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RELATIONSHIP BETWEEN THE STRUCTURE OF A FACTOR RING R/P AND DERIVATIONS OF R

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Abstract. The purpose of this paper is to study the relationship between the structure of a factor ring R/P and the behavior of some derivations of R. More precisely, we establish a connection between the commutativity of R/P and derivations of R satisfying specific identities involving the prime ideal P. Moreover, we provide an example to show that our results cannot be extended to semi-prime ideals.

1. Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$. Recall that R is 2-*torsion free* if whenever $2x = 0$ yields $x = 0$. A proper ideal P of R is said to be *prime* if for all $a, b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$. R is called a *prime ring* if (0) is a prime ideal of R; equivalently, R is prime if for any elements $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. For $a, b \in R$, the symbol $[a, b]$ will denote the *commutator* $ab - ba$ and the symbol $a \circ b$ will denote the *anti-commutator* $ab + ba$. An additive map $d : R \longrightarrow R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be *inner* if there exists $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A map $f : R \longrightarrow R$ is said to be *centralizing* on a subset S of R if $[f(x), x] \in Z(R)$ for all $x \in S$; in the particular case where $[f(x), x] = 0$ for all $x \in S$, the map f is said *commuting* on S. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [10] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later Posner [16] has proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative (Posner's second theorem). Over the last few decades, several authors have subsequently refined and extended these results in various directions (viz. [6, 14], where further references can be found). In [13], Idrissi and Oukhtite showed that if P is a prime ideal of a ring R and F is a generalized derivation of R associated with a derivation d such that $\overline{[F(x), x]} \in Z(R/P)$ for all $x \in R$, then $d(R) \subseteq P$

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or R/P is a commutative integral domain. Motivated by the above result, the authors in [7] considered a similar algebraic identity involving the Jordan product instead of the Lie product. More precisely, they proved that if P is a prime ideal of a ring R and F is a generalized derivation of R such that $char(R/P) \neq 2$ and $\overline{F(x) \circ x} \in Z(R/P)$ for all $x \in R$, then $F(R) \subseteq P$ or R/P is an integral domain.

In the seventies Herstein [11] established that a 2-torsion free prime ring R must be a commutative integral domain if it admits a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. Motivated by this result, Bell and Daif [5] showed that if R is a prime ring admitting a nonzero derivation d such that $d(xy) = d(yx)$ for all $x, y \in U$, where U is a nonzero two-sided ideal of R, then R is commutative. Daif [8] extended this result for semi-prime rings. Further, in 1992 Daif and Bell [9] established commutativity of semi-prime ring satisfying $d([x, y]) = [x, y]$ for all x, y in a nonzero ideal of R, and d a derivation of R. These results are generalized, in [15], for $*$-prime ring R satisfying one of the properties: (i) $d[x, y] = 0$, (ii) $d([x, y]) - [x, y] \in Z(R)$, (iii) $d([x, y]) + [x, y] \in Z(R)$, (iv) $d(x \circ y) = 0$, (v) $d(x \circ y) - x \circ y \in Z(R)$, (vi) $d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in J$, where J is a nonzero Jordan ideal of R. More recently, many authors (viz. for example, [3, 4, 6, 13, 14]) have investigated the connection between the commutativity of the category of prime (semi-prime) rings and some concrete additive mappings satisfying certain differential identities.

In this paper we continue this line of investigation by considering a more general concept that consists to establish a relationship between the structure of quotient rings R/P and derivations of R satisfying algebraic identities involving the prime ideal P.

2. Main results

We will make frequent use of the following facts which are very crucial for developing the proofs of our main results. Due to their ease, the proofs are left to the reader.

Fact 2.1. *Let R be a ring, I a nonzero ideal of R and P $(\neq I)$ a prime ideal of R such that $P \subset I$. If $aIb \subseteq P$, where $a, b \in R$, then $a \in P$ or $b \in P$.*

Fact 2.2. *Let R be a ring, I a nonzero ideal of R and P $(\neq I)$ a prime ideal of R such that $P \subset I$. If $[I, I] \subseteq P$, then R/P is a commutative ring.*

In [3], it is proved that a prime ring R must be commutative if it admits a derivation d satisfying one of the following conditions

$$\big(d(xy) \pm xy \in Z(R) \text{ for all } x, y \in I\big) \text{ or } \big(d(xy) \pm yx \in Z(R) \text{ for all } x, y \in I\big),$$

where I is a nonzero ideal I of R.

The question arises whether a similar result holds if we consider algebraic identities involving two derivations. In this context, it is also interesting to examine if the result of [3] can be extended to semi-prime rings.

The above questions lead to the following conjecture.

Conjecture 2.3. *Let R be a semi-prime ring and I a nonzero ideal of R. If R admits derivations d and g such that $\big(d(x)y+xg(y)\pm xy \in Z(R)$ for all $x, y \in I\big)$ or $\big(d(x)y + xg(y) \pm yx \in Z(R)$ for all $x, y \in I\big)$, then R is commutative.*

To solve this conjecture, the basic concept upon which our approach is based is to study the structure of a quotient ring R/P, where P is a prime ideal of R, equipped with derivations d and g satisfying

$$\overline{d(x)y + xg(y) \pm xy} \in Z(R/P) \text{ for all } x, y \in I,$$

where I is an ideal of R such that $P \subset I$ and $P \neq I$.

Theorem 2.4. *Let R be a ring, I a nonzero ideal of R and P ($\neq I$) a prime ideal of R such that $P \subset I$. If d and g are derivations of R, then the following assertions are equivalent*:

(1) $\overline{d(x)y + xg(y) \pm xy} \in Z(R/P)$ *for all* $x, y \in I$;
(2) $\overline{d(x)y + xg(y) \pm yx} \in Z(R/P)$ *for all* $x, y \in I$;
(3) $\overline{d(x)y + xg(y) \pm [x, y]} \in Z(R/P)$ *for all* $x, y \in I$;
(4) $\overline{d(x)y + xg(y) \pm x \circ y} \in Z(R/P)$ *for all* $x, y \in I$;
(5) R/P *is a commutative integral domain.*

Proof. We need to prove that (1) ⇒ (5), (2) ⇒ (5), (3) ⇒ (5), and (4) ⇒ (5).

(1) ⇒ (5) We are given that

$$\overline{d(x)y + xg(y) \pm xy} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.1}$$

Substituting yr for y in (2.1), one can see that

$$\overline{\big(d(x)y + xg(y) \pm xy\big)r + xyg(r)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

thereby obtaining

$$[xyg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{2.2}$$

Replacing x by tx in (2.2), we get

$$[t, r]Iyg(r) \subseteq P \quad \text{for all } y, r, t \in I.$$

Using Fact 2.1, we obviously obtain $[t, r] \in P$ or $yg(r) \in P$ for all $r \in I$. Consequently, I is a union of two additive subgroups I_1 and I_2, where

$$I_1 = \{r \in I \,/\, [I, r] \subseteq P\} \quad \text{and} \quad I_2 = \{r \in I \,/\, Ig(r) \subseteq P\}.$$

Since a group cannot be a union of two of its proper subgroups, then we have either $I = I_1$ or $I = I_2$.

If $I = I_1$, then $[I, I] \subseteq P$ and Fact 2.2 assures that R/P is a commutative integral domain.

If $I = I_2$, then $Ig(r) \subseteq P$ for all $r \in I$ and therefore $g(r) \in P$ for all $r \in I$. Hence the relation (2.1) yields $\big[(d(x) \pm x)y, r\big] \in P$ for all $x, y, r \in I$. Replacing x by xt in the last expression, it follows that

$$[xd(t)y, r] \in P \quad \text{for all } x, y, r, t \in I. \tag{2.3}$$

Substituting ux for x in (2.3), we arrive at $[u,r]xd(t)y \in P$ so that

$$[u,r]Id(t)yd(t) \subseteq P \quad \text{for all } \ y,r,t,u \in I.$$

Invoking Fact 2.1, the above relation assures that either $d(R) \subseteq P$ or $[I,I] \subseteq P$ in which case R/P is commutative by Fact 2.2. Now if $d(R) \subseteq P$ and $g(R) \subseteq P$, then our hypothesis becomes $\overline{xy} \in Z(R/P)$ for all $x,y \in I$, which implies that R/P is a commutative integral domain.

(2) $\Rightarrow$ (5) Suppose that

$$\overline{d(x)y + xg(y) + yx} \in Z(R/P) \quad \text{for all } \ x,y \in I. \tag{2.4}$$

Putting yr instead of y in the expression (2.4), we obtain

$$\overline{(d(x)y + xg(y) + yx)r + xyg(r) + y[r,x]} \in Z(R/P) \quad \text{for all } \ x,y \in I$$

so that

$$\big[xyg(r) + y[r,x], r\big] \in P \quad \text{for all } \ x,y,r \in I \tag{2.5}$$

which leads to

$$[x,r]yg(r) + x[yg(r),r] + \big[y[r,x],r\big] \in P \quad \text{for all } \ x,y,r \in I.$$

Substituting xy for y in the preceding relation and applying (2.5), one can see that

$$[x,r]xyg(r) - [x,r]y[x,r] \in P \quad \text{for all } \ x,y,r \in I. \tag{2.6}$$

Replacing r by $r+x$ in (2.6), we arrive at

$$[x,r]xyg(x) \in P \quad \text{for all } \ x,y,r \in I. \tag{2.7}$$

Writing rt for r in (2.7), one can easily verify that

$$[x,r]I[x,r]Ig(x) \subseteq P \quad \text{for all } \ x,r \in I.$$

Consequently, either $[x,r] \in P$ or $g(x) \in P$ for all $x,r \in I$. The sets of r for which theses conditions hold are additive subgroups of I with union equal I; thus by Brauer's trick either $[I,I] \subseteq P$ or $g(R) \subseteq P$. But, in the later case, equation (2.6) reduces to

$$[x,r]y[x,r] \in P \quad \text{for all } \ x,y,r \in I. \tag{2.8}$$

Invoking the primeness of P, equation (2.8) assures that $[I,I] \subseteq P$. Consequently, in both cases we arrive at $[I,I] \subseteq P$, so again an appeal to Fact 2.2 gives R/P is a commutative integral domain.

For the remaining implications, that is (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5), using the same reasoning as above we get the required result. □

The following theorem gives an affirmative answer to Conjecture 2.3.

Theorem 2.5. *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent*:
(1) $d(x)y + xg(y) \pm xy \in Z(R)$ *for all* $x, y \in I$;
(2) $d(x)y + xg(y) \pm yx \in Z(R)$ *for all* $x, y \in I$;
(3) $d(x)y + xg(y) \pm [x, y] \in Z(R)$ *for all* $x, y \in I$;
(4) $d(x)y + xg(y) \pm x \circ y \in Z(R)$ *for all* $x, y \in I$;
(5) *R is commutative.*

Proof. For the nontrivial sense, we will proceed by only proving (1) ⇒ (5) and the same technique remains valid for the other implications.

Assume that

$$d(x)y + xg(y) \pm xy \in Z(R) \text{ for all } x, y \in I.$$

By view of the semi-primeness of the ring R, there exists a family $\mathcal{P}$ of prime ideals $\mathcal{P} = \{P_\alpha \,/\, \alpha \in \Lambda\}$ such that $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$, thereby obtaining

$$[d(x)y + xg(y) \pm xy, r] \in P_\alpha \quad \text{for all} \quad \alpha \in \Lambda.$$

Invoking Theorem 2.4 we conclude that R/P_α is a commutative integral domain for all $\alpha \in \Lambda$ which, because of $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$, assures that R is commutative. □

Remark 2.6. If we set $g = d$ in Theorem 2.5, then we get a generalization of ([3], Theorems 2.1 and 2.3) for semi-prime rings.

In [1] it is proved that if d is a derivation of a 2-torsion free semi-prime ring R satisfying the following statements

$$\big(d(x)d(y) \pm xy = 0 \quad \text{for all} \quad x, y \in I\big) \quad \text{or} \quad \big(d(x)d(y) \pm yx = 0 \quad \text{for all} \quad x, y \in I\big),$$

where I is an ideal of R, then d is commuting on I. Furthermore, if $d(I) \neq 0$, then R has a nonzero central ideal.

Using a similar approach as in the preceding theorem, our next goal is to generalize the above result in two directions. First of all we will treat algebraic identities involving two derivations rather that one derivation. Secondly, we will consider the case when the underlying identity in not necessarily null but belongs to the center of the ring R, that is

$$\big(d(x)g(y) \pm xy \in Z(R) \text{ for all } x, y \in I\big) \text{ or } \big(d(x)g(y) \pm yx \in Z(R) \text{ for all } x, y \in I\big).$$

Theorem 2.7. *Let R be a ring, I a nonzero ideal of R and P $(\neq I)$ a prime ideal of R such that $P \subset I$. If d and g are derivations of R, then the following assertions are equivalent*:
(1) $\overline{d(x)g(y) \pm xy} \in Z(R/P)$ *for all* $x, y \in I$;
(2) $\overline{d(x)g(y) \pm yx} \in Z(R/P)$ *for all* $x, y \in I$;
(3) *R/P is a commutative integral domain.*

Moreover, if R/P is 2-torsion free and the derivations d and g are such that either $\overline{d(x)g(y) \pm [x,y]} \in Z(R/P)$ or $\overline{d(x)g(y) \pm x \circ y} \in Z(R/P)$ for all $x, y \in I$, then R/P is a commutative integral domain.

Proof. (1) ⇒ (3) Suppose that

$$\overline{d(x)g(y) \pm xy} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.9}$$

Substituting yr for y in (2.9), we get

$$\overline{\big(d(x)g(y) \pm xy\big)r + d(x)yg(r)} \in Z(R/P) \quad \text{for all } x, y, r \in I$$

thereby obtaining

$$[d(x)yg(r), r] \in P \quad \text{for all } x, y, r \in I. \tag{2.10}$$

Replacing x by xt in the above relation and subtracting it from (2.10), we get

$$[xd(t)yg(r), r] \in P \quad \text{for all } x, y, r, t \in I. \tag{2.11}$$

Writing ux for x in (2.11), we obtain

$$[u, r]Id(t)yg(r) \subseteq P \quad \text{for all } y, r, t, u \in I.$$

Using Fact 2.1, the above relation implies that $[u,r] \in P$ or $d(t)Ig(r) \subseteq P$ for all $r, t, u \in I$. In the later case, one again invoking Fact 2.1, we get $g(r) \in P$ or $d(I) \subseteq P$. If $d(I) \subseteq P$ or $g(I) \subseteq P$, then our hypothesis leads to $\overline{xy} \in Z(R/P)$ for all $x, y \in I$ so that R/P is a commutative integral domain. If $[I,I] \subseteq P$, then Fact 2.2 yields that R/P is commutative.

(2) ⇒ (3) We are given that

$$\overline{d(x)g(y) + yx} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.12}$$

Suppose that $Z(R/P) = \{\overline{0}\}$, then R/P is a non-commutative ring. Hence equation (2.12) forces

$$d(x)g(y) + yx \in P \quad \text{for all } x, y \in I. \tag{2.13}$$

Substituting yx for y in (2.13), we arrive at

$$d(x)yg(x) \in P \quad \text{for all } x, y \in I. \tag{2.14}$$

Therefore $d(I) \subseteq P$ or $g(I) \subseteq P$ in which case equation (2.13) assures that $P = I$, a contradiction. Consequently, we conclude that $Z(R/P) \neq \{\overline{0}\}$.

Substituting yz for y in (2.12), where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, we easily get

$$[d(x)yg(z), r] \in P \quad \text{for all } x, y, r \in I. \tag{2.15}$$

Since equation (2.15) is the same as equation (2.10), arguing as before we find that $[I,I] \subseteq P$ or $d(I) \subseteq P$ or $g(z) \in P$. In all that follows, R/P is a commutative integral domain.

Now if $\overline{d(x)g(y) - yx} \in Z(R/P)$ for all $x, y \in I$, then using similar techniques with essential variations, we get the required result.

Assume that

$$\overline{d(x)g(y) \pm [x,y]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.16}$$

Taking $x = y$ in (2.16), we see that

$$\overline{d(x)g(x)} \in Z(R/P) \quad \text{for all } x \in I. \tag{2.17}$$

Replacing y by yx in (2.16), one obtains

$$[d(x)yg(x), x] \in P \quad \text{for all } x, y \in I. \tag{2.18}$$

Substituting $g(x)y$ for y in (2.18) and invoking (2.17), we find that

$$d(x)g(x)[yg(x), x] \in P \quad \text{for all } x, y \in I.$$

Since P is prime, we conclude that

$$\big(d(x)g(x) \in P\big) \text{ or } \big([yg(x), x] \in P\big) \text{ for all } x, y \in I. \tag{2.19}$$

If $[yg(x), x] \in P$ for all $y \in I$, then putting ry instead of y, we thereby get $[r, x]yg(x) \in P$ for all $y, r \in I$ so that either $g(x) \in P$ and a fortiori $d(x)g(x) \in P$ or $\overline{x} \in Z(R/P)$. On the other hand, if $\overline{x} \in Z(R/P)$, then our hypothesis becomes $\overline{d(x)g(y)} \in Z(R/P)$ for all $y \in I$. Replacing y by yx in the last relation, we get $\overline{d(x)yg(x)} \in Z(R/P)$ for all $y \in I$. In particular, $\overline{d(x)g(x)yd(x)g(x)} \in Z(R/P)$. The primeness of P implies that $d(x)g(x) \in P$ or $\overline{yd(x)g(x)} \in Z(R/P)$ for all $y \in I$. Whence, either R/P is commutative or $d(x)g(x) \in P$.

Accordingly, equation (2.19) reduces to $d(x)g(x) \in P$ for all $x \in I$. Linearizing this expression, we obtain

$$d(x)g(y) + d(y)g(x) \in P \quad \text{for all } x, y \in I. \tag{2.20}$$

Writing yr for y in (2.20), it is obvious to see that

$$\big(d(x)g(y)+d(y)g(x)\big)r+d(x)yg(r)+d(y)rg(x)+yd(r)g(x)-d(y)g(x)r \in P \tag{2.21}$$

thereby obtaining

$$d(x)yg(r) + d(y)[r, g(x)] + yd(r)g(x) \in P \quad \text{for all } x, y, r \in I. \tag{2.22}$$

Replacing y by $d(x)y$ in (2.22) and combining it with (2.22), one can verify that

$$d^2(x)y[r, g(x)] \in P \quad \text{for all } x, y, r \in I.$$

Using Fact 2.1 together with Brauer's trick (that is a group cannot be a union of two of its proper subgroups), we conclude that $[r, g(x)] \in P$ for all $x, r \in I$ or $d^2(x) \in P$ for all $x \in I$. Substituting xy for x in the last relation, we find that

$$d^2(x)y + 2d(x)d(y) + xd^2(y) \in P \quad \text{for all } x, y \in I.$$

Applying the fact that $d^2(R) \subseteq P$, we get $2d(x)d(y) \in P$ for all $x, y \in I$. Writing xr for x and using 2-torsion freeness hypothesis, we arrive at $d(x)rd(y) \in P$. By virtue of Fact 2.1, the last equation implies that $d(R) \subseteq P$ and by expression (2.16), we have necessarily $\overline{[x, y]} \in Z(R/P)$ for all $x, y \in I$. Clearly, we conclude that R/P is commutative.

Now if $[r, g(x)] \in P$ for all $x, r \in I$; putting xr instead of x, one can verify that

$$[r, x]g(r) \in P \text{ for all } x, r \in I.$$

By appropriate expansion, obtain

$$[r, x]Ig(r) \subseteq P \text{ for all } x, r \in I.$$

Therefore, we claim that $g(R) \subseteq P$ or R/P is an integral domain. In light of the first case, the hypothesis forces that R/P is a commutative integral domain. Now consider the case

$$\overline{d(x)g(y) \pm x \circ y} \in Z(R/P) \quad \text{for all} \quad x, y \in I. \tag{2.23}$$

Analogously, if $Z(R/P) = \{\overline{0}\}$, then using the same techniques as in the proof of assertion (2), it is obvious that R/P is an integral domain. Hence, it follows that $Z(R/P) \neq \{\overline{0}\}$.

Returning to the main equation and writing yz for y, where $\overline{z} \in Z(R/P) \setminus \{\overline{0}\}$, then we get $d(R) \subseteq P$ or $g(z) \in P$ or R/P is an integral domain. On the other hand, the hypothesis leads to $\pm\overline{2xz} \in Z(R/P)$ for all $x \in I$. By view of 2-torsion freeness, it is easy to see that $\overline{x} \in Z(R/P)$ for all $x \in I$. Whence, R/P is a commutative integral domain. □

As an application of Theorem 2.7, we get the desired generalization of [1].

Corollary 2.8. *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent*:

(1) *$d(x)g(y) \pm xy \in Z(R)$ for all $x, y \in I$;*
(2) *$d(x)g(y) \pm yx \in Z(R)$ for all $x, y \in I$;*
(3) *R is commutative.*

Proposition 2.9. *Let R be a semi-prime ring and I a nonzero ideal of R. If d and g are derivations of R such that $d(x)g(y) \pm [x, y] \in Z(R)$ or $d(x)g(y) \pm x \circ y \in Z(R)$ for all $x, y \in I$, then R is commutative or there exist a minimal prime ideal P of R such that char$(R/P) = 2$.*

Proof. Suppose on the contrary that char $(R/P) \neq 2$ for any minimal prime ideal P of R. By hypothesis, we have $d(x)g(y) \pm [x, y] \in Z(R)$ for all $x, y \in I$, then $[d(x)g(y) \pm [x, y], r] = 0$ for all $x, y, r \in I$. Since R is semi-prime, there exists a family $\mathcal{P}$ of prime ideals P such that $\bigcap_{P \in \mathcal{P}} P = (0)$ and therefore $[d(x)g(y) \pm [x, y], r] \in P$ for all $P \in \mathcal{P}$. That is

$$\overline{d(x)g(y) \pm [x, y]} \in Z(R/P) \quad \text{for all} \quad x, y \in I \quad \text{and for all} \quad P \in \mathcal{P}.$$

Applying Theorem 2.7, it follows that R/P is commutative. Which, because of $\bigcap_{P \in \mathcal{P}} P = (0)$, assures that R is commutative.

Now we aim to prove that $d(x)g(y) \pm x \circ y \in Z(R)$ for all $x, y \in I$, then using similar arguments we get the required result. □

In [9] it is proved that a prime ring R must be commutative if it admits a derivation d such that either $xy+d(xy)=yx+d(yx)$ or $xy-d(xy)=yx-d(yx)$ for all x,y in a nonzero ideal I of R. Motivated by this result, Ashraf and Rehman [4] showed that if a 2-torsion free prime ring R admits a derivation d satisfying $d(x\circ y)\pm(x\circ y)=0$ for all x,y in a nonzero ideal I of R, then R is commutative. However this result is incorrect because if R is commutative, then the hypothesis $d(x\circ y)\pm(x\circ y)=0$ for all $x,y\in I$ leads to $R=\{0\}$, a contradiction. In fact, one can prove that if R is a 2-torsion free prime ring, then there is no derivation d of R such that $d(x\circ y)\pm(x\circ y)=0$ for all $x,y\in I$.

Inspired by the result of [9], our next aim is to study the relationship between the commutativity of a semi-prime ring and the existence of derivations d and g such that either

$$\big(d(xy)-g(yx)\pm[x,y]\in Z(R)\big) \quad \text{or} \quad \big(d(xy)-g(yx)\pm x\circ y\in Z(R)\big)$$

for all x,y in a nonzero ideal I of R.

Theorem 2.10. *Let R be a ring, I a nonzero ideal of R and P $(\neq I)$ a prime ideal of R such that $P\subset I$ and R/P is 2-torsion free. If d and g are derivations of R, then the following assertions are equivalent*:

(1) $\overline{d(xy)-g(yx)\pm[x,y]}\in Z(R/P)$ *for all* $x,y\in I$;

(2) $\overline{d(xy)-g(yx)\pm x\circ y}\in Z(R/P)$ *for all* $x,y\in I$;

(3) R/P *is a commutative integral domain.*

Proof. (1) ⇒ (2) Assuming that

$$\overline{d(xy)-g(yx)\pm[x,y]}\in Z(R/P) \quad \text{for all } x,y\in I. \tag{2.24}$$

If $Z(R/P)=\{\overline{0}\}$, then the relation (2.24) becomes

$$d(xy)-g(yx)\pm[x,y]\in P \quad \text{for all } x,y\in I. \tag{2.25}$$

Replacing y by yx in (2.25), we get

$$xyd(x)-yxg(x)\in P \quad \text{for all } x,y\in I. \tag{2.26}$$

Substituting ry for y in (2.26) and combining it with (2.26), we arrive at

$$[x,r]yd(x)\in P \quad \text{for all } x,y,r\in I$$

so Fact 2.1 yields $d(R)\subseteq P$ or R/P is commutative, a contradiction. Now if $d(R)\subseteq P$, then (2.26) reduces to $yg(x)x\in P$ and using the fact that a prime ideal is proper, we conclude that $g(x)x\in P$ for all $x\in I$. A linearization of this expression yields

$$g(x)y+g(y)x\in P \quad \text{for all } x,y\in I. \tag{2.27}$$

Writing yr for y in (2.27), we find that

$$g(y)[r,x]+yg(r)x\in P \quad \text{for all } x,y,r\in I. \tag{2.28}$$

If we replace y by ty in (2.28), then one can see that

$$g(t)I[r,x]\subseteq P \quad \text{for all } x,r,t\in I.$$

In light of primeness, it follows that $g(R) \subseteq P$ and thus the main equation reduces to $[x, y] \in P$ for all $x, y \in I$. Therefore R/P is an integral domain, a contradiction.

Now if $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in (2.24), we obtain

$$[xyd(z) - yxg(z), r] \in P \quad \text{for all } x, y, r \in I. \tag{2.29}$$

Putting ry instead of y in (2.29) and subtracting it from (2.29), we arrive at

$$\big[[x, r]yd(z), r\big] \in P \quad \text{for all } x, y, r \in I. \tag{2.30}$$

Substituting xt for x in (2.30) and using it, we can see that

$$\big[x[t, r]yd(z), r\big] \in P \quad \text{for all } x, y, r, t \in I. \tag{2.31}$$

Replacing x by tx in (2.31), one obtains

$$[t, r]x[t, r]yd(z) \in P \quad \text{for all } x, y, r, t \in I.$$

Accordingly, R/P is an integral domain or $\overline{d(z)} = \overline{0}$. In the later case, the expression (2.29) forces that $[yxg(z), r] \in P$ for all $x, y, r \in I$. If we replace y by ty, then we get $[t, r]yxg(z) \in P$. In particular, $[t, r]yg(z)xg(z) \in P$. By view of Fact 2.1, we find that either R/P is commutative or $\overline{g(z)} = \overline{0}$. Now setting $y = z$ in our hypothesis, one can easily verify that $\overline{\big(d(x) - g(x)\big)z} \in Z(R/P)$ for all $x \in I$. Then, we have $\overline{(d - g)(R)} \subseteq Z(R/P)$. So, the last equation assures that

$$\overline{g[x, y] \pm [x, y]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.32}$$

Replacing x by $[x, r]$, we get

$$\overline{\big[g[x, r] \pm [x, r], y\big] + \big[[x, r], g(y)\big]} \in Z(R/P) \quad \text{for all } x, y, r \in I.$$

Invoking equation (2.32), one can see that

$$\overline{\big[[x, g(y)], g(y)\big]} \in Z(R/P) \quad \text{for all } x, y \in I. \tag{2.33}$$

Substituting $xg(y)$ for x in (2.33), we obtain

$$\overline{\big[[x, g(y)], g(y)\big]g(y)} \in Z(R/P) \quad \text{for all } x, y \in I$$

in such a way that

$$\overline{\big[[x, g(y)], g(y)\big]} = \overline{0} \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x, y \in I.$$

Putting rx instead of x, we find that

$$\overline{[r, g(y)][x, g(y)]} = \overline{0} \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x, y, r \in I.$$

Writing tx for x in the above relation, we arrive at

$$[x, g(y)]I[x, g(y)] \subseteq P \quad \text{or} \quad \overline{g(y)} \in Z(R/P) \quad \text{for all } x, y \in I$$

and thus $\overline{g(R)} \subseteq Z(R/P)$. Consequently, expression (2.32) becomes $\overline{[x, y]} \in Z(R/P)$ for all $x, y \in I$, proving that R/P is a commutative integral domain.

The proof of (2) ⇒ (3) is similar and requires only slight modifications. □

Corollary 2.11. *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If d and g are derivations of R, then the following assertions are equivalent*:

(1) $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;

(2) $d(xy) \pm g(yx) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;

(3) *R is a commutative integral domain.*

The following proposition gives an improved version of ([2], Theorem 3.8) and ([12], Theorem 1).

Proposition 2.12. *Let R be a semi-prime ring, I a nonzero ideal of R. If d and g are derivations of R satisfying one of the following conditions*:

(1) $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ *for all* $x,y \in I$;

(2) $d(xy) \pm g(yx) \pm x \circ y \in Z(R)$ *for all* $x,y \in I$;

then either R is commutative or there exists a minimal prime ideal P of R such that $char(R/P) = 2$.

Proof. We need only to prove assertion (1) because assertion (2) can be proved similarly.

Suppose that $d(xy) \pm g(yx) \pm [x,y] \in Z(R)$ for all $x,y \in I$ and char $(R/P) \neq 2$ for any minimal prime ideal P of R. Then we obtain $\big[d(xy) \pm g(yx) \pm [x,y], r\big] = 0$ for all $x,y,r \in I$. According to semi-primeness, there exists a family $\mathcal{P}$ of prime ideals P such that $\bigcap\limits_{P \in \mathcal{P}} P = (0)$ and therefore $\big[d(xy) \pm g(yx) \pm [x,y], r\big] \in P$ for all $P \in \mathcal{P}$. Hence

$$\overline{d(xy) \pm g(yx) \pm [x,y]} \in Z(R/P) \quad \text{for all } \ x,y \in I \quad \text{and for all } \ P \in \mathcal{P}$$

and Theorem 2.10 assures that R/P is an integral domain. Accordingly, for all $x,y \in I$ we get $[x,y] \in P$ (for all $P \in \mathcal{P}$) so that $[x,y] = 0$ proving that R is commutative. ☐

Using the same approach as in the proof of Theorem 2.10, one can prove the following result.

Theorem 2.13. *Let R be a ring, I a nonzero ideal of R and P $(\neq I)$ a prime ideal of R such that $P \subset I$ and R/P is 2-torsion free. If d and g are derivations of R satisfying $\overline{d(xy) \pm g(yx)} \in Z(R/P)$ for all $x,y \in I$, then $\big(d(R) \subseteq P$ and $g(R) \subseteq P\big)$ or R/P is a commutative integral domain.*

Corollary 2.14. *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If d is a nonzero derivation of R, then the following assertions are equivalent*:

(1) $d[x,y] \in Z(R)$ *for all* $x,y \in I$;

(2) $d(x \circ y) \in Z(R)$ *for all* $x,y \in I$;

(3) *R is a commutative integral domain.*

The following example demonstrates that the `primeness` hypothesis imposed on the ideal P in our theorems is not superfluous.

Example 2.15. Let us consider the ring $R = \left\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \,\middle|\, a, b, c \in \mathbb{Z}\right\}$ and set $I = \left\{\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \,\middle|\, a \in \mathbb{Z}\right\}$. Since R is not prime, the ideal $P = \{0\}$ is not prime. Moreover, the mappings d and g defined from R into itself by

$$d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a+b-c \\ 0 & 0 \end{pmatrix} \text{ and } g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-b-c \\ 0 & 0 \end{pmatrix}$$

are derivations of R which satisfy conditions of Theorems 2.4, 2.5, 2.7, 2.10 and 2.13; but R is a noncommutative ring.

References

[1] S. Ali and S. L. Huang, *On derivations in semiprime rings*, Algebr. Represent. Theory **15** (2012), no. 6, 1023–1033. https://doi.org/10.1007/s10468-011-9271-9

[2] N. Argaç, *On prime and semiprime rings with derivations*, Algebra Colloq. **13** (2006), no. 3, 371–380. https://doi.org/10.1142/S1005386706000320

[3] M. Ashraf and N. Rehman, *On derivation and commutativity in prime rings*, East-West J. Math. **3** (2001), no. 1, 87–91.

[4] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math. **42** (2002), no. 1-2, 3–8. https://doi.org/10.1007/BF03323547

[5] H. E. Bell and M. N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995), no. 4, 337–343. https://doi.org/10.1007/BF01876049

[6] H. E. Bell and W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987), no. 1, 92–101. https://doi.org/10.4153/CMB-1987-014-x

[7] K. Bouchannafa, M. Abdallah Idrissi, and L. Oukhtite, *Relationship between the structure of a quotient ring and the behavior of certain additive mappings*, Commun. Korean Math. Soc. **37** (2022), no. 2, 359–370. https://doi.org/10.4134/CKMS.c210126

[8] M. N. Daif, *Commutativity results for semiprime rings with derivations*, Int. J. Math. Math. Sci. **21** (1998), no. 3, 471–474. https://doi.org/10.1155/S0161171298000660

[9] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci. **15** (1992), no. 1, 205–206. https://doi.org/10.1155/S0161171292000255

[10] N. J. Divinsky, *On commuting automorphisms of rings*, Trans. Roy. Soc. Canada Sect. III **49** (1955), 19–22.

[11] I. N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978), no. 3, 369–370. https://doi.org/10.4153/CMB-1978-065-x

[12] M. Hongan, *A note on semiprime rings with derivation*, Int. J. Math. Math. Sci. **20** (1997), no. 2, 413–415. https://doi.org/10.1155/S0161171297000562

[13] M. A. Idrissi and L. Oukhtite, *Structure of a quotient ring R/P with generalized derivations acting on the prime ideal P and some applications*, Indian J. Pure Appl. Math. **53** (2022), no. 3, 792–800. https://doi.org/10.1007/s13226-021-00173-x

[14] L. Oukhtite, *Posner's second theorem for Jordan ideals in rings with involution*, Expo. Math. **29** (2011), no. 4, 415–419. https://doi.org/10.1016/j.exmath.2011.07.002

[15] L. Oukhtite, A. Mamouni, and M. Ashraf, *Commutativity theorems for rings with differential identities on Jordan ideals*, Comment. Math. Univ. Carolin. **54** (2013), no. 4, 447–457.

[16] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100. https://doi.org/10.2307/2032686

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