Korean J. Math. **31** (2023), No. 3, pp. 259–267
https://dx.doi.org/10.11568/kjm.2023.31.3.259

PARTIAL SUMS AND NEIGHBORHOODS OF JANOWSKI-TYPE SUBCLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. The paper presents the introduction of a novel linear derivative operator for meromorphic functions that are linked with q-calculus. Using the linear derivative operator, a new category of meromorphic functions is generated in the paper. We obtain sufficient conditions and show some properties of functions belonging to these subclasses. The partial sums of its sequence and the q-neighborhoods problem are solved.

1. Introduction

Quantum calculus known as q-calculus is sometimes described as limitless calculus. It substitutes a difference operator for the classical derivative, allowing for the manipulation of sets of non-differentiable functions. Quantum difference operators play an intriguing role in a variety of mathematical fields, including the geometric function theory, calculus of variations, and relativity theory (see [2], [9], [27]). Kac & Cheung's and Gasper & Rahman's books [7,13] cover a large number of fundamental aspects of q-calculus.

We use the symbol Σ to represent the set of functions f that takes the following form

$$f(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} a_{\mathsf{n}} \mathsf{z}^{\mathsf{n}}, \qquad (a_{\mathsf{n}} > 0) \tag{1}$$

that are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{\mathsf{z} : \mathsf{z} \in \mathbb{C} : 0 < |\mathsf{z}| < 1\} = \mathbb{U} \setminus \{0\}.$$

Tang et al. [28] introduced the q-derivative $\partial_q(f(\mathsf{z}))$ for meromorphic functions, defined as follows:

$$\partial_q f(\mathsf{z}) = \frac{f(\mathsf{z}) - f(q\mathsf{z})}{(1-q)\mathsf{z}} = -\frac{1}{q\mathsf{z}^2} + \sum_{\mathsf{n}=1}^{\infty} [\mathsf{n}]_q a_{\mathsf{n}} \mathsf{z}^{\mathsf{n}-1}, \quad (0 < q < 1), \tag{2}$$

Received May 9, 2023. Revised June 21, 2023. Accepted August 21, 2023.
2010 Mathematics Subject Classification: 32A20, 30D30, 30C45.
Key words and phrases: Analytic function, Meromorphic functions, Starlike function, neighborhood, Partial sums, q-calculus.
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where

$$[\mathsf{n}]_q = \begin{cases} \frac{1-q^{\mathsf{n}}}{1-q}, & \mathsf{n} \in \mathbb{C} \setminus \mathbb{N} \\ 1 + q + q^2 + \dots, & \mathsf{n} \in \mathbb{N}. \end{cases} \tag{3}$$

If $q \to 1-$, then $[\mathsf{n}]_q \to \mathsf{n}$ and $\lim_{q\to 1-} \partial_q f = f'$.

More recently, Alatawi et al. [1] defined the q-derivative operator $\mathsf{D}_q^k : \Sigma \to \Sigma$ as below

$$\mathsf{D}_q^0 f(\mathsf{z}) = f(\mathsf{z})$$

$$\mathsf{D}_q^k f(\mathsf{z}) = q\beta \mathsf{z}\partial_q \mathsf{D}_q^{k-1} f(\mathsf{z}) + (1+\beta)\mathsf{D}_q^{k-1} f(\mathsf{z}).$$

$$\mathsf{D}_q^k f(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} (1 + \beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}} \mathsf{z}^{\mathsf{n}}, \quad (\beta \le 0, k \in \mathbb{N}). \tag{4}$$

Building on the research conducted in [19, 24], we introduce a subset denoted as $\mathcal{MS}_q[k; \mathsf{A}, \mathsf{B}]$, which is defined using the operator $\mathsf{D}_q^k f$ in the following manner:

DEFINITION 1.1. A function $f \in \Sigma$ is said to belong to the class $\Sigma_q[k; \mathsf{A}, \mathsf{B}]$, if

$$\left| \frac{q\mathsf{z}\partial_q \left(\mathsf{D}_q^k f(\mathsf{z})\right) + \mathsf{D}_q^k f(\mathsf{z})}{\mathsf{B}q\mathsf{z}\partial_q \left(\mathsf{D}_q^k f(\mathsf{z})\right) + \mathsf{A}\mathsf{D}_q^k f(\mathsf{z})} \right| < 1, \quad (\mathsf{z} \in \mathbb{U}^*), \tag{5}$$

where $k \in \mathbb{N}_0, -1 \le \mathsf{B} < \mathsf{A} \le 1$ and $q \in (0, 1)$.

Furthermore, a function

$$f(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} a_{\mathsf{n}} \mathsf{z}^{\mathsf{n}} \qquad (a_{\mathsf{n}} > 0, \mathsf{z} \in \mathbb{U}^*), \tag{6}$$

belongs to the class $\mathcal{T}\Sigma_q[k; \mathsf{A}, \mathsf{B}]$ if it meets the requirement stated in equation (5).

It's worth noting that the previous definition is primarily inspired by the latest research by Morga [19] and Srivastava et al. [24]

This paper's primary goal is to introduce neighborhoods and partial sums of functions that belong to the classes $\Sigma_q[k; \mathsf{A}, \mathsf{B}]$ and $\mathcal{T}\Sigma_q[k; \mathsf{A}, \mathsf{B}]$. Unless specified otherwise, we'll assume that $-1 \le \mathsf{B} < \mathsf{A} \le 1$ and $q \in (0, 1)$ in this paper.

2. Coefficient Bonds

This section outlines the process of the sufficient conditions based on coefficient estimates for functions f that are part of the subclasses $\Sigma_q[k; \mathsf{A}, \mathsf{B}]$ and $\mathcal{T}\Sigma_q[k; \mathsf{A}, \mathsf{B}]$.

THEOREM 2.1. *Let $f \in \Sigma$ as in (1) and satisfies the inequality*

$$\sum_{n=1}^{\infty} \big((\mathsf{B}+1)q[n]_q + \mathsf{A} + 1\big)(1 + \beta[n+1]_q)^k a_n \le (\mathsf{A} - \mathsf{B}), \tag{7}$$

then $f \in \Sigma_q[k; \mathsf{A}, \mathsf{B}]$.

Proof. To prove that $f \in \Sigma_q[k;\mathsf{A},\mathsf{B}]$ satisfies a certain condition, it's enough to demonstrate that

$$\Re e\left\{\frac{q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{D}_q^k f(\mathsf{z})}{\mathsf{B}q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{A}\mathsf{D}_q^k f(\mathsf{z})}\right\}>-1, \qquad (\mathsf{z}\in\mathbb{U}^*). \tag{8}$$

If the inequality (7) true and holds, we must show that

$$\left|q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{D}_q^k f(\mathsf{z})\right|-\left|\mathsf{B}q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{A}\mathsf{D}_q^k f(\mathsf{z})\right|\le 0. \tag{9}$$

Now

$$\begin{aligned}
&\left|q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{D}_q^k f(\mathsf{z})\right|-\left|\mathsf{B}q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{A}\mathsf{D}_q^k f(\mathsf{z})\right|\\
&=\left|\sum_{\mathsf{n}=1}^{\infty}(q[\mathsf{n}]_q+1)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}}\right|-\left|(\mathsf{A}-\mathsf{B})\frac{1}{\mathsf{z}}+\sum_{\mathsf{n}=1}^{\infty}(\mathsf{A}+\mathsf{B}q[\mathsf{n}]_q)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}}\right|
\end{aligned}$$

(10)

$$\le\sum_{\mathsf{n}=1}^{\infty}\big((\mathsf{B}+1)q[\mathsf{n}]_q+\mathsf{A}+1\big)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}|\mathsf{z}|^{\mathsf{n}+1}-(\mathsf{A}-\mathsf{B}).$$

Since (10) holds for all $r=|\mathsf{z}|$, $0<r<1$, we have

$$\sum_{\mathsf{n}=1}^{\infty}\big((\mathsf{B}+1)q[\mathsf{n}]_q+\mathsf{A}+1\big)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}-(\mathsf{A}-\mathsf{B})\le 0,$$

by (7), which yields (9). □

THEOREM 2.2. *Let $f=\frac{1}{\mathsf{z}}+\sum\limits_{\mathsf{n}=1}^{\infty}|a_{\mathsf{n}}|\mathsf{z}^{\mathsf{n}}$ be analytic in $\mathbb{U}^*$. Then $f\in\mathcal{T}\Sigma_q[k;\mathsf{A},\mathsf{B}]$, if and only if inequality (7) is satisfied. The result is sharp for the function $f(\mathsf{z})$, which is defined as*

$$f(\mathsf{z})=\frac{\mathsf{A}-\mathsf{B}}{\big((\mathsf{B}+1)q[\mathsf{n}]_q+\mathsf{A}+1\big)(1+\beta[\mathsf{n}+1]_q)^k}\mathsf{z}^{\mathsf{n}}, \qquad (\mathsf{n}\ge 1). \tag{11}$$

Proof. Considering Theorem 2.1, it's enough to prove the validity of the "if" component.
Assume that $f\in\mathcal{T}\Sigma_q[k;A,B]$. Then, we have

$$\Re e\left\{\frac{q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{D}_q^k f(\mathsf{z})}{\mathsf{B}q\mathsf{z}\partial_q\left(\mathsf{D}_q^k f(\mathsf{z})\right)+\mathsf{A}\mathsf{D}_q^k f(\mathsf{z})}\right\}>-1, \qquad (\mathsf{z}\in\mathbb{U}^*). \tag{12}$$

Since $\Re e\{f(\mathsf{z})\}\le|f(\mathsf{z})|$ for all $\mathsf{z}\in\mathbb{U}^*$, then

$$\Re e\left\{\frac{\sum\limits_{\mathsf{n}=1}^{\infty}(q[\mathsf{n}]_q+1)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}+1}}{\mathsf{A}-\mathsf{B}+\sum\limits_{\mathsf{n}=1}^{\infty}(\mathsf{A}+\mathsf{B}q[\mathsf{n}]_q)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}+1}}\right\}<1, \tag{13}$$

for all z and the above equation is true. By letting $\mathsf{z}\to 1-$ on the real axis, we have the following inequality

$$\sum_{\mathsf{n}=1}^{\infty}(q[\mathsf{n}]_q+1)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}\le\mathsf{A}-\mathsf{B}+\sum_{\mathsf{n}=1}^{\infty}(\mathsf{A}+\mathsf{B}q[\mathsf{n}]_q)(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}}.$$

Thus, we get the required inequality

$$\sum_{\mathsf{n}=1}^{\infty}(q[\mathsf{n}]_q(\mathsf{B}+1)+(\mathsf{A}+1))(1+\beta[\mathsf{n}+1]_q)^k a_{\mathsf{n}} \leq \mathsf{A}-\mathsf{B}.$$

This concludes the demonstration of our theorem. □

THEOREM 2.3. *Let f given by (6), $0 \leq \gamma_j \leq 1$ and $\sum_{j=0}^{\infty}\gamma_j = 1$. Then $f \in \mathcal{T}\Sigma_q[k;\mathsf{A},\mathsf{B}]$, if and only if*

$$f(\mathsf{z}) = \sum_{j=0}^{\infty}\gamma_j f_j, \tag{14}$$

where
(15)

$$f_0(\mathsf{z}) = \frac{1}{\mathsf{z}}, \quad f_j(\mathsf{z}) = \frac{1}{\mathsf{z}}+\left(\frac{\mathsf{A}-\mathsf{B}}{\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k}\right)\mathsf{z}^j, \quad j = 1,2,\cdots.$$

Proof. Let

$$f(\mathsf{z}) = \sum_{j=0}^{\infty}\gamma_j f_j$$
$$= \gamma_0 f_0 + \sum_{j=1}^{\infty}\gamma_j f_j = \frac{\gamma_0}{\mathsf{z}} + \sum_{j=1}^{\infty}\gamma_j\left\{\frac{1}{\mathsf{z}} + \left(\frac{\mathsf{A}-\mathsf{B}}{\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k}\right)\mathsf{z}^j\right\},$$

by applying condition we get (7),

$$\sum_{j=1}^{\infty}\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k\left(\frac{\mathsf{A}-\mathsf{B}}{\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k}\gamma_j\right)$$
$$= (1-\alpha)\sum_{j=1}^{\infty}\gamma_j = (1-\alpha)(1-\gamma_1) \leq \mathsf{A}-\mathsf{B}.$$

This means that $f \in \mathcal{T}\Sigma_q[k;\mathsf{A},\mathsf{B}]$.

Conversely, Let $f \in \mathcal{T}\Sigma_q[k;\mathsf{A},\mathsf{B}]$. By setting

$$\gamma_j = \sum_{j=1}^{\infty}\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k a_j \qquad (0 \leq \gamma_j \leq 1),$$
$$\gamma_0 == 1 - \sum_{j=1}^{\infty}\gamma_j\,.$$

As a result, the function f can be represented as

$$\begin{aligned} f(\mathsf{z}) =& \frac{1}{\mathsf{z}} + \sum_{j=1}^{\infty} a_j \mathsf{z}^j \\ =& \frac{1}{\mathsf{z}} + \sum_{j=1}^{\infty} \left(\frac{\mathsf{A}-\mathsf{B}}{\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k}\gamma_j \right) \mathsf{z}^j \\ =& \mathsf{z}_0 \left(\frac{1}{\mathsf{z}}\right) + \sum_{j=1}^{\infty} \left(\frac{1}{\mathsf{z}} + \frac{\mathsf{A}-\mathsf{B}}{\big(q[j]_q(\mathsf{B}+1)+(\mathsf{A}+1)\big)(1+\beta[j+1]_q)^k}\mathsf{z}^j \right) \gamma_j \\ =& \sum_{j=0}^{\infty} \gamma_j f_j, \end{aligned}$$

this concludes the demonstration of our theorem. □

3. Neighborhoods and partial sums

Inspired by previous studies that used the conventional idea of neighborhoods for analytic functions, such as Goodman [8], Silverman [25, 26], Murugusundaramoorthy and Velayudam [15], Darus and Ibrahim [5], and recently Altıntas and Owa [3], and Elhaddad and Darus [6], we present the following:

For $\mu \geq 0$, $-1 \leq \mathsf{B} < \mathsf{A} \leq 1$ and $f = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} |a_\mathsf{n}|\mathsf{z}^\mathsf{n}$, we define the δ-nieghbourhoods of $f(\mathsf{z})$ by

$$\mathcal{N}_{\mu,q}(f) = \left\{ g(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=2}^{\infty} |b_\mathsf{n}|\mathsf{z}^\mathsf{n} \text{ and} \sum_{\mathsf{n}=2}^{\infty} \frac{\big((\mathsf{B}+1)q[\mathsf{n}]_q + \mathsf{A}+1\big)(1+\beta[\mathsf{n}+1]_q)^k}{\mathsf{A}-\mathsf{B}} \big||a_\mathsf{n}| - |b_\mathsf{n}|\big| \leq \mu \right\}. \tag{16}$$

THEOREM 3.1. *Let $\mu > 0$ and $-1 < \mathsf{A} \leq 0$. If $f \in \Sigma$ of the form (1) satisfies the condition*

$$\frac{f(\mathsf{z}) + \frac{\epsilon}{\mathsf{z}}}{1+\epsilon} \in \Sigma_q[k; \mathsf{A}, \mathsf{B}] \tag{17}$$

for any complex number ϵ satisfying $|\epsilon| < \delta$, we have

$$\mathcal{N}_{\mu,q}(f) \subset \Sigma_q[k; \mathsf{A}, \mathsf{B}].$$

Proof. It is obvious from (5) and $g(\mathsf{z}) \in \Sigma_q[k; \mathsf{A}, \mathsf{B}]$ if and only if for any complex number η with $|\eta| = 1$

$$\frac{\mathsf{z}\partial_q\left(\mathsf{D}_q^k g(\mathsf{z})\right) + \mathsf{D}_q^k g(\mathsf{z})}{\mathsf{B}\mathsf{z}\partial_q\left(\mathsf{D}_q^k g(\mathsf{z})\right) + \mathsf{A}\mathsf{D}_q^k g(\mathsf{z})} \neq \eta \qquad (\mathsf{z} \in \mathbb{U}^*),$$

this is the same as saying

$$\frac{g(\mathsf{z}) * h(\mathsf{z})}{\mathsf{z}^{-1}} \neq 0 \qquad (\mathsf{z} \in \mathbb{U}^*), \tag{18}$$

where

$$h(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} c_{\mathsf{n}} \mathsf{z}^{\mathsf{n}}$$

$$= \frac{1}{\mathsf{z}} + \sum_{\mathsf{n}=1}^{\infty} \frac{\big((1+q[\mathsf{n}]_q) - \eta(\mathsf{A}+q[\mathsf{n}]_q\mathsf{B})\big)}{(\mathsf{B}-\mathsf{A})\,\eta}(1+\beta[\mathsf{n}+1]_q)^k \mathsf{z}^{\mathsf{n}}. \tag{19}$$

From (17), we have

$$|c_{\mathsf{n}}| = \left| \frac{\big((1+q[\mathsf{n}]_q) - \eta(\mathsf{A}+q[\mathsf{n}]_q\mathsf{B})\big)}{(\mathsf{B}-\mathsf{A})\,\eta}(1+\beta[\mathsf{n}+1]_q)^k \right|$$

$$\leq \frac{\big((1-\mathsf{A}) + q[\mathsf{n}]_q\eta(1-\mathsf{B})\big)}{(\mathsf{B}-\mathsf{A})\,\eta}(1+\beta[\mathsf{n}+1]_q)^k. \tag{20}$$

If $f(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum\limits_{\mathsf{n}=1}^{\infty} a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}}, (a_{\mathsf{n}} > 0)$ fulfills the condition (17), then (18) yields

$$\left| \frac{f(\mathsf{z}) * h(\mathsf{z})}{\mathsf{z}^{-1}} \right| \geq \delta \qquad (\mathsf{z} \in \mathbb{U}^*). \tag{21}$$

Now let $\rho(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum\limits_{\mathsf{n}=1}^{\infty} b_{\mathsf{n}}\mathsf{z}^{\mathsf{n}} \in \mathcal{N}_{mu,q}(f)$, then

$$\left| \frac{(\rho(\mathsf{z}) - f(\mathsf{z})) * h(\mathsf{z})}{\mathsf{z}^{-1}} \right| = \left| \sum_{\mathsf{n}=1}^{\infty} (b_{\mathsf{n}} - a_{\mathsf{n}}) c_{\mathsf{n}} \mathsf{z}^{\mathsf{n}+1} \right|$$

$$\leq |\mathsf{z}| \sum_{\mathsf{n}=1}^{\infty} \frac{\big((1-\mathsf{A}) + q[\mathsf{n}]_q\eta(1-\mathsf{B})\big)}{(\mathsf{B}-\mathsf{A})\,\eta}(1+\beta[\mathsf{n}+1]_q)^k |b_{\mathsf{n}} - a_{\mathsf{n}}|$$

$$< \delta.$$

Thus for any complex number η satisfying $|\eta| = 1$, then

$$\frac{\rho(\mathsf{z}) * h(\mathsf{z})}{\mathsf{z}^{-1}} \neq 0 \qquad (z \in \mathbb{U}^*),$$

this leads to the conclusion that $\rho(\mathsf{z}) \in \Sigma_q[k; \mathsf{A}, \mathsf{B}]$. □

THEOREM 3.2. *Let* $-1 < \mathsf{A} \leq 0$. *If* $f \in \Sigma$ *of the form (1) and* $\mathfrak{s}_k(\mathsf{z}) = \frac{1}{\mathsf{z}} + \sum\limits_{\mathsf{n}=1}^{k-1} a_{\mathsf{n}}\mathsf{z}^{\mathsf{n}} \quad (k \geq 2)$. *Suppose that*

$$\sum_{\mathsf{n}=1}^{\infty} c_{\mathsf{n}} |a_{\mathsf{n}}| \leq 1, \tag{22}$$

where

$$c_{\mathsf{n}} = \frac{\big((1-\mathsf{A}) + q[\mathsf{n}]_q\eta(1-\mathsf{B})\big)}{(\mathsf{B}-\mathsf{A})\,\eta}(1+\beta[\mathsf{n}+1]_q)^k.$$

Then, we have

$$1) f(z) \in \Sigma_q[k;\mathsf{A},\mathsf{B}].$$

$$2)\ \Re e \left\{ \frac{f(\mathsf{z})}{\mathfrak{s}_k(\mathsf{z})} \right\} > 1 - \frac{1}{c_{k-1}}. \tag{23}$$

$$3)\ \Re e \left\{ \frac{\mathfrak{s}_k(\mathsf{z})}{f(\mathsf{z})} \right\} > \frac{c_{k-1}}{1 + c_{k-1}}. \tag{24}$$

The estimates are sharp.

Proof. (1) It is obvious that $\frac{1}{\mathsf{z}} \in \Sigma_q[k;\mathsf{A},\mathsf{B}]$. Thus from Theorem 2.1, and the condition (22), we have $\mathcal{N}_{mu,q}(\frac{1}{\mathsf{z}}) \subseteq \Sigma_q[k;\mathsf{A},\mathsf{B}]$. This gives $f(\mathsf{z}) \in \Sigma_q[k;\mathsf{A},\mathsf{B}]$.

(2) It is easy to see that $1 < c_k < c_{k+1}$. Thus

$$\sum_{\mathsf{n}=1}^{k-2} |a_\mathsf{n}| + c_{k+1} \sum_{\mathsf{n}=k-1}^{\infty} |a_\mathsf{n}| \le \sum_{\mathsf{n}=1}^{\infty} c_\mathsf{n} |a_\mathsf{n}| \le 1. \tag{25}$$

Let

$$\mathfrak{h}_1 = c_{k-1} \left\{ \frac{f(\mathsf{z})}{\mathfrak{s}_k(\mathsf{z})} - \left(1 - \frac{1}{c_{k-1}}\right) \right\} = 1 + \frac{c_{k-1} \sum\limits_{\mathsf{n}=k-1}^{\infty} a_\mathsf{n} \mathsf{z}^{\mathsf{n}+1}}{1 + \sum\limits_{\mathsf{n}=1}^{k-2} a_\mathsf{n} \mathsf{z}^{\mathsf{n}+1}}.$$

It follows from (25) that

$$\left| \frac{\mathfrak{h}_1 - 1}{\mathfrak{h}_1 + 1} \right| \le \frac{c_{k-1} \sum\limits_{\mathsf{n}=k-1}^{\infty} |a_\mathsf{n}|}{2 - 2\sum\limits_{\mathsf{n}=1}^{k-2} |a_\mathsf{n}| - c_{k-1} \sum\limits_{\mathsf{n}=k-1}^{\infty} |a_\mathsf{n}|} \le 1, \qquad (\mathsf{z} \in \mathbb{U}^*).$$

From this we obtain the inequality (23).
If we take

$$f(\mathsf{z}) = \frac{1}{\mathsf{z}} - \frac{\mathsf{z}^{k-1}}{c_{k-1}}, \tag{26}$$

then

$$f(\mathsf{z}) = 1 - \frac{\mathsf{z}^k}{c_{k-1}} \longrightarrow 1 - \frac{1}{c_{k-1}} \qquad \text{as} \quad k \longrightarrow 1-.$$

This demonstrates that the bound in (23) is best possible for any k.

Similarly, assuming that we take

$$\mathfrak{h}_2 = (1 + c_{k+1}) \left\{ \frac{\mathfrak{s}_k(\mathsf{z})}{f(\mathsf{z})} - \left(\frac{c_{k-1}}{1 + c_{k-1}}\right) \right\} = 1 + \frac{(1 + c_{k-1}) \sum\limits_{\mathsf{n}=k-1}^{\infty} a_\mathsf{n} \mathsf{z}^{\mathsf{n}+1}}{1 + \sum\limits_{\mathsf{n}=0}^{\infty} a_\mathsf{n} \mathsf{z}^{\mathsf{n}+1}}.$$

Then, we deduce that

$$\left|\frac{\mathfrak{h}_2 - 1}{\mathfrak{h}_2 + 1}\right| \leq \frac{(1 + c_{k-1}) \sum\limits_{\mathsf{n}=k-1}^{\infty} |a_{\mathsf{n}}|}{2 - 2\sum\limits_{\mathsf{n}=1}^{k-2} |a_{\mathsf{n}}| + (1 - c_{k-1}) \sum\limits_{\mathsf{n}=k-1}^{\infty} |a_{\mathsf{n}}|} \leq 1, \qquad (\mathsf{z} \in \mathbb{U}^*),$$

which yields (24). The estimate (24) is sharp with the extremal function $f(\mathsf{z})$ given by (26). □

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