Bull. Korean Math. Soc. **60** (2023), No. 5, pp. 1265–1280
https://doi.org/10.4134/BKMS.b220645
pISSN: 1015-8634 / eISSN: 2234-3016

BETCHOV-DA RIOS EQUATION BY NULL CARTAN, PSEUDO NULL AND PARTIALLY NULL CURVE IN MINKOWSKI SPACETIME

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Abstract. The aim of this paper is to investigate Betchov-Da Rios equation by using null Cartan, pseudo null and partially null curve in Minkowski spacetime. Time derivative formulas of frame of s parameter null Cartan, pseudo null and partially null curve are examined, respectively. By using the obtained derivative formulas, new results are given about the solution of Betchov-Da Rios equation. The differential geometric properties of these solutions are obtained with respect to Lorentzian causal character of s parameter curve. For a solution of Betchov-Da Rios equation, it is seen that null Cartan s parameter curves are space curves in three-dimensional Minkowski space. Then all points of the soliton surface are flat points of the surface for null Cartan and partially null curve. Thus, it is seen from the results obtained that there is no surface corresponding to the solution of Betchov-Da Rios equation by using the pseudo null s parameter curve.

1. Introduction

The mathematical environment in which Einstein's theory of special relativity is most simply expressed is called Minkowski space or Minkowski spacetime. In this concept, a spacetime is represented by a four-dimensional manifold made up of the three regular dimensions of space and one dimension of time.

The most natural way to see how space and time are intimately linked is in a world representation with four dimensions, three spatial and one temporal, as provided by mathematics. This goes beyond math; the physics just makes more sense when viewed in four spatial dimensions with time as a parameter than in three spatial dimensions alone.

Received September 19, 2022; Accepted March 16, 2023.
2020 *Mathematics Subject Classification.* Primary 35Q55, 53A05, 53Z05.
Key words and phrases. Betchov-Da Rios equation, localized induction equation (LIE), null Cartan curve, pseudo null curve, partially null curve, Minkowski spacetime.
This work was funded by the National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Minkowski space and Euclidean space are frequently compared in theoretical physics. Minkowski space has one timelike dimension in addition to the spacelike dimensions that characterize Euclidean space. As a result, the Poincaré group is the symmetry group for Minkowski space and the Euclidean group is the symmetry group for Euclidean space.

In conventional methods to quantum gravity, specifying equal time commutation relation is a natural application of the background causal structure. The kinematical question of which events occur at equal times cannot be answered without a background causal structure of some kind until the dynamical problem is resolved. Additionally, any idea of causation that results will probably be unclear because the metric is itself. In this study, the Minkowski spacetime metric is represented as follows:

$$\langle u, v\rangle_L = -u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

for all $u, v \in \mathbb{E}_1^4$. According to this metric, regular curves in Minkowski spacetime are classified.

The motion of the vortex filament is one of the research areas that both mathematicians and physicists have been working on in recent years. The foundations of the geometric relationship between the motion of the inextensible curve and the soliton theory are based on the work of Da Rios in 1906 [5]. Levi Civita and Betchov both reexamined the significance of this work of Da Rios [4, 10]. Hasimoto also shown how Betchov's findings and nonlinear Schrödinger equation are related [9]. There are studies that include both physical and geometric applications of the solutions of nonlinear Schrödinger equation [6, 7, 16].

Smooth and expressive, the thin filament does not intersect on its own. The localized induction approximation, used by Da Rios, relates to the velocity caused by a vortex line at an outside point (LIA). The following equation

$$\Psi_t = \Psi_s \times \Psi_{ss} \times \Psi_{sss} \tag{1}$$

is stated the movement of a curve in different kind of space as the movement of a vortex in an inviscid fluid. This equation is also named as Betchov-Da Rios (BDR) equation [1–3, 8, 13, 14, 18–20].

In light of the geometric invariants and findings, soliton surface, which is the solution to BDR equation in Euclidean space, is analyzed, and some of its aspects are examined [12].

In [11], the geometric features of nonnull soliton surface related with BDR equation in Minkowski spacetime are investigated. It is demonstrated in this situation that the s parameter curves are nonnull for all t values of the soliton surface. For all t, we give derivative formulas of Frenet-Serret frame of the s parameter curves. The geometric structure of this nonnull soliton surface is discussed, and some results such as the mean curvature vector field, linear map of Weingarten type, Gaussian curvature and geometric invariants are presented. The existence conditions of the flat points of this soliton surface are obtained.

In this paper, we firstly give some brief information about null Cartan, pseudo null and partially null curve in Minkowski spacetime. Then, we investigate BDR equation by using null Cartan, pseudo null and partially null curve in Minkowski spacetime. Time derivative formulas of frames of s parameter null Cartan, pseudo null and partially null curve are examined, respectively. Then, it is proved that derivatives of Cartan and pseudo null frames according to time parameter t are all zero vectors. Moreover, it is obtained that derivatives of partially null frames according to time parameter t are different from zero vectors. The differential geometric properties of these solutions are obtained with respect to the causal character of s parameter curve. For a solution of BDR equation, it is seen that null Cartan s parameter curves lie on three dimensional Minkowski space. Then all points of the soliton surface are flat points of the surface for null Cartan and partially null curve. Thus, it is seen from results obtained that there is no surface corresponding to the solution of BDR equation by using pseudo null s parameter curve.

It is obtained in [11] that the s parameter curves of the nonnull soliton surface were all nonnull curves. This study investigated the cases in which the s parameter curves are, respectively, null Cartan, pseudo null, and partly null. Because the characters of the s parameter curve of the soliton surface are null rather than nonnull, different results have emerged. Unlike [11], all derivatives of Cartan and pseudo null frames with respect to time parameter t are zero vectors in this study. However, in addition to the results on the existence of flat points which are presented in [11], it is seen that all points for null Cartan and partial null curves were flat points. It is shown that there is no such soliton surface which has the pseudo null s parameter curve.

2. Preliminaries

Derivative formulas of pseudo orthonormal frames are given as following cases [15, 17].

Case 1. Let $\alpha : I \to \mathbb{E}_1^4$ be a null Cartan curve. Then derivative formulas of Cartan frame are given as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & -\sigma \\ \sigma & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{2}$$

where Cartan frame fields satisfy the equations

$$\begin{aligned} &\langle T, T\rangle_L = \langle N_2, N_2\rangle_L = \langle N_2, N_3\rangle_L = 0, \\ &\langle N_3, N_3\rangle_L = \langle N_1, N_1\rangle_L = 1, \quad \langle T, N_2\rangle_L = -1, \\ &\langle N_1, N_3\rangle_L = \langle T, N_1\rangle_L = \langle T, N_3\rangle_L = \langle N_1, N_2\rangle_L = 0. \end{aligned} \tag{3}$$

Moreover, the cross product of Cartan frame fields are obtained as follows:

$$T \times_L N_1 \times_L N_2 = -N_3,$$

$$T \times_L N_1 \times_L N_3 = -T,$$
$$T \times_L N_3 \times_L N_2 = N_1, \tag{4}$$
$$N_1 \times_L N_2 \times_L N_3 = -N_2.$$

Case 2. Let $\alpha : I \to \mathbb{E}_1^4$ be a pseudo null curve. Then derivative formulas of pseudo null frame are given as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\tau \\ -\kappa & 0 & -\sigma & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{5}$$

where T, N_1, N_2, N_3 satisfy the equations

$$\langle T,T\rangle_L = \langle N_1,N_3\rangle_L = \langle N_2,N_2\rangle_L = 1,$$
$$\langle N_3,N_3\rangle_L = \langle N_1,N_1\rangle_L = \langle T,N_3\rangle_L = 0, \tag{6}$$
$$\langle T,N_1\rangle_L = \langle T,N_2\rangle_L = \langle N_1,N_2\rangle_L = \langle N_2,N_3\rangle_L = 0.$$

Case 3. Let $\alpha : I \to \mathbb{E}_1^4$ be a partially null curve. Then derivative formulas of partially null frame are given as follows:

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & -\sigma \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{7}$$

where T, N_1, N_2, N_3 satisfy the equations

$$\langle T,T\rangle_L = \langle N_1,N_1\rangle_L = \langle N_2,N_2\rangle_L = 1,$$
$$\langle N_3,N_3\rangle_L = \langle N_2,N_2\rangle_L = \langle N_2,N_3\rangle_L = 0, \tag{8}$$
$$\langle T,N_1\rangle_L = \langle T,N_2\rangle_L = \langle T,N_3\rangle_L = \langle N_1,N_2\rangle_L = 0.$$

3. BDR equation by using null Cartan curve

This section contains the examination of the soliton surface $\mathcal{M} : \Psi = \Psi(s,t)$ associated with BDR equation by using derivative formulas for the Cartan frame of the null Cartan s parameter curve of $\mathcal{M}$.

3.1. Time derivative formulas of null Cartan frame

Theorem 3.1. *Let's assume that $\Psi = \Psi(s,t)$ is a solution of BDR equation, where $\Psi = \Psi(s,t)$ is a null Cartan curve for every t. Then, derivatives of Cartan frames according to time parameter t are all zero vectors.*

Proof. The metric coefficient matrix is obtained as follows:

$$I^* = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by Eq. (3). We define semi skew symmetric matrix A by equation $I^*AI^*+A^T = 0_{4\times 4}$. If we denote

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

then we get the following matrix equation

$$\begin{bmatrix} a_{11}+a_{33} & a_{12}-a_{23} & 2a_{13} & a_{14}-a_{43} \\ a_{21}-a_{32} & 2a_{22} & a_{23}-a_{12} & a_{24}+a_{42} \\ 2a_{31} & a_{32}-a_{21} & a_{11}+a_{33} & a_{34}-a_{41} \\ a_{41}-a_{34} & a_{24}+a_{42} & a_{43}-a_{14} & 2a_{44} \end{bmatrix} = 0_{4\times 4}.$$

This implies

$$\begin{aligned} &a_{11} = -a_{33} = \zeta_{11}, \quad a_{12} = a_{23} = \zeta_{12}, \quad a_{13} = 0, \\ &a_{14} = a_{43} = \zeta_{14}, \quad a_{21} = a_{32} = \zeta_{21}, \quad a_{22} = 0, \\ &a_{24} = -a_{42} = \zeta_{24}, \quad a_{31} = 0, \quad a_{34} = a_{41} = \zeta_{34}, \quad a_{44} = 0. \end{aligned}$$

There exist smooth functions: $\zeta_{11}, \zeta_{12}, \zeta_{14}, \zeta_{21}, \zeta_{24}$, and ζ_{34} such that

$$\frac{d}{dt}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & 0 & \zeta_{14} \\ \zeta_{21} & 0 & \zeta_{12} & \zeta_{24} \\ 0 & \zeta_{21} & -\zeta_{11} & \zeta_{34} \\ \zeta_{34} & -\zeta_{24} & \zeta_{14} & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

These functions are needed to find in terms of the curvature functions κ, τ and σ. By Eq. (2), we have

$$\begin{aligned} \Psi_s(s,t) &= T(s,t), \\ \Psi_{ss}(s,t) &= N_1(s,t), \\ \Psi_{sss}(s,t) &= -T(s,t) + \tau(s,t)N_2(s,t). \end{aligned}$$

By Eq. (4), BDR equation implies that

$$\begin{aligned} \Psi_t(s,t) &= \Psi_s(s,t) \times_L \Psi_{ss}(s,t) \times_L \Psi_{sss}(s,t) \\ &= -N_3(s,t). \end{aligned}$$

Then, we obtain

$$\begin{aligned} \frac{\partial}{\partial s}(\Psi_t(s,t)) &= \frac{\partial}{\partial s}(-N_3(s,t)) \\ &= -\sigma(s,t)T(s,t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\Psi_s(s,t)) &= \frac{\partial}{\partial t}(T(s,t)) \\ &= \zeta_{11}(s,t)T(s,t) + \zeta_{12}(s,t)N_1(s,t) + \zeta_{14}(s,t)N_3(s,t). \end{aligned}$$

By compatibility condition $\Psi_{st} = \Psi_{ts}$, we get

$$\begin{aligned}\zeta_{11}(s,t) &= -\sigma(s,t),\\ \zeta_{12}(s,t) &= 0,\\ \zeta_{14}(s,t) &= 0.\end{aligned}$$

By $T_{st} = T_{ts}$, we find

$$\begin{aligned}\frac{\partial}{\partial s}(T_t(s,t)) &= \frac{\partial}{\partial s}\left(-\sigma(s,t)T(s,t)\right)\\ &= -\frac{\partial}{\partial s}\sigma(s,t)T(s,t) - \sigma(s,t)N_1(s,t).\end{aligned}$$

Furthermore, we also have

$$\begin{aligned}\frac{\partial}{\partial t}(T_s(s,t)) &= \frac{\partial}{\partial t}(N_1(s,t))\\ &= \zeta_{21}(s,t)T(s,t) + \zeta_{24}(s,t)N_3(s,t).\end{aligned}$$

This implies that

$$\begin{aligned}\zeta_{24}(s,t) &= 0,\\ \sigma(s,t) &= 0,\\ \zeta_{21}(s,t) &= -\frac{\partial}{\partial s}\sigma(s,t) = 0.\end{aligned}$$

From $(N_1)_{st} = (N_1)_{ts}$, we get

$$\zeta_{34}(s,t) = 0.$$

We find that all smooth functions: $\zeta_{11}, \zeta_{12}, \zeta_{14}, \zeta_{21}, \zeta_{24}$, and ζ_{34} are zero. Then, we get derivatives of Cartan frames according to time parameter t are all zero vectors. □

Corollary 3.2. *Let's assume that $\Psi = \Psi(s,t)$ is a solution of BDR equation where $\Psi = \Psi(s,t)$ is a null Cartan curve for every t. Then we have*

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Here $\{T, N_1, N_2, N_3\}$ is Cartan frame, κ is the curvature function, τ is the first torsion function $\Psi = \Psi(s,t)$ for all t.

Proof. By $T_{st} = T_{ts}$, we obtain that $\sigma(s,t) = 0$ by proof of Theorem 3.1. □

3.2. Surface associated with BDR equation by using null Cartan curve

Let $\mathcal{M} : \Psi = \Psi(s,t)$ be a two-dimensional surface associated with BDR equation. Then $T_p(\mathcal{M}) = span\{\Psi_s = T,\ \Psi_t = -N_3\}$ is the tangent space of surface $\mathcal{M}$ at arbitrary point P. Given that the quadruple $\{\Psi_s, \Psi_t, N_1, N_2\}$ is positively oriented frame in $\mathbb{E}_1^4$, $N_p(\mathcal{M}) = span\{N_1, N_2\}$ is the normal space. The resulting decomposition is as follows:

$$\mathbb{E}_1^4 = T_p(\mathcal{M}) \oplus N_p(\mathcal{M}).$$

The below explains how the metric tensor's components are obtained:

$$\begin{aligned} g_{11}(s,t) &= \langle T(s,t), T(s,t)\rangle_L = 0, \\ g_{12}(s,t) &= \langle T(s,t), -N_3(s,t)\rangle_L = 0, \\ g_{22}(s,t) &= \langle -N_3(s,t), -N_3(s,t)\rangle_L = 1. \end{aligned}$$

This means that

$$g_{11}g_{22} - g_{12}^2 = 0.$$

For normal frame field $\{N_1, N_2\}$ of $\mathcal{M}$, we have the second derivative formulas:

$$\begin{aligned} D_{\Psi_s}\Psi_s &= \Psi_{ss} = \Gamma_{11}^1\Psi_s + \Gamma_{11}^2\Psi_t + c_{11}^1 N_1 + c_{11}^2 N_2, \\ D_{\Psi_s}\Psi_t &= \Psi_{st} = \Gamma_{12}^1\Psi_s + \Gamma_{12}^2\Psi_t + c_{12}^1 N_1 + c_{12}^2 N_2, \\ D_{\Psi_t}\Psi_t &= \Psi_{tt} = \Gamma_{22}^1\Psi_s + \Gamma_{22}^2\Psi_t + c_{22}^1 N_1 + c_{22}^2 N_2, \end{aligned}$$

where Γ_{ij}^k are Christoffel's symbols and c_{ij}^k are functions on $\mathcal{M}$ for $i,j,k = 1,2$. However, we also have

$$\Psi_{ss}(s,t) = N_1(s,t),\ \Psi_{st} = 0,\ \Psi_{tt} = 0.$$

So, we get Christoffel's symbols:

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^2 = \Gamma_{22}^1 = 0.$$

We obtain the functions c_{ij}^k as follows:

$$c_{11}^1 = 1, \quad c_{11}^2 = c_{12}^1 = c_{12}^2 = c_{22}^1 = c_{22}^2 = 0.$$

Due to the nondegenerate structure of the metric induced on $\mathcal{M}$, i.e., $c_{11}^1 = 1$, at least one of the coefficients c_{ij}^k is not zero. We get the second fundamental tensor of $\mathcal{M}$ as follows:

$$\Pi(\Psi_s, \Psi_s) = N_1, \quad \Pi(\Psi_s, \Psi_t) = 0, \quad \Pi(\Psi_t, \Psi_t) = 0.$$

The following functions are introduced:

$$\Delta_1 = 0, \quad \Delta_2 = 0, \quad \Delta_3 = 0.$$

Since $\Delta_1 = \Delta_2 = \Delta_3 = 0$, all points of the soliton surface are flat points.

Remark 3.3. We obtain that

$$\frac{\partial}{\partial s}N_3 = \frac{\partial}{\partial t}N_3 = 0$$

by Theorem 3.1 and Corollary 3.2. This means that the vector field N_3 of s parameter curve $\Psi = \Psi(s,t)$ should be totally constant vector for all t. Therefore, for a solution $\Psi = \Psi(s,t)$ of BDR equation, the fact that null Cartan s parameter curve lies on three dimensional Minkowski space.

4. BDR equation by using pseudo null curve

In this section, solution $\tilde{\Psi} = \tilde{\Psi}(s,t)$ of BDR equation by using pseudo null curve is investigated with obtaining derivative formulas of pseudo null s parameter curve of $\tilde{\Psi} = \tilde{\Psi}(s,t)$ for all t.

Since BDR equation is expressed using cross product, it is necessary to obtain the cross product of pseudo null frame fields. Thus, the cross product of pseudo null frame fields are obtained as follows:

$$\begin{aligned} T \times_L N_1 \times_L N_2 &= N_1, \\ T \times_L N_1 \times_L N_3 &= -N_2, \\ T \times_L N_3 \times_L N_2 &= N_3, \\ N_1 \times_L N_2 \times_L N_3 &= -T \end{aligned} \tag{9}$$

by relation $\langle T, N_1 \times_L N_2 \times_L N_3 \rangle_L = 1$.

Theorem 4.1. *Let's assume that $\Psi = \Psi(s,t)$ is a solution of BDR equation where $\Psi = \Psi(s,t)$ is a pseudo null curve for every t. Then, derivatives of pseudo null frames according to time parameter t are all zero vectors.*

Proof. By Eq. (6), the metric coefficient matrix is obtained as follows:

$$I^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

We define semi skew symmetric matrix $A = (a_{ij})$ by equation $I^*AI^* + A^T = 0_{4\times4}$. We get the following matrix equation

$$\begin{bmatrix} 2a_{11} & a_{21}+a_{14} & a_{13}+a_{31} & a_{12}+a_{41} \\ a_{12}+a_{41} & a_{22}+a_{44} & a_{32}+a_{43} & 2a_{42} \\ a_{13}+a_{31} & a_{23}+a_{34} & 2a_{33} & a_{32}+a_{43} \\ a_{21}+a_{14} & 2a_{24} & a_{23}+a_{34} & a_{22}+a_{44} \end{bmatrix} = 0_{4\times4}.$$

This implies

$$\begin{aligned} &a_{11} = 0, \quad a_{33} = 0, \quad a_{24} = 0 \quad a_{42} = 0, \\ &a_{13} = -a_{31} = \tilde{\zeta}_{13}, \quad a_{22} = -a_{44} = \tilde{\zeta}_{22}, \quad a_{21} = -a_{14} = \tilde{\zeta}_{21}, \\ &a_{32} = -a_{43} = \tilde{\zeta}_{32}, \quad a_{23} = -a_{34} = \tilde{\zeta}_{23}, \quad a_{12} = -a_{41} = \tilde{\zeta}_{12}. \end{aligned}$$

There exist smooth functions: $\tilde{\zeta}_{12}, \tilde{\zeta}_{13}, \tilde{\zeta}_{21}, \tilde{\zeta}_{22}, \tilde{\zeta}_{23}$, and $\tilde{\zeta}_{32}$ such that

$$\frac{d}{dt}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\zeta}_{12} & \tilde{\zeta}_{13} & -\tilde{\zeta}_{21} \\ \tilde{\zeta}_{21} & \tilde{\zeta}_{22} & \tilde{\zeta}_{23} & 0 \\ -\tilde{\zeta}_{13} & \tilde{\zeta}_{32} & 0 & -\tilde{\zeta}_{23} \\ -\tilde{\zeta}_{12} & 0 & -\tilde{\zeta}_{32} & -\tilde{\zeta}_{22} \end{bmatrix}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

These functions must be found in terms of the curvature functions. By Eq. (5), we find

$$\begin{aligned} \tilde{\Psi}_s(s,t) &= T(s,t), \\ \tilde{\Psi}_{ss}(s,t) &= N_1(s,t), \\ \tilde{\Psi}_{sss}(s,t) &= \tau(s,t)N_2(s,t). \end{aligned}$$

By Eq. (9), BDR equation implies that

$$\begin{aligned} \tilde{\Psi}_t(s,t) &= \tilde{\Psi}_s(s,t) \times_L \tilde{\Psi}_{ss}(s,t) \times_L \tilde{\Psi}_{sss}(s,t) \\ &= \tau(s,t)N_1(s,t). \end{aligned}$$

By $\tilde{\Psi}_{st} = \tilde{\Psi}_{ts}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial s}(\tilde{\Psi}_t(s,t)) &= \frac{\partial}{\partial s}(\tau(s,t)N_1(s,t)) \\ &= \frac{\partial}{\partial s}\tau(s,t)N_1(s,t) + \tau^2(s,t)N_2(s,t). \end{aligned}$$

Moreover, we get

$$\begin{aligned} \frac{\partial}{\partial t}(\tilde{\Psi}_s(s,t)) &= \frac{\partial}{\partial t}(T(s,t)) \\ &= \tilde{\zeta}_{12}(s,t)N_1(s,t) + \tilde{\zeta}_{13}(s,t)N_2(s,t) - \tilde{\zeta}_{21}(s,t)N_3(s,t). \end{aligned}$$

Then, we have

$$\begin{aligned} \tilde{\zeta}_{12}(s,t) &= \frac{\partial}{\partial s}\left(\tau(s,t)\right), \\ \tilde{\zeta}_{13}(s,t) &= \tau^2(s,t), \\ \tilde{\zeta}_{21}(s,t) &= 0. \end{aligned}$$

By $T_{st} = T_{ts}$, we have

$$\begin{aligned} \frac{\partial}{\partial s}(T_t(s,t)) &= \frac{\partial}{\partial s}(\tilde{\zeta}_{12}(s,t)N_1(s,t) + \tilde{\zeta}_{13}(s,t)N_2(s,t)) \\ &= \left(\frac{\partial}{\partial s}\tilde{\zeta}_{12}(s,t) + \tilde{\zeta}_{13}(s,t)\sigma(s,t)\right)N_1(s,t) \\ &\quad + \left(\tau(s,t)\tilde{\zeta}_{12}(s,t) + \frac{\partial}{\partial s}\tilde{\zeta}_{13}(s,t)\right)N_2(s,t) \\ &\quad - \tau(s,t)\tilde{\zeta}_{13}(s,t)N_3(s,t). \end{aligned}$$

Furthermore, we also have

$$\frac{\partial}{\partial t}(T_s(s,t)) = \tilde{\zeta}_{22}(s,t)N_1(s,t) + \tilde{\zeta}_{23}(s,t)N_2(s,t).$$

According to this, we have

$$\begin{aligned} \tau(s,t)\tilde{\zeta}_{13}(s,t) = 0, \\ \tilde{\zeta}_{22}(s,t) &= \frac{\partial}{\partial s}\tilde{\zeta}_{12}(s,t) + \tilde{\zeta}_{13}(s,t)\sigma(s,t), \\ \tilde{\zeta}_{23}(s,t) &= \tau(s,t)\tilde{\zeta}_{12}(s,t) + \frac{\partial}{\partial s}\tilde{\zeta}_{13}(s,t). \end{aligned}$$

By the above equations, we find that $\tau(s,t) = 0$, $\tilde{\zeta}_{12}(s,t) = 0$, $\tilde{\zeta}_{13}(s,t) = 0$, $\tilde{\zeta}_{22}(s,t) = 0$, and $\tilde{\zeta}_{23}(s,t) = 0$. Then, derivatives of frame fields according to time parameter t are all zero vectors. □

Corollary 4.2. *Let's assume that $\Psi = \Psi(s,t)$ is a solution of BDR equation where $\Psi = \Psi(s,t)$ is a pseudo null curve for every t. Then the followings are obtained*

$$\frac{d}{ds}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ -\kappa & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Proof. By compatibility conditions, we obtain that $\tau(s,t) = 0$ by proof of Theorem 4.1. □

Remark 4.3. We obtain that

$$\tau(s,t) = 0.$$

By Theorem 4.1 and Corollary 4.2, we have

$$\frac{d}{dt}\tilde{\Psi}(s,t) = 0.$$

Thus, it is seen from results obtained that there is no surface corresponding to the solution of BDR equation by using pseudo null s parameter curve.

5. BDR equation by using partially null curve

This section examines the soliton surface $\mathcal{M} : \bar{\Psi} = \bar{\Psi}(s,t)$ which is a solution of BDR equation by using partially null s parameter curve $\bar{\Psi} = \bar{\Psi}(s,t)$ for all t. Then, we get the cross product of partially null frame vectors as follows:

$$\begin{aligned} T \times_L N_1 \times_L N_2 &= N_2, \\ T \times_L N_1 \times_L N_3 &= -N_3, \\ T \times_L N_3 \times_L N_2 &= -N_1, \\ N_1 \times_L N_2 \times_L N_3 &= -T \end{aligned} \tag{10}$$

by the relation $\langle T, N_1 \times_L N_2 \times_L N_3 \rangle_L = 1$.

5.1. Time derivative formulas of partially null frame

Theorem 5.1. *Let's assume that $\Psi = \Psi(s,t)$ is a solution of BDR equation, where $\Psi = \Psi(s,t)$ is a partially null curve for every t. Then we have the following equations*

$$\frac{d}{dt}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \bar{\zeta}_{13} & 0 \\ 0 & 0 & \bar{\zeta}_{23} & 0 \\ 0 & 0 & \bar{\zeta}_{33} & 0 \\ -\bar{\zeta}_{13} & -\bar{\zeta}_{23} & 0 & -\bar{\zeta}_{33} \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix},$$

where

$$\bar{\zeta}_{13}(s,t) = \frac{\partial}{\partial s}\tau(s,t) + \tau(s,t)\sigma(s,t),$$

$$\bar{\zeta}_{23}(s,t) = \frac{\partial}{\partial s}\bar{\zeta}_{13}(s,t) + \bar{\zeta}_{13}(s,t)\sigma(s,t),$$

$$\bar{\zeta}_{33}(s,t) = \frac{1}{\tau(s,t)}[\frac{\partial}{\partial s}\bar{\zeta}_{23}(s,t) + \bar{\zeta}_{23}(s,t)\sigma(s,t) - \frac{\partial}{\partial s}\tau(s,t)\bar{\zeta}_{13}(s,t)].$$

Proof. By Eq. (8), the metric coefficient matrix is obtained as follows:

$$I^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We define semi skew symmetric matrix $A = (a_{ij})$ by equation $I^*AI^* + A^T = 0_{4\times 4}$. We get the following matrix equation

$$\begin{bmatrix} 2a_{11} & a_{12}+a_{21} & a_{13}+a_{41} & a_{31}+a_{14} \\ a_{12}+a_{21} & 2a_{22} & a_{23}+a_{42} & a_{32}+a_{24} \\ a_{31}+a_{14} & a_{32}+a_{24} & a_{33}+a_{44} & 2a_{34} \\ a_{13}+a_{41} & a_{23}+a_{42} & 2a_{43} & a_{33}+a_{44} \end{bmatrix} = 0_{4\times 4}.$$

This implies

$$a_{11} = 0, \quad a_{12} = -a_{21} = \bar{\zeta}_{12}, \quad a_{13} = -a_{41} = \bar{\zeta}_{13}, \quad a_{34} = 0,$$
$$a_{31} = -a_{14} = \bar{\zeta}_{31}, \quad a_{23} = -a_{42} = \bar{\zeta}_{23}, \quad a_{22} = 0, \quad a_{43} = 0,$$
$$a_{24} = -a_{42} = \bar{\zeta}_{24}, \quad a_{33} = -a_{44} = \bar{\zeta}_{33}.$$

There exist smooth functions; $\bar{\zeta}_{12}, \bar{\zeta}_{13}, \bar{\zeta}_{31}, \bar{\zeta}_{23}, \bar{\zeta}_{24}$, and $\bar{\zeta}_{33}$ such that

$$\frac{d}{dt}\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{\zeta}_{12} & \bar{\zeta}_{13} & -\bar{\zeta}_{31} \\ -\bar{\zeta}_{12} & 0 & \bar{\zeta}_{23} & \bar{\zeta}_{24} \\ \bar{\zeta}_{31} & -\bar{\zeta}_{24} & \bar{\zeta}_{33} & 0 \\ -\bar{\zeta}_{13} & -\bar{\zeta}_{23} & 0 & -\bar{\zeta}_{33} \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

These functions are needed to find in terms of the curvature functions. By Eq. (7), we have

$$\bar{\Psi}_s(s,t) = T(s,t),$$

$$\begin{aligned}\bar{\Psi}_{ss}(s,t) &= N_1(s,t),\\ \bar{\Psi}_{sss}(s,t) &= -T(s,t)+\tau(s,t)N_2(s,t).\end{aligned}$$

By using Eq. (10), BDR equation implies that

$$\begin{aligned}\bar{\Psi}_t(s,t) &= \bar{\Psi}_s(s,t)\times_L \bar{\Psi}_{ss}(s,t)\times_L \bar{\Psi}_{sss}(s,t)\\ &= \tau(s,t)N_2(s,t).\end{aligned}$$

By $\bar{\Psi}_{st}=\bar{\Psi}_{ts}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial s}(\bar{\Psi}_t(s,t)) &= \frac{\partial}{\partial s}(\tau(s,t)N_2(s,t))\\ &= \frac{\partial}{\partial s}\tau(s,t)N_2(s,t)+\tau(s,t)\sigma(s,t)N_2(s,t)\\ &= (\frac{\partial}{\partial s}\tau(s,t)+\tau(s,t)\sigma(s,t))N_2(s,t).\end{aligned}$$

Then, we get

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{\Psi}_s(s,t)) &= \frac{\partial}{\partial t}(T(s,t))\\ &= \bar{\zeta}_{12}(s,t)N_1(s,t)+\bar{\zeta}_{13}(s,t)N_2(s,t)-\bar{\zeta}_{31}(s,t)N_3(s,t).\end{aligned}$$

Thus, we have

$$\begin{aligned}\bar{\zeta}_{12}(s,t) &= 0,\\ \bar{\zeta}_{13}(s,t) &= \frac{\partial}{\partial s}\tau(s,t)+\tau(s,t)\sigma(s,t),\\ \bar{\zeta}_{31}(s,t) &= 0.\end{aligned}$$

By $T_{st}=T_{ts}$, we find

$$\begin{aligned}\frac{\partial}{\partial s}(T_t(s,t)) &= \frac{\partial}{\partial s}(\bar{\zeta}_{13}(s,t)N_2(s,t))\\ &= \frac{\partial}{\partial s}\bar{\zeta}_{13}(s,t)N_2(s,t)+\bar{\zeta}_{13}(s,t)\sigma(s,t)N_2(s,t)\\ &= (\frac{\partial}{\partial s}\bar{\zeta}_{13}(s,t)+\bar{\zeta}_{13}(s,t)\sigma(s,t))N_2(s,t).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(T_s(s,t)) &= \frac{\partial}{\partial t}(N_1(s,t))\\ &= \bar{\zeta}_{23}N_2(s,t)+\bar{\zeta}_{24}(s,t)N_3(s,t).\end{aligned}$$

Thus, we get

$$\begin{aligned}\bar{\zeta}_{23}(s,t) &= \frac{\partial}{\partial s}\bar{\zeta}_{13}(s,t)+\zeta_{13}(s,t)\sigma(s,t),\\ \bar{\zeta}_{24}(s,t) &= 0.\end{aligned}$$

By $(N_1)_{st} = (N_1)_{ts}$, we have

$$\begin{aligned}\frac{\partial}{\partial s}(N_{1t}(s,t)) &= \frac{\partial}{\partial s}(\bar{\zeta}_{23}N_2(s,t)) \\ &= \frac{\partial}{\partial s}\bar{\zeta}_{23}(s,t)N_2(s,t) + \bar{\zeta}_{23}(s,t)\sigma(s,t)N_2(s,t) \\ &= (\frac{\partial}{\partial s}\bar{\zeta}_{23}(s,t) + \bar{\zeta}_{23}(s,t)\sigma(s,t))N_2(s,t).\end{aligned}$$

Furthermore, we also have

$$\begin{aligned}\frac{\partial}{\partial t}(N_{1s}(s,t)) &= \frac{\partial}{\partial t}(-T(s,t) + \tau(s,t)N_2(s,t)) \\ &= -\bar{\zeta}_{13}(s,t)N_2(s,t) + \frac{\partial}{\partial t}\tau(s,t)N_2(s,t) + \tau(s,t)\bar{\zeta}_{33}(s,t)N_2(s,t) \\ &= (-\bar{\zeta}_{13}(s,t) + \frac{\partial}{\partial t}\tau(s,t) + \tau(s,t)\bar{\zeta}_{33}(s,t))N_2(s,t).\end{aligned}$$

This implies that

$$\frac{\partial}{\partial s}\bar{\zeta}_{23}(s,t) + \bar{\zeta}_{23}(s,t)\sigma(s,t) = -\bar{\zeta}_{13}(s,t) + \frac{\partial}{\partial t}\tau(s,t) + \tau(s,t)\bar{\zeta}_{33}(s,t).$$

Thus, it is found

$$\bar{\zeta}_{33}(s,t) = \frac{1}{\tau(s,t)}[\frac{\partial}{\partial s}\bar{\zeta}_{23}(s,t) + \bar{\zeta}_{23}(s,t)\sigma(s,t) - \frac{\partial}{\partial s}\tau(s,t)\bar{\zeta}_{13}(s,t)].$$

□

5.2. Surface associated with BDR equation by using partially null curve

Let $\mathcal{M} : \bar{\Psi} = \bar{\Psi}(s,t)$ be a two-dimensional surface associated with BDR equation. Then $T_p(\mathcal{M}) = span\{\bar{\Psi}_s = T,\ \bar{\Psi}_t = \tau N_2\}$ is the tangent space of surface $\mathcal{M}$ at arbitrary point P. Given that the quadruple $\{\bar{\Psi}_s, \bar{\Psi}_t, N_1, N_3\}$ is positively oriented frame in $\mathbb{E}^4_1$, $N_p(\mathcal{M}) = span\{N_1, N_3\}$ is the normal space. The resulting decomposition is as follows

$$\mathbb{E}^4_1 = T_p(\mathcal{M}) \oplus N_p(\mathcal{M}).$$

The metric tensor's components are as follows:

$$\begin{aligned}g_{11} &= \langle T(s,t), T(s,t)\rangle_L = 1, \\ g_{12} &= \langle T(s,t), \tau(s,t)N_2(s,t)\rangle_L = 0, \\ g_{22} &= \langle \tau(s,t)N_2(s,t), \tau(s,t)N_2(s,t)\rangle_L = 0.\end{aligned}$$

Therefore, we obtain

$$g_{11}g_{22} - g_{12}^2 = 0.$$

For normal frame fields $\{N_1, N_3\}$ of $\mathcal{M}$, we have second derivative formulas

$$\begin{aligned}D_{\bar{\Psi}_s}\bar{\Psi}_s &= \bar{\Psi}_{ss} = \Gamma^1_{11}\bar{\Psi}_s + \Gamma^2_{11}\bar{\Psi}_t + c^1_{11}N_1 + c^2_{11}N_3, \\ D_{\bar{\Psi}_s}\bar{\Psi}_t &= \bar{\Psi}_{st} = \Gamma^1_{12}\bar{\Psi}_s + \Gamma^2_{12}\bar{\Psi}_t + c^1_{12}N_1 + c^2_{12}N_3, \\ D_{\bar{\Psi}_t}\bar{\Psi}_t &= \bar{\Psi}_{tt} = \Gamma^1_{22}\bar{\Psi}_s + \Gamma^2_{22}\bar{\Psi}_t + c^1_{22}N_1 + c^2_{22}N_3,\end{aligned}$$

where Γ_{ij}^k are Christoffel's symbols and c_{ij}^k are functions on $\mathcal{M}$ for $i,j,k=1,2$. Then, we have

$$\begin{aligned}\bar{\Psi}_{ss}(s,t) &= N_1(s,t),\\ \bar{\Psi}_{st}(s,t) &= (\frac{\partial}{\partial s}\tau(s,t)+\tau(s,t)\sigma(s,t))N_2(s,t),\\ \bar{\Psi}_{tt}(s,t) &= (\frac{\partial}{\partial t}\tau(s,t)+\tau(s,t)\bar{\zeta}_{33}(s,t))N_2(s,t).\end{aligned}$$

Thus, we find Christoffel's symbols as follows:

$$\begin{aligned}\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^2 &= \Gamma_{22}^2 = \Gamma_{22}^1 = 0,\\ \Gamma_{12}^1(s,t) &= \frac{1}{\tau(s,t)}(\frac{\partial}{\partial s}\tau(s,t))+\sigma(s,t),\\ \Gamma_{22}^2(s,t) &= \frac{1}{\tau(s,t)}(\frac{\partial}{\partial t}\tau(s,t))+\bar{\zeta}_{33}(s,t).\end{aligned}$$

Finally, we obtain the functions c_{ij}^k as follows:

$$\begin{aligned}&c_{11}^1 = 1,\\ &c_{11}^2 = c_{12}^1 = c_{12}^2 = c_{22}^1 = c_{22}^2 = 0.\end{aligned}$$

The metric induced on $\mathcal{M}$ is nondegenerate because at least one of the coefficients c_{ij}^k is not zero, which means that $c_{11}^1 = 1$. We get the second fundamental tensor of $\mathcal{M}$ as follows:

$$\Pi(\Psi_s,\Psi_s) = N_1(s,t),\quad \Pi(\Psi_s,\Psi_t)=0,\quad \Pi(\Psi_t,\Psi_t)=0.$$

The following functions are introduced:

$$\Delta_1 = 0,\quad \Delta_2 = 0,\quad \Delta_3 = 0.$$

Since $\Delta_1=\Delta_2=\Delta_3=0$, all points of the soliton surface are flat points.

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