17), we have

$$\Omega(\mathcal{L}\mathcal{M}) = -\mathcal{L}\Omega(\mathcal{M}) - \Omega(\mathcal{L})\mathcal{M}$$

Taking $\mathcal{M} = \mathcal{I}$, we see that $\Omega(\mathcal{L}) = -\Omega(\mathcal{L})$ which gives $\Omega(\mathcal{L}) = 0$ and hence $\Omega = 0$. This completes the proof. \square

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