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MIXED RADIAL-ANGULAR INTEGRABILITIES FOR HARDY TYPE OPERATORS

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Abstract. In this paper, we are devoted to studying the mixed radial-angular integrabilities for Hardy type operators. As an application, the upper and lower bounds are obtained for the fractional Hardy operator. In addition, we also establish the sharp weak-type estimate for the fractional Hardy operator.

1. Introduction

The Hardy operator, as the most fundamental averaging operator, is defined by

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(t)dt,$$

where the function f is a nonnegative integrable function on $\mathbb{R}^+$ and $x > 0$. A celebrated integral inequality, due to Hardy [7], states that

$$\|\mathcal{H}f\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}$$

holds for $1 < p < \infty$, and the constant $\frac{p}{p-1}$ is the best possible.

For the multidimensional case $n \geq 2$, generally speaking, there exist two different definitions. One is the rectangle averaging operator, and its norm depends on the dimensions. Another version is the n-dimensional spherical averaging operator, which was introduced by Christ and Grafakos in [2] as follows:

$$\mathcal{H}f(x) = \frac{1}{\nu_n|x|^n}\int_{|t|\leq|x|} f(t)dt, \quad x \in \mathbb{R}^n \setminus \{0\},$$

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where ν_n is the volume of the unit ball in $\mathbb{R}^n$. However, the norm of $\mathcal{H}$ is different from the rectangle averaging operator, its norm on $L^p(\mathbb{R}^n)$ was evaluated and found to be equal to that of the 1-dimensional averaging operator. $\|\mathcal{H}\|_{L^p\to L^p}$ $(p>1)$, that is to say, does not depend on the dimension of the space.

In 2013, Lu et al. [15] gave the definition of Hardy type operator as follows:

$$\mathcal{H}_m(f)(x):=\Big(\prod_{i=1}^{m}\frac{1}{|B(0,|x_i|)|}\Big)\int_{|y_1|<|x_1|}\cdots\int_{|y_m|<|x_m|}f(y_1,\dots,y_m)dy_m\cdots dy_1,$$

where f is a nonnegative measurable function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$, $m\in\mathbb{N}$, $n_i\in\mathbb{N}$, $x=(x_1,x_2,\dots,x_m)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$, $x_i\in\mathbb{R}^{n_i}$ and $\prod\limits_{i=1}^{m}|x_i|\neq 0$. Furthermore, the corresponding operator norm on the Lebesgue product spaces with power weights was obtained by some classical techniques.

Theorem A ([15])**.** *Let $1<q<\infty$, $m\in\mathbb{N}$, $n_i\in\mathbb{N}$, $x_i\in\mathbb{R}^{n_i}$, $i=1,\dots,m$. If $f\in L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m},|x|^{\vec{\alpha}})$, where $|x|^{\vec{\alpha}}:=|x|^{\alpha_1}\times|x|^{\alpha_2}\cdots\times|x|^{\alpha_m}$ and $\alpha_i<(q-1)n_i$, then the Hardy type operator $\mathcal{H}_m$ is bounded on $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m},|x|^{\vec{\alpha}})$. Moreover, the norm of $\mathcal{H}_m$ can be obtained as follows*:

$$\|\mathcal{H}_m\|_{L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m},|x|^{\vec{\alpha}})\to L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m},|x|^{\vec{\alpha}})}=\prod_{i=1}^{m}\frac{q}{q-1-\alpha_i/n_i}.$$

Later on, Liu and Zhou extended the above result to the case of the p-adic Hardy type operators in [14]. In addition, as one of the important operators, the research of Hardy type operators have made rich results. For other recent works about this topic, see [5,6,16] and related references therein.

On the other hand, our second aim is to consider the mixed radial-angular integrability for the fractional Hardy operator. Recall that, for a nonnegative measurable function f on $\mathbb{R}^n$, the n-dimensional fractional Hardy operator H_β with spherical mean is defined by

$$\mathrm{H}_\beta(f)(x)=\frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\int_{|y|\le|x|}f(y)dy,\quad x\in\mathbb{R}^n\setminus\{0\}\text{ and }0<\beta<n.$$

Clearly,

$$\mathrm{H}_\beta(f)(x)\le C\mathrm{M}_\beta(f)(x),$$

where

$$\mathrm{M}_\beta(f)(x)=\sup_{r>0}\frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}}\int_{|y-x|\le r}|f(y)|dy.$$

From [17], we know that the operator M_β is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $0<\beta<n$, $1<p<\frac{\beta}{n}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$. Therefore, the fractional Hardy operator H_β has also this property. Furthermore, Lu et al. derived the following result.

Theorem B ([15])**.** *Suppose that* $0 < \beta < n$, $1 < p, q < \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. *Then we have*

$$\|\mathrm{H}_\beta(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

where,

$$\Big(\frac{p}{q}\Big)^{1/q}\Big(\frac{p}{p-1}\Big)^{1/q}\Big(\frac{q}{q-1}\Big)^{1-1/q}\Big(1-\frac{p}{q}\Big)^{1/p-1/q} \leq C \leq \Big(\frac{p}{p-1}\Big)^{\frac{p}{q}}.$$

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with Lebesgue measure $d\sigma = d\sigma(\cdot)$. For any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, applying the spherical coordinate formula, we write

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &= \Big(\int_0^\infty \int_{S^{n-1}} |f(r\theta)|^p d\sigma(\theta) r^{n-1} dr\Big)^{1/p} \\ &= \Big(\int_0^\infty \|f(r\cdot)\|^p_{L^p(S^{n-1})} r^{n-1} dr\Big)^{1/p}.\end{aligned}$$

Therefore, from the perspective of radial and angular integrability, Lebesgue norms can be interpreted as certain special norms with the same integrability in the radial and angular directions. Inspired by this version, we naturally consider the case of Lebesgue norms with different integrability in the radial and angular directions, namely,

$$\|f\|_{L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)} := \left(\int_0^\infty \|f(r\cdot)\|^p_{L^{\tilde{p}}(S^{n-1})} r^{n-1} dr\right)^{1/p}, \quad 1 \leq p, \tilde{p} \leq \infty,$$

and when $p = \infty$ or $\tilde{p} = \infty$, we just need to make the usual modifications in the above definition, but we do not use these cases in the current work.

In addition, the mixed radial-angular space $L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)$, as a formal extension of the Lebesgue spaces $L^p(\mathbb{R}^n)$, was introduced to study of regularity and some important estimates, such as angular regularity and Strichartz estimates (see [1,3,4,18,19] etc.). Recently, the author et al. also established the boundedness of some classical operators with rough kernels on mixed radial-angular spaces in [8–13].

For any $1 \leq p, \tilde{p} < \infty$, the weighted mixed radial-angular spaces $L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n, \omega)$ are defined by

$$L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n, \omega) = \big\{f : \|f\|_{L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n,\omega)} < \infty\big\},$$

where

$$\|f\|_{L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n,\omega)} := \left(\int_0^\infty \|f(r\cdot)\|^p_{L^{\tilde{p}}(S^{n-1})} r^{n-1}\omega(r) dr\right)^{1/p}.$$

The main purpose in this paper is twofold, the first is to extend the results of the operator $\mathcal{H}_m$ on the product Lebesgue spaces with power weights to the product mixed radial-angular integrability cases. The second is to establish the upper and lower bounds of the fractional Hardy operator H_β on the mixed radial-angular spaces $L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)$ by adopting some classical ideas.

To achieve our goals, we first give the definition of the product mixed radial-angular integrability with power weights as follows:

$$L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}} := \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}, |x|^{\vec{\alpha}}) = \big\{f : \|f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},|x|^{\vec{\alpha}})} < \infty\big\},$$

where

$$L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}}, |x|^{\vec{\alpha}}) := L^{p_1}_{rad_1}L^{q_2}_{ang_1}\cdots L^{p_m}_{rad_m}L^{q_m}_{ang_m}(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}, r^{\vec{\alpha}}),$$

$$r^{\vec{\alpha}} = r_1^{\alpha_1}\times\cdots\times r_2^{\alpha_m}, \quad 1\le p_i, q_i < \infty,\ i = 1,\dots,m,$$

and

$$\begin{aligned}&\|f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},|x|^{\vec{\alpha}})}\\ &= \Bigg(\int_0^\infty\Big(\int_{S^{n_1-1}}\cdots\Big(\int_0^\infty\big(\int_{S^{n_m-1}}|f(r_1\theta_1,\dots,r_m\theta_m)|^{q_m}d\sigma(\theta_m)\big)^{p_m/q_m}\\ &\quad\times r_m^{\alpha_m+n_m-1}dr_m\Big)^{q_{m-1}/p_m}\cdots d\sigma(\theta_1)\Big)^{p_1/q_1} r_1^{\alpha_1+n_1-1}dr_1\Bigg)^{1/p_1}.\end{aligned}$$

Our results can be formulated as follows.

Theorem 1.1. *Let $1 < p_i, q_i, \tilde{q}_i < \infty$, $n_i \in \mathbb{N}$, $x_i \in \mathbb{R}^{n_i}$, $r_i \in (0,\infty)$, $i = 1,\dots,m$. If $f \in L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}}, r^{\vec{\alpha}})$, where $r^{\vec{\alpha}} = r_1^{\alpha_1}\times\cdots\times r_2^{\alpha_m}$ and $\alpha_i < (p_i-1)n_i$. Then the Hardy type operator $\mathcal{H}_m$ is bounded from $L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}}, r^{\vec{\alpha}})$ to $L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}}, r^{\vec{\alpha}})$. Moreover, the norm of $\mathcal{H}_m$ can be obtained as follows*:

$$\|\mathcal{H}_m\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})\to L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})} = \prod_{i=1}^m {\omega_i}^{\frac{1}{q_i}-\frac{1}{\tilde{q}_i}}\left(\frac{p_i}{p_i-1-\frac{\alpha_i}{n_j}}\right).$$

Theorem 1.2. *Suppose that $0 < \beta < n$, $1 < p_1, q_1, p_2, q_2 < \infty$ and $\frac{1}{p_1}-\frac{1}{p_2} = \frac{\beta}{n}$. Then we have*

$$\|\mathrm{H}_\beta(f)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}.$$

Moreover, the constant C satisfies the following inequality

$$\begin{aligned}&{\omega_n}^{1/q_2-1/q_1+\beta/n}\Big(\frac{p_1}{p_2}\Big)^{1/p_2}\Big(\frac{p_1}{p_1-1}\Big)^{1/p_2}\Big(\frac{p_2}{p_2-1}\Big)^{1-1/p_2}\Big(1-\frac{p_1}{p_2}\Big)^{1/p_1-1/p_2}\\ &\le C \le {\omega_n}^{1/q_2-1/q_1+\beta/n}\Big(\frac{p_1}{p_1-1}\Big)^{\frac{p_1}{p_2}}.\end{aligned}$$

Remark 1.3. Taking $q_i = \tilde{q}_i = p_i = q$ for $i = 1,\dots,m$ in Theorem 1.1, then the result reduces to Theorem A. Similarly, Theorem 1.2 returns to Theorem B by taking $p_1 = q_1$ and $p_2 = q_2$ in Theorem 1.2.

Remark 1.4. Using L'Hospital's rule, we have

$$\lim_{p_2\to p_1}\Big(1-\frac{p_1}{p_2}\Big)^{1/p_1-1/p_2} = 1.$$

This deduces that

$$\|\mathrm{H}\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)\to L^{p_1}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/q_2-1/q_1}\frac{p_1}{p_1-1}.$$

It is easy to see that this conclusion happens to be Theorem 1.1 in the case of $m=1, \alpha_i=0$.

The rest of this paper is organized as follows. In Section 2, we will provide the proofs of Theorems 1.1 and 1.2. Finally, we establish the weak-type estimate for the fractional Hardy operator on the weak mixed radial-angular spaces.

2. Proofs of main results

Proof of Theorem 1.1. In fact, we only provide the proof with the case $m=2$, and the similar procedure leads to the general case $m>2$. We follow some strategies in [15]. Set

$$g_f = \frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}} f(|x_1|\xi_1, |x_2|\xi_2)d\sigma(\xi_1)d\sigma(\xi_2).$$

Obviously, the function g is a nonnegative radial function with respect to variables x_1 and x_2, respectively. Then

$$\mathcal{H}_2(g_f)(x_1,x_2) = \mathcal{H}_2(f)(x_1,x_2).$$

For the angular integral, using the Minkowski inequality gives rise to the following

$$\left(\int_{S^{n_2-1}_{x_2}}\Big(\frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}} f(|x_1|\xi_1, |x_2|\xi_2)d\sigma(\xi_1)d\sigma(\xi_2)\Big)^{\tilde{q}_2}d\sigma(x_2')\right)^{1/\tilde{q}_2}$$
$$\leq \frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}}\left(\int_{S^{n_1-1}_{x_2}}|f(|x_1|\xi_1, |x_2|\xi_2)|^{\tilde{q}_2}d\sigma(x_2')\right)^{1/\tilde{q}_2} d\sigma(\xi_1)d\sigma(\xi_2).$$

For the radial integral, using the Minkowski inequality again, we have

$$\left(\int_0^\infty\left(\int_{S^{n_2-1}_{x_2}}\Big(\frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}} f(|x_1|\xi_1, |x_2|\xi_2)d\sigma(\xi_1)d\sigma(\xi_2)\Big)^{\tilde{q}_2}\right.\right.$$
$$\left.\left. d\sigma(x_2')\right)^{p_2/\tilde{q}_2} \times r_2^{n_2-1+\alpha_1}dr_2\right)^{1/p_2}$$
$$\leq \frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}}\left(\int_0^\infty\left(\int_{S^{n_1-1}_{x_2}}|f(|x_1|\xi_1, |x_2|\xi_2)|^{\tilde{q}_2}d\sigma(x_2')\right)^{p_2/\tilde{q}_2}\right.$$
$$\left.\times r_2^{n_2-1+\alpha_1}dr_2\right)^{1/p_2} d\sigma(\xi_1)d\sigma(\xi_2).$$

Similarly, we perform the same operation repeatedly for the first variable, we have

$$\begin{aligned}\|g_f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})} &\leq \frac{1}{\omega_{n_1}}\frac{1}{\omega_{n_2}}\int_{S^{n_1-1}_{\xi_1}}\int_{S^{n_2-1}_{\xi_2}}\|f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}d\sigma(\xi_1)d\sigma(\xi_2)\\ &= \|f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}.\end{aligned}$$

Therefore,

$$\frac{\|\mathcal{H}_2(f)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}}{\|f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}} \leq \frac{\|\mathcal{H}_2(g_f)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}}{\|g_f\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}}.$$

This implies that the norm of the operator of $\mathcal{H}_2$ is equal to the norm of the operator of $\mathcal{H}_2$ restricts to the set of nonnegative radial functions. Therefore, it suffices to show the case that the function f is a nonnegative radial function.

For a radial function f, it is not hard to see that $\mathcal{H}_2 f(x_1, x_2)$ is also a radial function. Consequently, we conclude that

$$\begin{aligned}&\|\mathcal{H}_2(f)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}\\ &= \Bigg(\int_0^\infty \Bigg(\int_{S^{n_1-1}_{x_1}} \Bigg(\int_0^\infty \Big(\int_{S^{n_2-1}_{x_2}} |\mathcal{H}_2 f((r_1,\theta_1),(r_2,\theta_2))|^{q_2}\\ &\qquad \times d\sigma(\theta_2)\Big)^{p_2/q_2} r_2^{n_2-1+\alpha_2}dr_2\Bigg)^{q_1/p_2} d\sigma(\theta_1)\Bigg)^{p_1/q_1} r_1^{n_1-1+\alpha_1}dr_1\Bigg)^{1/p_1}\\ &= {\omega_{n_1}}^{1/q_1}{\omega_{n_2}}^{1/q_2}\Bigg(\int_0^\infty\Bigg(\int_0^\infty |\mathcal{H}_2 f(r_1,r_2)|^{p_2} r_2^{n_2-1+\alpha_2}dr_2\Bigg)^{q_1/p_2} r_1^{n_1-1+\alpha_1}dr_1\Bigg)^{1/p_1}.\end{aligned}$$

By changing variables, we derive that

$$\mathcal{H}_2 f(r_1,r_2) = \frac{1}{\nu_{n_1}}\frac{1}{\nu_{n_2}}\int_{B_{n_1}(1)}\int_{B_{n_2}(1)} f(r_1y_1, r_2y_2)dy_1dy_2.$$

From this and using Minkowski's inequality, we have

$$\begin{aligned}&\|\mathcal{H}_2(f)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}\\ &= \frac{{\omega_{n_1}}^{1/q_1}}{\nu_{n_1}}\frac{{\omega_{n_2}}^{1/q_2}}{\nu_{n_2}}\Bigg(\int_0^\infty\Bigg(\int_0^\infty \Big|\int_{B_{n_1}(1)}\int_{B_{n_2}(1)} f(r_1|y_1|, r_2|y_2|)dy_1dy_2\Big|^{p_2}\\ &\qquad \times r_2^{n_2-1+\alpha_2}dr_2\Bigg)^{p_1/p_2} r_1^{n_1-1+\alpha_1}dr_1\Bigg)^{1/p_1}\\ &\leq \frac{{\omega_{n_1}}^{1/q_1}}{\nu_{n_1}}\frac{{\omega_{n_2}}^{1/q_2}}{\nu_{n_2}}\int_{B_{n_1}(1)}\int_{B_{n_2}(1)}\Bigg(\int_0^\infty\Bigg(\int_0^\infty |f(r_1|y_1|, r_2|y_2|)|^{p_2}\end{aligned}$$

$$\times r_2^{n_2-1+\alpha_2}dr_2\Big)^{p_1/p_2} r_1^{n_1-1+\alpha_1}dr_1\Big)^{1/p_1} dy_1dy_2$$

$$= \frac{{\omega_{n_1}}^{1/q_1-1/\tilde{q}_1}}{\nu_{n_1}} \frac{{\omega_{n_2}}^{1/q_2-1/\tilde{q}_2}}{\nu_{n_2}} \int_{B_{n_1}(1)} \int_{B_{n_2}(1)} \Bigg(\int_0^\infty {\omega_{n_1}}^{p_1/\tilde{q}_1} \Bigg(\int_0^\infty {\omega_{n_2}}^{p_2/\tilde{q}_2}$$

$$\times |f(r_1|y_1|, r_2|y_2|)|^{p_2} r_2^{n_2-1+\alpha_2}dr_2\Bigg)^{p_1/p_2} r_1^{n_1-1+\alpha_1}dr_1\Bigg)^{1/p_1}$$

$$\times |y_1|^{-(n_1+\alpha_1)/p_1}|y_2|^{-(n_2+\alpha_2)/p_2}dy_1dy_2$$

$$= {\omega_{n_1}}^{1/\tilde{q}_1-1/q_1}{\omega_{n_2}}^{1/\tilde{q}_2-1/q_2} \frac{\omega_{n_1}}{\nu_{n_1}} \frac{\omega_{n_2}}{\nu_{n_2}} \frac{p_1}{n_1(p_1-1-\alpha_1/n_1)} \frac{p_2}{n_2(p_2-1-\alpha_2/n_2)}$$

$$\times \|f\|_{L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}$$

$$= {\omega_{n_1}}^{1/q_1-1/\tilde{q}_1}{\omega_{n_2}}^{1/q_2-1/\tilde{q}_2} \frac{p_1}{p_1-1-\alpha_1/n_1} \frac{p_2}{p_2-1-\alpha_2/n_2} \|f\|_{L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}.$$

To prove the above constant is best possible, we need to prove the converse inequality. Set

$$f_\varepsilon(x_1,x_2) = |x_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}|x_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}\chi_{\{|x_1|>1,|x_2|>1\}}(x_1,x_2).$$

It follows from the elementary calculation that

$$\|f_\varepsilon\|_{L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})} = \frac{{\omega_{n_1}}^{1/\tilde{q}_1}}{(p\varepsilon)^{1/p_1}} \frac{{\omega_{n_2}}^{1/\tilde{q}_2}}{(p\varepsilon)^{1/p_2}}.$$

On the other hand,

$$\mathcal{H}_2(f_\varepsilon)(x_1,x_2)$$

$$= \frac{|x_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}}{\nu_{n_1}} \frac{|x_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}}{\nu_{n_2}}$$

$$\times \int_{\{|z_1|<1,|z_1|<1/|x_1|\}} \int_{\{|z_2|<1,|z_2|<1/|x_2|\}} |z_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}|z_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}dz_1dz_2.$$

Therefore, we have

$$\|\mathcal{H}_2(f_\varepsilon)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}$$

$$= \frac{\omega_{n_1}^{1/q_1}}{\nu_{n_1}} \frac{\omega_{n_2}^{1/q_2}}{\nu_{n_2}} \Bigg(\int_1^\infty \Big(\Big| {r_1}^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon} \int_{\frac{1}{r_1}<|z_1|<1} |z_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}dz_1 \Big|$$

$$\times \int_1^\infty \Big| {r_2}^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon} \int_{\frac{1}{r_2}<|z_2|<1} |z_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}dz_2 \Big|^{p_2} r_2^{n_2+\alpha_2-1}dr_2\Big)^{p_1/p_2} r_1^{n_1+\alpha_1-1}dr_1\Bigg)^{1/p_1}$$

$$\geq \frac{\omega_{n_1}^{1/q_1}}{\nu_{n_1}} \frac{\omega_{n_2}^{1/q_2}}{\nu_{n_2}} \Bigg(\int_{1/\varepsilon}^\infty \Big(\Big| {r_1}^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon} \int_{\varepsilon<|z_1|<1} |z_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}dz_1 \Big|$$

$$\times \int_{1/\varepsilon}^{\infty}\Big|r_2{}^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}\int_{\varepsilon<|z_2|<1}|z_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}dz_2\Big|^{p_2}r_2^{n_2+\alpha_2-1}dr_2\Big)^{p_1/p_2}r_1^{n_1+\alpha_1-1}dr_1\Big)^{1/p_1}$$

$$= \frac{\omega_{n_1}^{1/q_1}}{\nu_{n_1}}\frac{\omega_{n_2}^{1/q_2}}{\nu_{n_2}}\frac{\varepsilon^{\varepsilon}}{(p_1\varepsilon)^{1/p_1}}\frac{\varepsilon^{\varepsilon}}{(p_2\varepsilon)^{1/p_2}}\int_{\varepsilon<|z_1|<1}|z_1|^{-\frac{n_1+\alpha_1}{p_1}+\varepsilon}dz_1\int_{\varepsilon<|z_2|<1}|z_2|^{-\frac{n_2+\alpha_2}{p_2}+\varepsilon}dz_2$$

$$= \frac{\omega_{n_1}^{1+1/q_1}}{\nu_{n_1}}\frac{\omega_{n_2}^{1+1/q_2}}{\nu_{n_2}}\frac{\varepsilon^{\varepsilon}}{(p_1\varepsilon)^{1/p_1}}\frac{\varepsilon^{\varepsilon}}{(p_2\varepsilon)^{1/p_2}}\int_{\varepsilon}^{1}r_1^{n_1-1-(\frac{n_1+\alpha_1}{p_1}-\varepsilon)}dr_1\int_{\varepsilon}^{2}r_2^{n_2-1-(\frac{n_2+\alpha_2}{p_2}-\varepsilon)}dr_2$$

$$= \frac{\omega_{n_1}^{1+1/q_1}}{\nu_{n_1}}\frac{\omega_{n_2}^{1+1/q_2}}{\nu_{n_2}}\frac{\varepsilon^{\varepsilon}}{(p_1\varepsilon)^{1/p_1}}\frac{\varepsilon^{\varepsilon}}{(p_2\varepsilon)^{1/p_2}}\frac{1-\varepsilon^{\frac{n_1+\alpha_1}{p_1}+\varepsilon}}{n_1-\frac{n_1+\alpha_1}{p_1}+\varepsilon}\frac{1-\varepsilon^{\frac{n_2+\alpha_2}{p_2}+\varepsilon}}{n_2-\frac{n_2+\alpha_2}{p_2}+\varepsilon}$$

$$= n_1\varepsilon^{\varepsilon}\frac{1-\varepsilon^{\frac{n_1+\alpha_1}{p_1}+\varepsilon}}{n_1-\frac{n_1+\alpha_1}{p_1}+\varepsilon}\omega_{n_1}^{1/q_1-1/\tilde{q}_1}n_2\varepsilon^{\varepsilon}\omega_{n_2}^{1/q_2-1/\tilde{q}_2}\frac{1-\varepsilon^{\frac{n_2+\alpha_2}{p_2}+\varepsilon}}{n_2-\frac{n_2+\alpha_2}{p_2}+\varepsilon}\|f_\varepsilon\|_{L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}.$$

This implies that

$$\frac{\|\mathcal{H}_2(f_\varepsilon)\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}}{\|f_\varepsilon\|_{L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}}$$
$$\geq n_1\varepsilon^{\varepsilon}\frac{1-\varepsilon^{\frac{n_1+\alpha_1}{p_1}+\varepsilon}}{n_1-\frac{n_1+\alpha_1}{p_1}+\varepsilon}\omega_{n_1}^{1/q_1-1/\tilde{q}_1}n_2\varepsilon^{\varepsilon}\omega_{n_2}^{1/q_2-1/\tilde{q}_2}\frac{1-\varepsilon^{\frac{n_2+\alpha_2}{p_2}+\varepsilon}}{n_2-\frac{n_2+\alpha_2}{p_2}+\varepsilon}.$$

Consequently, using the definition of the norm of the operator and letting $\varepsilon\to 0$, we conclude that

$$\|\mathcal{H}_2\|_{L^{\vec{p}}_{rad}L^{\vec{q}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})\to L^{\vec{p}}_{rad}L^{\vec{\tilde{q}}}_{ang}(\mathbb{R}^{\vec{n}},r^{\vec{\alpha}})}$$
$$\geq \omega_{n_1}^{1/q_1-1/\tilde{q}_1}\omega_{n_2}^{1/q_2-1/\tilde{q}_2}\frac{p_1}{p_1-1-\alpha_1/n_1}\frac{p_2}{p_2-1-\alpha_2/n_2}.$$

This proves Theorem 1.1. □

Now we turn to prove Theorem 1.2 in this position.

Proof of Theorem 1.2. Let

$$g_f(y)=\frac{1}{\omega_n}\int_{S_\xi^{n-1}}f(|y|\xi)d\sigma(\xi),\quad y\in\mathbb{R}^n.$$

Clearly, g_f is a radial function, and

$$\|g_f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}=\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)},\quad \mathrm{H}_\beta(g_f)(x)=\mathrm{H}_\beta(f)(x).$$

Therefore, we have

$$\frac{\|\mathrm{H}_\beta(f)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}}{\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}}\leq\frac{\|\mathrm{H}_\beta(g_f)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}}{\|g_f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}},$$

and

$$\|\mathrm{H}_\beta(f)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}$$
$$=\Big(\int_0^\infty\Big(\int_{S^{n-1}}|\mathrm{H}_\beta f(r,\theta)|^{q_2}d\sigma(\theta)\Big)^{p_2/q_2}r^{n-1}dr\Big)^{1/p_2}$$

$$= \left(\int_0^\infty \Big(\int_{S^{n-1}} \Big| \frac{1}{(\nu_n r^n)^{1-\beta/n}} \int_{|y|<r} f(y)dy \Big|^{q_2} d\sigma(\theta) \Big)^{p_2/q_2} r^{n-1} dr \right)^{1/p_2}$$
$$= {\nu_n}^{\beta/n-1} \left(\int_0^\infty \Big(\int_{S^{n-1}} \Big| \frac{1}{r^{n-\beta}} \int_{|y|<r} f(y)dy \Big|^{q_2} d\sigma(\theta) \Big)^{p_2/q_2} r^{n-1} dr \right)^{1/p_2}$$
$$= {\nu_n}^{\beta/n-1} {\omega_n}^{1/q_2} \left(\int_0^\infty \Big| \int_{|y|<r} f(y)dy \Big|^{p_2} r^{(\beta-n)p_2+n-1} dr \right)^{1/p_2}$$
$$= {\nu_n}^{\beta/n-1} {\omega_n}^{1/q_2} \left(\int_0^\infty \Big| \int_{|y|<r} f(y)dy \Big|^{p_2-p_1} \Big| \int_{|y|<r} f(y)dy \Big|^{p_1} r^{(\beta-n)p_2+n-1} dr \right)^{1/p_2}$$
$$= {\nu_n}^{\beta/n-1+p_1/p_2} {\omega_n}^{1/q_2} \left(\int_0^\infty \Big| \int_{|y|<r} f(y)dy \Big|^{p_2-p_1} |\mathrm{H}f(r,\theta)|^{p_1} r^{np_1+(\beta-n)p_2+n-1} dr \right)^{1/p_2}.$$

The Hölder inequality deduces that

$$\begin{aligned}
&\Big| \int_{|y|<r} f(y)dy \Big| \\
&\leq \int_0^r \int_{S^{n-1}} |f(\rho,\theta)| d\sigma(\theta) \rho^{n-1} d\rho \\
&\leq r^{n(1-1/p_1)} {\omega_n}^{1-1/q_1} \left(\int_0^r \Big(\int_{S^{n-1}} |f(\rho,\theta)|^{q_1} d\sigma(\theta) \Big)^{p_1/q_1} \rho^{n-1} d\rho \right)^{1/p_1} \\
&= n^{(1/p_1-1)} r^{n(1-1/p_1)} {\omega_n}^{1-1/q_1} \|f\|_{L^{p_1}_{rad} L^{q_1}_{ang}(\mathbb{R}^n)}.
\end{aligned}$$

This, together with Theorem 1.1 in the case of $m=1$ and $\alpha_i = 0$, gives that

$$\begin{aligned}
&\|\mathrm{H}_\beta(f)\|_{L^{p_2}_{rad} L^{q_2}_{ang}(\mathbb{R}^n)} \\
&\leq {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} {\omega_n}^{1/q_2+(1-1/q_1)(1-p_1/p_2)} \|f\|^{\frac{p_2-p_1}{p_2}}_{L^{p_1}_{rad} L^{q_1}_{ang}(\mathbb{R}^n)} \\
&\quad \times \left(\int_0^\infty |\mathrm{H}f(r,\theta)|^{p_1} r^{n(1-1/p_1)(p_2-p_1)+np_1+(\beta-n)p_2+n-1} dr \right)^{1/p_2} \\
&= {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} {\omega_n}^{1/q_2+(1-1/q_1)(1-p_1/p_2)} \|f\|^{\frac{p_2-p_1}{p_2}}_{L^{p_1}_{rad} L^{q_1}_{ang}(\mathbb{R}^n)} \\
&\quad \times \left(\int_0^\infty |\mathrm{H}f(r,\theta)|^{p_1} r^{n-1} dr \right)^{1/p_2} \qquad \Big(\frac{1}{p_1} - \frac{1}{p_2} = \frac{\beta}{n} \Big) \\
&\leq {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} {\omega_n}^{1/q_2-p_1/\tilde{q}_1 p_2+(1-1/q_1)(1-p_1/p_2)} \|f\|^{\frac{p_2-p_1}{p_2}}_{L^{p_1}_{rad} L^{q_1}_{ang}(\mathbb{R}^n)} \\
&\quad \times \left(\int_0^\infty \Big(\int_{S^{n-1}} |\mathrm{H}f(r,\theta)|^{\tilde{q}_1} d\sigma(\theta) \Big)^{p_1/\tilde{q}_1} r^{n-1} dr \right)^{1/p_2} \\
&= {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} {\omega_n}^{1/q_2-p_1/\tilde{q}_1 p_2+(1-1/q_1)(1-p_1/p_2)} \\
&\quad \times \|f\|^{\frac{p_2-p_1}{p_2}}_{L^{p_1}_{rad} L^{q_1}_{ang}(\mathbb{R}^n)} \|\mathrm{H}(f)\|^{\frac{p_1}{p_2}}_{L^{p_1}_{rad} L^{\tilde{q}_1}_{ang}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} {\omega_n}^{1/q_2-p_1/\tilde{q}_1 p_2+(1-1/q_1)(1-p_1/p_2)} \|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}^{\frac{p_2-p_1}{p_2}}$$

$$\times \Big(\frac{p_1}{p_1-1}{\omega_n}^{1/\tilde{q}_1-1/q_1}\Big)^{\frac{p_1}{p_2}} \|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}^{\frac{p_1}{p_2}}$$

$$= {\nu_n}^{\beta/n-1+p_1/p_2} n^{(1/p_1-1)(1-p_1/p_2)} \Big(\frac{p_1}{p_1-1}\Big)^{\frac{p_1}{p_2}} {\omega_n}^{1/q_2-1/q_1+1-p_1/p_2} \|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}$$

$$= \Big(\frac{\omega_n}{n}\Big)^{\beta/n-1+p_1/p_2} \Big(\frac{p_1}{p_1-1}\Big)^{\frac{p_1}{p_2}} {\omega_n}^{1/q_2-1/q_1+1-p_1/p_2} \|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}$$

$$= {\omega_n}^{1/q_2-1/q_1+\beta/n} \Big(\frac{p_1}{p_1-1}\Big)^{\frac{p_1}{p_2}} \|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}.$$

To obtain the lower bound of $\|\mathrm{H}_\beta\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)\to L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}$, we set

$$f_0(x) = |x|^{\frac{nC_0}{p_1}} \chi_{\{|x|<1\}}(x), \quad C_0 > -1.$$

Then

$$\|f_0\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/q_1} n^{-1/p_1} \Big(\frac{1}{C_0+1}\Big)^{1/p_1}.$$

Also,

$$\|\mathrm{H}_\beta(f_0)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}$$

$$= \Big(\int_0^\infty \Big(\int_{S^{n-1}} |\mathrm{H}_\beta f_0(r,\theta)|^{q_2} d\sigma(\theta)\Big)^{p_2/q_2} r^{n-1} dr\Big)^{1/p_2}$$

$$= \Big(\int_0^\infty \Big(\int_{S^{n-1}} \Big|\frac{1}{(\nu_n r^n)^{1-\beta/n}} \int_{|y|<r} |y|^{\frac{nC_0}{p_1}} \chi_{\{|y|<1\}}(y) dy\Big|^{q_2} d\sigma(\theta)\Big)^{p_2/q_2} r^{n-1} dr\Big)^{1/p_2}$$

$$\leq \Big(\int_1^\infty \Big(\int_{S^{n-1}} \Big|\frac{1}{(\nu_n r^n)^{1-\beta/n}} \int_{|y|<1} |y|^{\frac{nC_0}{p_1}} dy\Big|^{q_2} d\sigma(\theta)\Big)^{p_2/q_2} r^{n-1} dr\Big)^{1/p_2}$$

$$+ \Big(\int_0^1 \Big(\int_{S^{n-1}} \Big|\frac{1}{(\nu_n r^n)^{1-\beta/n}} \int_{|y|<r} |y|^{\frac{nC_0}{p_1}} dy\Big|^{q_2} d\sigma(\theta)\Big)^{p_2/q_2} r^{n-1} dr\Big)^{1/p_2}$$

$$= \frac{{\omega_n}^{1/q_2+1}}{\nu_n^{(1-\beta/n)}(nC_0/p_1+n)} \Big(\int_1^\infty r^{(\beta-n)p_2+n-1} dr\Big)^{1/p_2}$$

$$+ \frac{{\omega_n}^{1/q_2+1}}{\nu_n^{(1-\beta/n)}(nC_0/p_1+n)} \Big(\int_1^\infty r^{(\beta+nC_0/p_1)p_2+n-1} dr\Big)^{1/p_2}$$

$$= \frac{{\omega_n}^{1/q_2+1}}{\nu_n^{(1-\beta/n)}(nC_0/p_1+n)} \Big(\frac{p_1}{np_2(p_1-1)} + \frac{p_1}{np_2(1+C_0)}\Big)^{1/p_2}$$

$$= \frac{{\omega_n}^{1/q_2+1}}{\nu_n^{(1-\beta/n)}(nC_0/p_1+n)} \Big(\frac{p_1}{np_2}\Big)^{1/p_2} \Big(\frac{1}{p_1-1} + \frac{1}{1+C_0}\Big)^{1/p_2}.$$

Therefore, we have

$$\begin{aligned}
&\frac{\|\mathrm{H}_\beta(f_0)\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}}{\|f_0\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}}\\
&= \frac{{\omega_n}^{1/q_2-1/q_1+1}}{\nu_n^{(1-\beta/n)}(nC_0/p_1+n)}\Big(\frac{p_1}{np_2}\Big)^{1/p_2}\Big(\frac{1}{p_1-1}+\frac{1}{1+C_0}\Big)^{1/p_2} n^{1/p_1}(1+C_0)^{1/p_1}\\
&= \Big(\frac{p_1}{p_2}\Big)^{1/p_2}\frac{{\omega_n}^{1/q_2-1/q_1+1}n^{1/p_1-1/p_2-1}}{\nu_n^{(1-\beta/n)}}\Big(\frac{p_1}{p_1-1}\Big)^{\frac{1}{p_2}}\frac{(1+C_0)^{1/p_1-1/p_2}}{(1+C_0/p_1)^{1-1/p_2}}\\
&= {\omega_n}^{1/q_2-1/q_1+\beta/n}\Big(\frac{p_1}{p_2}\Big)^{1/p_2}\Big(\frac{p_1}{p_1-1}\Big)^{\frac{1}{p_2}}\frac{(1+C_0)^{1/p_1-1/p_2}}{(1+C_0/p_1)^{1-1/p_2}}.
\end{aligned}$$

We now take

$$f_1(x)=\frac{(1+x)^{1/p_1-1/p_2}}{(1+x/p_1)^{1-1/p_2}},\quad x>-1.$$

It is not hard to check the function $f_1(x)$ defined over the open interval $(-1,\infty)$ has a unique global maximum at the point $x_0=-p_1/p_2$. This implies that

$$\begin{aligned}
&\|\mathrm{H}_\beta(f)\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)\to L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&\geq {\omega_n}^{1/q_2-1/q_1+\beta/n}\Big(\frac{p_1}{p_2}\Big)^{1/p_2}\Big(\frac{p_1}{p_1-1}\Big)^{1p_2}\Big(\frac{p_2}{p_2-1}\Big)^{1-1/p_2}\Big(1-\frac{p_1}{p_2}\Big)^{1/p_1-1/p_2}.
\end{aligned}$$

Therefore, the proof of Theorem 1.2 is finished. □

3. Sharp weak-type estimate for the fractional Hardy operator

In this section, we will consider the weak-type estimate for the fractional Hardy operator H_β. In 2013, Lu et al. established the following sharp estimate.

Theorem C ([15]). *Suppose that $0<\beta<n$, $1<p,q<\infty$ and $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$. Then for $\lambda>0$, we have*

$$\Big|\Big\{x\in\mathbb{R}^n:|\mathrm{H}_\beta(f)(x)|>\lambda\Big\}\Big|\leq\Bigg(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{\lambda}\Bigg)^{\frac{n}{n-\beta}}.$$

Moreover,

$$\|\mathrm{H}_\beta\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\beta},\infty}(\mathbb{R}^n)}=1.$$

Here, the norm $\|f\|_{L^{p,\infty}(\mathbb{R}^n)}$ (weak $L^p(\mathbb{R}^n)$ norm), defined by

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)}=\sup_{\lambda>0}\lambda\Big|\Big\{x\in\mathbb{R}^n:|f(x)|>\lambda\Big\}\Big|^{1/p}.$$

It can be rewrite as

$$\begin{aligned}
\|f\|_{L^{p,\infty}(\mathbb{R}^n)}&=\sup_{\lambda>0}\lambda\Big(\int_{\mathbb{R}^n}\chi_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}(x)dx\Big)^{1/p}\\
&=\sup_{\lambda>0}\lambda\big\|\chi_{\{\cdot\,\in\mathbb{R}^n\,:\,|f(\cdot)|>\lambda\}}\big\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Inspired by this, we also give the definition of the weak mixed radial-angular spaces $\mathcal{W}L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)$ as follows:

$$\|f\|_{\mathcal{W}L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)} = \sup_{\lambda>0}\lambda\big\|\chi_{\{\cdot\in\mathbb{R}^n:|f(\cdot)|>\lambda\}}\big\|_{L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)}, \quad 1\le p,\tilde{p}<\infty.$$

Next, we will extend Theorem C to the weak mixed radial-angular spaces $\mathcal{W}L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)$.

Theorem 3.1. *Suppose that* $0<\beta<n$, $1\le p_1,q_1,p_2,q_2<\infty$ *and* $\frac{1}{p_1}-\frac{1}{p_2}=\frac{\beta}{n}$. *Then the fractional Hardy operator* H_β *is bounded from* $L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)$ *to* $\mathcal{W}L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)$. *Moreover,*

$$\|\mathrm{H}_\beta\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)\to\mathcal{W}L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/q_2-1/q_1+\beta/n}.$$

Proof. Note that

$$\begin{aligned}
&\frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}}\Big|\int_{|y|<|x|}f(y)dy\Big|\\
&\le \frac{1}{(\nu_n|x|^n)^{1-\frac{\beta}{n}}}\int_0^{|x|}\int_{S^{n-1}}|f(\rho,\theta)|d\sigma(\theta)\rho^{n-1}d\rho\\
&\le \frac{1}{(\nu_n|x|^n)^{1-\frac{\beta}{n}}}|x|^{n(1-1/p_1)}{\omega_n}^{1-1/q_1}\left(\int_0^r\Big(\int_{S^{n-1}}|f(\rho,\theta)|^{q_1}d\sigma(\theta)\Big)^{p_1/q_1}\rho^{n-1}d\rho\right)^{1/p_1}\\
&= \frac{1}{\nu_n^{1-\frac{\beta}{n}}}\frac{1}{|x|^{\frac{n}{p_2}}}n^{(1/p_1-1)}{\omega_n}^{1-1/q_1}\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big\|\chi_{\{x\in\mathbb{R}^n:|\mathrm{H}_\beta f(x)|>\lambda\}}\big\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&\le \Big\|\chi_{\{x\in\mathbb{R}^n:\frac{1}{\nu_n^{1-\frac{\beta}{n}}}\frac{1}{|x|^{\frac{n}{p_2}}}n^{(1/p_1-1)}{\omega_n}^{1-1/q_1}\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}>\lambda\}}\Big\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= \Big\|\chi_{\{x\in\mathbb{R}^n:|x|^{\frac{n}{p_2}}<\frac{n^{(1/p_1-1)}{\omega_n}^{1-1/q_1}\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}}{\lambda\nu_n^{1-\frac{\beta}{n}}}\}}\Big\|_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= {\omega_n}^{1/q_2}\left(\int_0^{\left[\frac{n^{(1/p_1-1)}{\omega_n}^{1-1/q_1}\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}}{\lambda\nu_n^{1-\frac{\beta}{n}}}\right]^{\frac{p_2}{n}}} r^{n-1}dr\right)^{1/p_2}\\
&= \frac{{\omega_n}^{1/q_2-1/q_1+\beta/n}}{\lambda}\|f\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}.
\end{aligned}$$

Next, we will show that the constant ${\omega_n}^{1/q_2-1/q_1+\beta/n}$ is sharp. For this purpose, we set the function

$$f_0(x)=\chi_{\{|x|<1\}}(x).$$

Then we have

$$\|f_0\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/q_1}n^{-1/p_1}.$$

It follows from [15] that

$$\mathrm{H}_\beta(f_0)(x) = \frac{|\{y : |y| < \min\{1,|x|\}\}|}{(\nu_n|x|^n)^{1-\frac{\beta}{n}}} \le {\nu_n}^{\beta/n}.$$

Hence, for $0 < \lambda < {\nu_n}^{\beta/n}$, we have

$$\begin{aligned}
&\left\|\chi_{\{x\in\mathbb{R}^n:|\mathrm{H}_\beta f_0(x)|>\lambda\}}\right\|^{p_2}_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= \left\|\chi_{B(0,1)\cap\{x\in\mathbb{R}^n:|\mathrm{H}_\beta f_0(x)|>\lambda\}}\right\|^{p_2}_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&\quad + \left\|\chi_{B(0,1)^c\cap\{x\in\mathbb{R}^n:\frac{{\nu_n}^{n/\beta}}{|x|^{n-\beta}}>\lambda\}}\right\|^{p_2}_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= \left\|\chi_{\{x\in\mathbb{R}^n:|x|<1\}}\right\|^{p_2}_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&\quad + \left\|\chi_{\{x\in\mathbb{R}^n:1<|x|<\left(\frac{{\nu_n}^{\beta/n}}{\lambda}\right)^{\frac{1}{n-\beta}}\}}\right\|^{p_2}_{L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= {\omega_n}^{p_2/q_2}\frac{1}{n}\frac{{\nu_n}^{\beta/(n-\beta)}}{\lambda^{n/(n-\beta)}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\|\chi_{\{x\in\mathbb{R}^n:|\mathrm{H}_\beta f_0(x)|>\lambda\}}\right\|_{\mathcal{W}L^{p_2}_{rad}L^{q_2}_{ang}(\mathbb{R}^n)}\\
&= {\omega_n}^{1/q_2-1/q_1}n^{\beta/n}\sup_{0<\lambda<{\nu_n}^{\beta/n}}\lambda\Big(\frac{{\nu_n}^{\beta/(n-\beta)}}{\lambda^{n/(n-\beta)}}\Big)^{1/p_2}\|f_0\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}\\
&= {\omega_n}^{1/q_2-1/q_1+\beta/n}\|f_0\|_{L^{p_1}_{rad}L^{q_1}_{ang}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, the proof of Theorem 3.1 is completed. □

Remark 3.2. If we take some suitable indexes for $1 \le p_1, q_1, p_2, q_2 < \infty$ in Theorem 3.1, then the result reduces to Theorem C. As a consequence, this represents a weak type generalization of Theorem C.

4. Bounds of weighted Hardy-Littlewood averages

Suppose $\psi : [0,1] \to [0,\infty)$ is a function. For a measurable complex valued function f on $\mathbb{R}^n$, Xiao in [20] defined the weighted Hardy-Littlewood average $U_\psi(f)$ and the weighted Cesàro average $V_\psi(f)$ as

$$U_\psi(f)(x) = \int_0^1 f(tx)\psi(t)dt, \quad V_\psi(f)(x) = \int_0^1 f(x/t)t^{-n}\psi(t)dt.$$

It is easy to see that if $\psi = 1$ and $n = 1$, then

$$U_\psi = \mathcal{H}, \quad V_\psi = \int_x^\infty \frac{f(y)}{y}dy, \quad x > 0,$$

and

$$U_\psi(f)(x) + V_\psi(f)(x) = \frac{1}{x}\int_0^x f(y)dy + \int_x^\infty \frac{f(y)}{y}dy, \quad x > 0$$

becomes the Calderón maximal operator.

In [20], the author established the following results for the operators U_ψ and V_ψ.

Theorem D. *Let $\psi : [0,1] \to [0,\infty)$ be a function and let $p \in [1,\infty)$. Then*

(i) $U_\psi(f) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *exists as a bounded operator if and only if*

$$\int_0^1 t^{-n/p}\psi(t)dt < \infty.$$

Moreover, the operator norm of U_ψ on $L^p(\mathbb{R}^n)$ is given by

$$\|U_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 t^{-n/p}\psi(t)dt.$$

(ii) $V_\psi(f) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *exists as a bounded operator if and only if*

$$\int_0^1 t^{-n(1-1/p)}\psi(t)dt < \infty.$$

Moreover, the operator norm of V_ψ on $L^p(\mathbb{R}^n)$ is given by

$$\|V_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 t^{-n(1-1/p)}\psi(t)dt.$$

Next, we will extend the above results to the mixed radial-angular spaces $L^p_{rad}L^{\tilde{p}}_{ang}(\mathbb{R}^n)$.

Theorem 4.1. *Let $\psi : [0,1] \to [0,\infty)$ be a function and let $1 < p < \infty$, $1 < p_2 \le p_1 < \infty$. Then we have*

(i) $U_\psi(f) : L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n) \to L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)$ *exists as a bounded operator if and only if*

$$\int_0^1 t^{-n/p}\psi(t)dt < \infty.$$

Moreover, the operator norm of U_ψ is given by

$$\|U_\psi\|_{L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)\to L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/p_2-1/p_1} \int_0^1 t^{-n/p}\psi(t)dt.$$

(ii) $V_\psi(f) : L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n) \to L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)$ *exists as a bounded operator if and only if*

$$\int_0^1 t^{-n(1-1/p)}\psi(t)dt < \infty.$$

Moreover, the operator norm of V_ψ is given by

$$\|V_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = {\omega_n}^{1/p_2-1/p_1} \int_0^1 t^{-n(1-1/p)}\psi(t)dt.$$

Proof. (i) Suppose $\int_0^1 t^{-n/p}\psi(t)dt < \infty$. Then

$$\begin{aligned}
&\|U_\psi(f)\|_{L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)} \\
&= \Big(\int_0^\infty \Big(\int_{S^{n-1}} |U_\psi(f)(r\theta)|^{p_2} d\sigma(\theta)\Big)^{p/p_2} r^{n-1}dr\Big)^{1/p} \\
&= \Big(\int_0^\infty \Big(\int_{S^{n-1}} \Big|\int_0^1 f(tr\theta)\psi(t)dt\Big|^{p_2} d\sigma(\theta)\Big)^{p/p_2} r^{n-1}dr\Big)^{1/p} \\
&\le \Big(\int_0^\infty \Big(\int_0^1 \Big(\int_{S^{n-1}} |f(tr\theta)|^{p_2} d\sigma(\theta)\Big)^{1/p_2} \psi(t)dt\Big)^p r^{n-1}dr\Big)^{1/p} \\
&\le \int_0^1 \Big(\int_0^\infty \Big(\int_{S^{n-1}} |f(tr\theta)|^{p_2} d\sigma(\theta)\Big)^{p/p_2} r^{n-1}dr\Big)^{1/p} \psi(t)dt \\
&\le {\omega_n}^{1/p_2-1/p_1} \int_0^1 t^{-n/p}\psi(t)dt \|f\|_{L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)}.
\end{aligned}$$

Thus U_ψ is bounded from $L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)$ to $L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)$.

Now we consider the converse. For $0 < \varepsilon < 1$, we take

$$f_\varepsilon(x) = |x|^{-\frac{n}{p}-\varepsilon}\chi_{\{|x|>1\}}(x).$$

Then

$$\|f_\varepsilon\|_{L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)} = \frac{{\omega_n}^{1/p_1}}{(p\varepsilon)^{1/p}},$$

and

$$U_\psi(f_\varepsilon)(x) = |x|^{-\frac{n}{p}-\varepsilon}\chi_{\{|x|>1\}}(x)\int_{1/|x|}^1 t^{-\frac{n}{p}-\varepsilon}\psi(t)dt.$$

Therefore, we have

$$\begin{aligned}
\|U_\psi(f_\varepsilon)\|_{L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)} &= {\omega_n}^{1/p_2}\Big(\int_1^\infty \Big(r^{-\frac{n}{p}-\varepsilon}\int_{1/|x|}^1 t^{-\frac{n}{p}-\varepsilon}\psi(t)dt\Big)^p r^{n-1}dr\Big)^{1/p} \\
&\ge {\omega_n}^{1/p_2}\Big(\int_{1/\varepsilon}^\infty \Big(r^{-\frac{n}{p}-\varepsilon}\int_{\varepsilon}^1 t^{-\frac{n}{p}-\varepsilon}\psi(t)dt\Big)^p r^{n-1}dr\Big)^{1/p} \\
&= {\omega_n}^{1/p_2}\varepsilon^\varepsilon \int_\varepsilon^1 t^{-\frac{n}{p}-\varepsilon}\psi(t)dt \\
&= \|f_\varepsilon\|_{L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)}{\omega_n}^{1/p_2-1/p_1}\varepsilon^\varepsilon \int_\varepsilon^1 t^{-\frac{n}{p}-\varepsilon}\psi(t)dt.
\end{aligned}$$

Letting $\varepsilon \to 0$, we finish the proof Theorem 4.1(i).

Next, we turn to prove (ii), by adopting similar ideas. Suppose

$$\int_0^1 t^{-n(1-1/p)}\psi(t)dt < \infty.$$

Then

$$\|V_\psi(f)\|_{L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)}$$

$$= \left(\int_0^\infty \left(\int_{S^{n-1}} |V_\psi(f)(r\theta)|^{p_2} d\sigma(\theta)\right)^{p/p_2} r^{n-1}dr\right)^{1/p}$$
$$= \left(\int_0^\infty \left(\int_{S^{n-1}} \Big|\int_0^1 f(r\theta/t)t^{-n}\psi(t)dt\Big|^{p_2} d\sigma(\theta)\right)^{p/p_2} r^{n-1}dr\right)^{1/p}$$
$$\leq \left(\int_0^\infty \left(\int_0^1 \left(\int_{S^{n-1}} |f(r\theta/t)|^{p_2} d\sigma(\theta)\right)^{1/p_2} t^{-n}\psi(t)dt\right)^{p} r^{n-1}dr\right)^{1/p}$$
$$\leq \int_0^1 \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r\theta/t)|^{p_2} d\sigma(\theta)\right)^{p/p_2} r^{n-1}dr\right)^{1/p} t^{-n}\psi(t)dt$$
$$\leq {\omega_n}^{1/p_2-1/p_1} \int_0^1 t^{-n(1-1/p)}\psi(t)dt\|f\|_{L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)}.$$

Thus V_ψ is bounded from $L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)$ to $L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)$.

For the converse, we take f_ε the same as (i). Then

$$V_\psi(f_\varepsilon)(x) = |x|^{-\frac{n}{p}-\varepsilon}\chi_{\{|x|>1\}}(x)\int_0^1 t^{-n+\frac{n}{p}+\varepsilon}\psi(t)dt.$$

Therefore, we get

$$\|V_\psi(f_\varepsilon)\|_{L^p_{rad}L^{p_2}_{ang}(\mathbb{R}^n)} = {\omega_n}^{1/p_2}\left(\int_1^\infty \left(r^{-\frac{n}{p}-\varepsilon}\int_0^1 t^{-n+\frac{n}{p}+\varepsilon}\psi(t)dt\right)^p r^{n-1}dr\right)^{1/p}$$
$$\geq {\omega_n}^{1/p_2}\left(\int_{1/\varepsilon}^\infty \left(r^{-\frac{n}{p}-\varepsilon}\int_0^1 t^{-n+\frac{n}{p}+\varepsilon}\psi(t)dt\right)^p r^{n-1}dr\right)^{1/p}$$
$$= {\omega_n}^{1/p_2}\varepsilon^\varepsilon \int_0^1 t^{-n+\frac{n}{p}+\varepsilon}\psi(t)dt$$
$$= \|f_\varepsilon\|_{L^p_{rad}L^{p_1}_{ang}(\mathbb{R}^n)}{\omega_n}^{1/p_2-1/p_1}\varepsilon^\varepsilon \int_0^1 t^{-n+\frac{n}{p}+\varepsilon}\psi(t)dt.$$

Letting $\varepsilon \to 0$, we complete the proof Theorem 4.1(ii). □

References

[1] F. Cacciafesta and R. Lucà, *Singular integrals with angular integrability*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3413–3418. https://doi.org/10.1090/proc/13123

[2] M. Christ and L. Grafakos, *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123** (1995), no. 6, 1687–1693. https://doi.org/10.2307/2160978

[3] P. D'Ancona and R. Lucà, *Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities with angular integrability*, J. Math. Anal. Appl. **388** (2012), no. 2, 1061–1079. https://doi.org/10.1016/j.jmaa.2011.10.051

[4] P. D'Ancona and R. Lucà, *On the regularity set and angular integrability for the Navier-Stokes equation*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1255–1284. https://doi.org/10.1007/s00205-016-0982-2

[5] Z. Fu, S. L. Gong, S. Z. Lu, and W. Yuan, *Weighted multilinear Hardy operators and commutators*, Forum Math. **27** (2015), no. 5, 2825–2851. https://doi.org/10.1515/forum-2013-0064

[6] Z. Fu, S. Z. Lu, and S. Shi, *Two characterizations of central BMO space via the commutators of Hardy operators*, Forum Math. **33** (2021), no. 2, 505–529. https://doi.org/10.1515/forum-2020-0243

[7] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317. https://doi.org/10.1007/BF01199965

[8] F. Liu and D. Fan, *Weighted estimates for rough singular integrals with applications to angular integrability*, Pacific J. Math. **301** (2019), no. 1, 267–295. https://doi.org/10.2140/pjm.2019.301.267

[9] R. Liu, F. Liu, and H. Wu, *Mixed radial-angular integrability for rough singular integrals and maximal operators*, Proc. Amer. Math. Soc. **148** (2020), no. 9, 3943–3956. https://doi.org/10.1090/proc/15037

[10] R. Liu, F. Liu, and H. Wu, *On the mixed radial-angular integrability of Marcinkiewicz integrals with rough kernels*, Acta Math. Sci. Ser. B (Engl. Ed.) **41** (2021), no. 1, 241–256. https://doi.org/10.1007/s10473-021-0114-4

[11] R. Liu, S. Tao, and H. Wu, *Characterizations of the mixed radial-angular central Campanato space via the commutators of Hardy type*, Forum Math. 2023, (in press).

[12] R. Liu and H. Wu, *Rough singular integrals and maximal operator with radial-angular integrability*, Proc. Amer. Math. Soc. **150** (2022), no. 3, 1141–1151. https://doi.org/10.1090/proc/15705

[13] R. Liu and H. Wu, *Mixed radial-angular integrability for rough maximal singular integrals and Marcinkiewicz integrals with mixed homogeneity*, Math. Nachr. (2023), 1–16. https://doi.org/10.1002/mana.202100253

[14] R. Liu and J. Zhou, *Sharp estimates for the p-adic Hardy type operators on higher-dimensional product spaces*, J. Inequal. Appl. **2017** (2017), Paper No. 219, 13 pp. https://doi.org/10.1186/s13660-017-1491-z

[15] S. Z. Lu, D. Yan, and F. Zhao, *Sharp bounds for Hardy type operators on higher-dimensional product spaces*, J. Inequal. Appl. **2013** (2013), 148, 11 pp. https://doi.org/10.1186/1029-242X-2013-148

[16] S. Shi, Z. Fu, and S. Z. Lu, *On the compactness of commutators of Hardy operators*, Pacific J. Math. **307** (2020), no. 1, 239–256. https://doi.org/10.2140/pjm.2020.307.239

[17] E. M. Stein, *Harmonic Analysis*: *Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43, Princeton Univ. Press, Princeton, NJ, 1993.

[18] J. Sterbenz, *Angular regularity and Strichartz estimates for the wave equation*, Int. Math. Res. Not. **2005** (2005), no. 4, 187–231. https://doi.org/10.1155/IMRN.2005.187

[19] T. C. Tao, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*, Comm. Partial Differential Equations **25** (2000), no. 7-8, 1471–1485. https://doi.org/10.1080/03605300008821556

[20] J. Xiao, *L^p and BMO bounds of weighted Hardy-Littlewood averages*, J. Math. Anal. Appl. **262** (2001), no. 2, 660–666. https://doi.org/10.1006/jmaa.2001.7594

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