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COMMUTATORS OF THE MAXIMAL FUNCTIONS ON BANACH FUNCTION SPACES

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Abstract. Let M and $M^{\#}$ be Hardy-Littlewood maximal operator and sharp maximal operator, respectively. In this article, we present necessary and sufficient conditions for the boundedness properties for commutator operators $[M, b]$ and $[M^{\#}, b]$ in a general context of Banach function spaces when b belongs to $\mathrm{BMO}(\mathbb{R}^n)$ spaces. Some applications of the results on weighted Lebesgue spaces, variable Lebesgue spaces, Orlicz spaces and Musielak–Orlicz spaces are also given.

1. Introduction

Let T be a Calderón–Zygmund singular integral operator. In 1976, Coifman, Rochberg and Weiss [6] studied the commutator generated by T and a function $b \in \mathrm{BMO}(\mathbb{R}^n)$ as follows:

$$[T, b]f(x) := T(bf)(x) - b(x)Tf(x). \tag{1}$$

A well-known result states that $[T, b]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Sufficiency was proved by Coifman, Rochberg and Weiss [6] and necessity part was obtained by Janson [24]. Unlike the classical theory, $[T, b]$ fails to be weak type $(1, 1)$ and enjoys weak $L(1 + \log^+ L)$ type estimate (see [30]).

The Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all the cubes containing x with sides parallel to the coordinate axes and where f is any locally integrable function.

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Define a basis $\mathcal{B}$ to be a collection of open sets $B \subset \mathbb{R}^n$. A basis $\mathcal{B}$ is called a Muckenhoupt basis if for each p, $1 < p < \infty$, and every $w \in A_{p,\mathcal{B}}$, the maximal operator $M_{\mathcal{B}}$ associated with $\mathcal{B}$ is bounded on $L^p(w)$.

Throughout the paper, all cubes are assumed to have their sides parallel to the coordinate axes which is an example of the Muckenhoupt basis (see [10]).

We recall that the sharp maximal operator $M^{\#}$ of Fefferman–Stein is given by

$$M^{\#}f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where f_Q denotes the usual average of f over Q, namely $f_Q = \frac{1}{|Q|}\int_Q f$, and, also, the supremum is taken over all cubes containing x.

The spaces of functions of bounded mean oscillation, $\mathrm{BMO}(\mathbb{R}^n)$, consist of all the locally integrable functions f such that $M^{\#}f \in L^\infty(\mathbb{R}^n)$. In other words

$$\|f\|_{\mathrm{BMO}} = \|M^{\#}f\|_\infty := \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy < \infty.$$

Definition 1. For a fixed cube Q_0, the Hardy–Littlewood maximal function with respect to Q_0 of a function f is given by

$$M_{Q_0}f(x) := \sup_{Q_0 \supseteq Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all the cubes Q with $Q_0 \supseteq Q$ and $Q \ni x$.

For a function b defined on $\mathbb{R}^n$, $b^+(x)$ and $b^-(x)$ is defined by $b^+(x) := \max\{b(x), 0\}$ and $b^-(x) := -\min\{b(x), 0\}$. As a result $b(x) = b^+(x) - b^-(x)$ and $|b(x)| = b^+(x) + b^-(x)$.

Lemma 1.1 ([2])**.** *Let b be a locally integrable function on $\mathbb{R}^n$. If*

$$\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)| dx < \infty, \tag{2}$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

In order to investigate $[M, b]$, we start with the consideration of maximal commutators $M_b f$. Given a measurable function b, the maximal commutators $M_b f$ are defined by

$$M_b(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| dy, \qquad (x \in \mathbb{R}^n).$$

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see [14, 28, 34, 35]). In [14], García–Cuerva et al. proved that M_b is bounded on $L^p(\mathbb{R}^n)$, $p > 1$, if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. In general, M_b fails to be weak type (1,1) when $b \in \mathrm{BMO}(\mathbb{R}^n)$ (see [14]). Instead, an endpoint theory was provided for this operator such as weak type $L(1 + \log^+ L)$, see for instance [1, 20, 21].

Similar to (1), we can give the definitions of commutators of maximal operators as follows:

Let b be a locally integrable function on $\mathbb{R}^n$. We define the commutator operators $[M,b]f$ and $[M^{\#},b]f$ as

$$[M,b]f := M(bf) - bMf,$$
$$[M^{\#},b]f := M^{\#}(bf) - bM^{\#}(f),$$

respectively. The commutator of the maximal function $[M,b]$ arises, for example, when one tries to give a meaning to the product of a function in Hardy spaces H^1 and a function in $\mathrm{BMO}(\mathbb{R}^n)$ (which may not be a locally integrable function, see [4]). In 1990, Milman and Schonbek [29] proved by using the real interpolation method that the commutator is a bounded map from $L^p(\mathbb{R}^n)$ onto itself for $p > 1$ if a nonnegative symbol b belongs to $\mathrm{BMO}(\mathbb{R}^n)$. In 2000, Bastero, Milman and Ruiz [2] studied the necessary and sufficient conditions for the boundedness of $[M,b]$ and $[M^{\#},b]$ on $L^p(\mathbb{R}^n)$ spaces. A simple example shows that $[M,b]$ is not of weak type $(1,1)$ and enjoys weak type $L(1+\log^+ L)$ estimates (see [1]).

The mapping properties of these operators have been studied by several authors in many function spaces such as variable Lebesgue spaces, Morrey spaces, Orlicz spaces and some remarkable results have been obtained (see [16], [36], [39], [41]).

The purpose of this paper is to extend the above boundedness results in a general context of Banach function space X over $\mathbb{R}^n$ equipped with Lebesgue measure. For a Banach function space X, we denote by X' the associated space of it (see Section 2 for details). Our main results are the following:

Theorem 1.2. *Let X be a Banach function space. Assume that M is bounded on both X and X'. Then M_b is bounded from X to itself if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Theorem 1.3. *Let X be a Banach function space. Assume that M is bounded on both X and X'. Then the following assertions are equivalent:*

(i) *$b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*
(ii) *$[M,b]$ is bounded on X.*
(iii) *There exists a constant $c > 0$ such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_X}{\|\chi_Q\|_X} \le c < \infty.$$

Theorem 1.4. *Let X be a Banach function space. Assume that M is bounded on both X and X'. Then the following assertions are equivalent:*

(i) *$b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*
(ii) *$[M^{\#},b]$ is bounded on X.*

(iii) *There exists a constant $c > 0$ such that*

$$\sup_Q \frac{\|(b - 2M^{\#}(b\chi_Q))\chi_Q\|_X}{\|\chi_Q\|_X} \le c < \infty.$$

This paper is organized as follows: Section 2 contains some basic definitions and facts which will be used throughout the paper. In Section 3, we give the proofs of Theorems 1.2, 1.3, and 1.4. Finally in Section 4, we present some examples of Banach function spaces and reduce the results to these spaces and give some corollaries.

2. Some preliminaries

We begin with some definitions, lemmas and the theorems which we use to prove our main theorems.

Let Ω be any measurable subset of $\mathbb{R}^n$. Let $\mathcal{M}(\Omega)$ denote the set of all (Lebesgue) measurable functions on Ω and $\mathcal{M}^+(\Omega)$ the class of functions in $\mathcal{M}(\Omega)$ that are nonnegative a.e.. χ_Q is the characteristic function of Q and $|Q|$ is the Lebesgue measure of Q. Throughout the paper, we always denote by c or C a positive constant, which is independent of main parameters but it may vary from line to line.

2.1. Banach function spaces

Definition 2. Let $\rho : \mathcal{M}^+(X) \to [0, \infty)$ be a mapping satisfying, for all f, g, f_k ($k = 1, 2, 3, \ldots$), in $\mathcal{M}^+(X)$ and all constants $\alpha \in \mathbb{R}$,

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.; $\rho(\alpha f) = |\alpha|\rho(f)$, $\rho(f + g) \le \rho(f) + \rho(g)$,
(ii) if $|f| \le |g|$ a.e., then $\rho(f) \le \rho(g)$,
(iii) if $|f_k| \uparrow |f|$ a.e., then $\rho(f_k) \uparrow \rho(f)$,
(iv) if $E \subseteq \mathbb{R}^n$ is bounded, then $\rho(\chi_E) < \infty$,
(v) if $E \subseteq \mathbb{R}^n$ is bounded, then

$$\int_E |f| dx \le c_E \rho(f).$$

By a Banach function space X over $\mathbb{R}^n$ equipped with Lebesgue measure, we mean a collection of functions f in $\mathcal{M}(X)$ such that

$$\|f\|_X = \rho(|f|) < \infty.$$

A more common requirement is that E is a set of finite measure in (iv) and (v).

Given a Banach function space X, equipped with a norm $\|\cdot\|_X$, the associate space of X is defined by

$$X' = \left\{ f \in \mathcal{M}(X) : \|f\|_{X'} := \sup_{g \in X, \|g\|_X \le 1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| < \infty \right\}.$$

Moreover, we have the following generalization of Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_X \|g\|_{X'}. \tag{3}$$

Lemma 2.1 ([23]). *Let X be a Banach function space. If M is bounded on X', then*

$$\sup_Q \frac{1}{|Q|} \|\chi_Q\|_X \|\chi_Q\|_{X'} < \infty.$$

Remark 2.2 ([23]). Lemma 2.1 also holds if M is bounded on X because Banach function space is reflexive by Lorentz–Luxemburg Theorem [3, p. 10], that is, $(X')' = X$.

For more information about the theory of Banach function space, see for instance [3] and [10].

2.2. Weights and extrapolation

By a weight we mean a nonnegative and locally integrable function. There is a vast literature on weights and weighted norm inequalities. We refer the readers to [15] and their references for complete informations.

Central to the study of weights are the so-called A_p weights, $1 \le p \le \infty$. When $1 < p < \infty$, we say $w \in A_p$ if for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} \le C < \infty,$$

where p' is the conjugate of p which satisfies $1/p + 1/p' = 1$. It is obtained directly from the definition that $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.

We say that $w \in A_1$ if $Mw(x) \le Cw(x)$ for a.e. x. If $1 \le p < q < \infty$, then $A_p \subset A_q$. Hence it is natural to define A_∞ class by the union of the all the A_p classes, namely;

$$A_\infty := \bigcup_{p>1} A_p.$$

Hereafter, by $\mathcal{F}$ we will mean a family of pairs (f, g) of nonnegative, measurable functions that are not identically zero. In [10], the authors proved an extrapolation theorem as follows:

Theorem 2.3 ([10], Theorem 4.6). *Let $\mathcal{B}$ be a Muckenhoupt basis and X be a Banach function space. Suppose that for some p_0, $0 < p_0 < \infty$, and every weight $w \in A_{1,\mathcal{B}}$*

$$\left(\int_{\mathbb{R}^n} f(x)^{p_0} w(x)dx\right) \le c\left(\int_{\mathbb{R}^n} g(x)^{p_0} w(x)dx\right), \qquad (f,g) \in \mathcal{F}. \tag{4}$$

If there exists q_0, $p_0 \le q_0 < \infty$, such that $X^{\frac{1}{q_0}}$ is a Banach function space and $M_{\mathcal{B}}$ is bounded on $(X^{\frac{1}{q_0}})'$, then

$$\|f\|_X \le c\|g\|_X, \qquad (f,g) \in \mathcal{F}.$$

3. Proof of the theorems

In this section, we will prove our theorems. We first give an estimate of Coifman–Fefferman's type for commutators of the Hardy–Littlewood maximal operator.

According to a result of Coifman and Fefferman [5], singular integral operator T and the Hardy–Littlewood maximal operator M satisfy the following estimate

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx \le C[w]_{A_\infty}^p \int_{\mathbb{R}^n} [Mf(x)]^p\, w(x)dx$$

for every function f for which the left hand side is finite, where $0 < p < \infty$ and $w \in A_\infty$.

In 1997, Pérez [31] extended the result to commutators of singular integrals. Actually, this kind of estimates also hold for commutators of the Hardy–Littlewood maximal operator. Precisely, we have the following result.

Theorem 3.1. *Let $0 < p < \infty$, $w \in A_\infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a positive constant $C_{b,w}$, depending on b and w, such that*

$$\int_{\mathbb{R}^n} [M_b f(x)]^p\, w(x)dx \le C_{b,w} \int_{\mathbb{R}^n} [M^2 f(x)]^p w(x)dx. \tag{5}$$

The proof of this type of estimates for commutators is by now standard. We refer the readers to [31, Theorem 1], [32, Theorem 1.1] and [37, Theorem 1.1].

The second author proved the multilinear case for Theorem 3.1 in [38, Theorem 3.5] from which Theorem 3.1 can be deduced directly.

Moreover, we would like to note that inequality (5) is essentially implied, although not be directly stated, in the proof of Theorem 1.1 in [37]. Here we give the idea of the proof but omit the details. We let $v \in C^\infty[0,+\infty)$ such that $|v'(t)| \le Ct^{-1}$ and $\chi_{[0,1]}(t) \le v(t) \le \chi_{[0,2]}(t)$ and set

$$\widetilde{M}f(x) = \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \left| \frac{1}{\varepsilon^n} v\left(\frac{|x|}{\varepsilon}\right) f(y) \right| dy$$

and

$$\widetilde{M_b}f(x) = \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \left| \frac{1}{\varepsilon^n} v\left(\frac{|x|}{\varepsilon}\right) [b(x)-b(y)] f(y) \right| dy.$$

The arguments for $V_b^* f$ in [37] can be followed step by step in our case for $\widetilde{M_b}f$ almost without any changes. Then (5) follows from $\widetilde{M_b}f \approx M_b f$.

Proof of Theorem 1.2. We will make use of Theorem 2.3 for the pair $(M_b f, M^2 f)$ with $p_0 = q_0 = 1$, and note that Theorem 3.1 holds for all $w \in A_1 \subset A_\infty$. We get the 'if' part of Theorem 1.2 by using the boundedness of M on X:

$$\begin{aligned}\|M_b f\|_X &\le c\|M^2 f\|_X \\ &\le c\|f\|_X.\end{aligned}$$

Suppose that M_b is bounded on X. We will prove $b \in \mathrm{BMO}(\mathbb{R}^n)$. For any cube Q_0 and $x \in Q_0$, we have obviously,

$$M_b(\chi_{Q_0})(x) \ge |b(x) - b_{Q_0}|\chi_{Q_0}(x) \text{ for } x \in \mathbb{R}^n.$$

Then, by (3) and the boundedness of M_b on X, we get

$$\begin{aligned}\frac{1}{|Q_0|}\int_{Q_0} |b(x) - b_{Q_0}|\, dx &= \frac{1}{|Q_0|}\int_{Q_0} |b(x) - b_{Q_0}|\chi_{Q_0}(x)\, dx \\ &\le \frac{1}{|Q_0|}\int_{Q_0} M_b(\chi_{Q_0})(x)dx \\ &= \frac{1}{|Q_0|}\int_{\mathbb{R}^n} M_b(\chi_{Q_0})(x)\chi_{Q_0}(x)dx \\ &\le \frac{1}{|Q_0|}\|M_b(\chi_{Q_0})\|_X\|\chi_{Q_0}\|_{X'} \\ &\le \frac{1}{|Q_0|}\|\chi_{Q_0}\|_X\|\chi_{Q_0}\|_{X'} \\ &\le C,\end{aligned}$$

where we can use Remark 2.2 in the last step since M is bounded on X. Since Q_0 is arbitrary cube, taking supremum over all cubes $Q_0 \subset \mathbb{R}^n$, we have $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

Proof of Theorem 1.3. (i) $\Rightarrow$ (ii). The following pointwise estimate for any locally integrable function b holds:

$$|[M, b]f(x)| \le M_b f(x) + 2b^-(x)Mf(x), \tag{6}$$

(see [1, Lemma 3.2]). Applying Definition 2(ii) and (i), hypothesis and Theorem 1.2, respectively, we have

$$\begin{aligned}\|[M, b]f\|_X &\le c\|M_b f + b^- Mf\|_X \\ &\le c\|M_b f\|_X + \|b^-\|_\infty\|Mf\|_X \\ &\le c\|f\|_X.\end{aligned}$$

(ii) $\Rightarrow$ (iii). For any $Q \subset \mathbb{R}^n$, it is easy to check that for any $x \in Q$,

$$\begin{aligned}&M(\chi_Q)(x) = M_Q(\chi_Q)(x) = \chi_Q(x), \\ &M(b\chi_Q)(x) = M_Q(b\chi_Q)(x) = M_Q(b)(x),\end{aligned}$$

see [2, p. 3331] and (2.4) in [40]. Then it follows from the boundedness of $[M, b]$ on X that

$$\begin{aligned} \|(b - M_Q(b))\chi_Q\|_X &= \|b\chi_Q(x) - M_Q(b)\chi_Q(x)\|_X \\ &\leq \|bM(\chi_Q) - M(b\chi_Q)\|_X \\ &= \|[M, b](\chi_Q)\|_X \\ &\leq c\,\|\chi_Q\|_X\,. \end{aligned}$$

Then, by Definition 2(iv) $\|\chi_Q\|_X$ and taking supremum over all cubes $Q \subseteq \mathbb{R}^n$, we get

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_X}{\|\chi_Q\|_X} \leq C < \infty. \tag{7}$$

(iii) $\Rightarrow$ (i). From (3), Remark 2.2 and Lemma 2.1, we have

$$\begin{aligned} \frac{1}{|Q|}\int_Q |b(x) - M_Q(b)(x)|dx &= \frac{1}{|Q|}\int_{\mathbb{R}^n} |b(x) - M_Q(b)(x)|\chi_Q(x)dx \\ &\leq \frac{1}{|Q|}\,\|(b - M_Q(b))\chi_Q\|_X\,\|\chi_Q\|_{X'} \\ &\leq \frac{c}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'} \\ &\leq c. \end{aligned} \tag{8}$$

Then, we achieve $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$ by Lemma 1.1. □

Proof of Theorem 1.4. (i) $\Rightarrow$ (ii). The following inequality can be obtained, see [41, p. 1382]

$$|[M^\#, b](f)(x)| \leq M^\#(2b^- f)(x) + 2b^-(x)M^\# f(x) + |[M^\#, |b|](f)(x)|\,.$$

This, together with $M^\# f \leq 2Mf$ and $|[M^\#, |b|](f)| \leq 2M_{|b|}f$ (see (3.2) in [41]) gives

$$|[M^\#, b](f)(x)| \leq cM_b(f)(x) + b^-(x)Mf(x) + M(b^- f)(x).$$

Then by using Definition 2(ii) and (i), Theorem 1.2 and the boundedness of M on X, we get

$$\begin{aligned} \|[M^\#, b](f)\|_X &\leq c\,\|M_b f\|_X + \|b^-\|_\infty\,\|Mf\|_X \\ &\leq c\,\|f\|_X\,. \end{aligned}$$

(ii) $\Rightarrow$ (iii). For all $x \in \mathbb{R}^n$, it is calculated that $(M^\#(\chi_Q)(x))\chi_Q(x) = \frac{1}{2}(\chi_Q)(x)$ (see [2, p. 3333] for details). Note that $\chi_Q(x) \in X$ by Definition 2(iv). Then using the hypothesis (ii), we get

$$\begin{aligned} \|(b - 2M^\#(b\chi_Q))\chi_Q\|_X &= \left\| 2\left(\frac{1}{2}b\chi_Q - M^\#(b\chi_Q)\right)\chi_Q \right\|_X \\ &= \left\| 2\left(bM^\#(\chi_Q)\chi_Q - M^\#(b\chi_Q)\right)\chi_Q \right\|_X \end{aligned}$$

$$
\begin{aligned}
&= \left\|2\left(bM^{\#}(\chi_Q) - M^{\#}(b\chi_Q)\right)\chi_Q\right\|_X \\
&\leq 2\left\|[M^{\#}, b](\chi_Q)\right\|_X \\
&\leq C\|\chi_Q\|_X.
\end{aligned}
$$

Hence the conclusion is proved.

(iii) $\Rightarrow$ (i). Bastero, Milman and Ruiz obtained the following inequality (see [2, p. 3333]):

$$
|b_Q| \leq 2M^{\#}(b\chi_Q)(x), \quad x \in Q. \tag{9}
$$

Now, we can achieve that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Indeed, let $E = \{x \in Q : b(x) \leq b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. Then

$$
\int_E |b(t) - b_Q| dt = \int_F |b(t) - b_Q| dt.
$$

Since for any $x \in E$, $b(x) \leq b_Q \leq |b_Q| \leq 2M^{\#}(b\chi_Q)(x)$. Then

$$
|b(x) - b_Q| \leq |b(x) - 2M^{\#}(b\chi_Q)(x)| \quad \text{for any } x \in E.
$$

By (3), hypothesis (iii) and Lemma 2.1, we have

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - b_Q| dx &= \frac{2}{|Q|}\int_E |b(x) - b_Q| dx \\
&\leq \frac{2}{|Q|}\int_E |b(x) - 2M^{\#}(b\chi_Q)(x)| dx \\
&\leq \frac{2}{|Q|}\int_Q |b(x) - 2M^{\#}(b\chi_Q)(x)| dx \\
&\leq \frac{2}{|Q|}\int_{\mathbb{R}^n} |b(x) - 2M^{\#}(b\chi_Q)(x)|\chi_Q(x) dx \\
&\leq \frac{2}{|Q|}\left\|(b - 2M^{\#}(b\chi_Q))\chi_Q\right\|_X \|\chi_Q\|_{X'} \\
&\leq \frac{C}{|Q|}\|\chi_Q\|_X \|\chi_Q\|_{X'} \\
&\leq C.
\end{aligned}
$$

This means that $b \in \mathrm{BMO}(\mathbb{R}^n)$. To show that $b^- \in L^\infty(\mathbb{R}^n)$, by (9), we get

$$
2M^{\#}(b\chi_Q)(x) - b(x) \geq |b_Q| - b(x) \geq |b_Q| - b^+(x) + b^-(x)
$$

for $x \in Q$. Then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q |2M^{\#}(b\chi_Q)(x) - b(x)| dx \\
&\geq \frac{1}{|Q|}\int_Q (|b_Q| - b^+(x) + b^-(x)) dx \\
&= |b_Q| - \frac{1}{|Q|}\int_Q b^+(x) dx + \frac{1}{|Q|}\int_Q b^-(x) dx.
\end{aligned} \tag{10}
$$

On the other hand applying (3), hypothesis (iii) and Lemma 2.1 again, we obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q |2M^{\#}(b\chi_Q)(x)-b(x)|dx &\le \frac{2}{|Q|}\left\|(b-2M^{\#}(b\chi_Q))\chi_Q\right\|_X \|\chi_Q\|_{X'}\\ &\le \frac{C}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'}\\ &\le C.\end{aligned}$$

This, together with (10), gives

$$|b_Q| - \frac{1}{|Q|}\int_Q b^+(x)dx + \frac{1}{|Q|}\int_Q b^-(x)dx \le C$$

Let $|Q|\to 0$ with $x\in Q$, Lebesgue Differentiation Theorem assures that

$$C \ge |b(x)| - b^+(x) + b^-(x) = 2b^-(x)$$

and the desired result follows. Therefore, the proof is completed. □

4. Applications

Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces, Orlicz spaces and Musielak–Orlicz (generalized Orlicz) spaces are known as some examples of Banach function spaces (see [27]). In this final section, basic definitions and facts about Banach function space examples will be summarized and applications on the boundedness results of commutators of maximal operators and maximal commutators will be presented.

4.1. Weighted Lebesgue spaces

In this subsection, we apply Theorems 1.2 and 1.3 and 1.4 to the weighted Lebesgue spaces. Given a weight w and $1<p<\infty$, we define the space $L^p(w)$ to be set of all measurable functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n}|f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

Further, the Hardy–Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w\in A_p$ (see [15, p. 400]). Moreover, it is easy to see that the A_p condition is equivalent to

$$\frac{1}{|Q|}\|\chi_Q\|_{L^p(w)}\|\chi_Q\|_{L^{p'}(w^{1-p'})} < \infty$$

for all cubes Q.

Then $L^p(w)$ is a Banach function space (see [10, p. 70]), and it is well known that the associated space is $L^{p'}(w^{1-p'})$.

Corollary 4.1. *Let $1<p<\infty$ and $w\in A_p$. Then M_b is bounded from $L^p(w)$ to itself if and only if $b\in \mathrm{BMO}(\mathbb{R}^n)$.*

Corollary 4.2. *Let $1 < p < \infty$ and $w \in A_p$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M, b]$ *is bounded on* $L^p(w)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_{L^p(w)}}{\|\chi_Q\|_{L^p(w)}} \leq c < \infty.$$

Corollary 4.3. *Let $1 < p < \infty$ and $w \in A_p$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M^{\#}, b]$ *is bounded on* $L^p(w)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - 2M^{\#}(b\chi_Q))\chi_Q\|_{L^p(w)}}{\|\chi_Q\|_{L^p(w)}} \leq c < \infty.$$

4.2. Variable Lebesgue spaces

Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \{f \in \mathcal{M}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty\}$$

for some constant $\lambda > 0$. This set is a Banach space with respect to Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \, m(f/\lambda, p) = \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1 \right\}.$$

If a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ satisfies

$$1 < p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x), \qquad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x) < \infty, \tag{11}$$

then the function $p'(x) = p(x)/(p(x) - 1)$ is well defined and satisfies (11). For a complete information of variable Lebesgue spaces see [7] and [12].

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that (11) holds.

Theorem 4.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Suppose further that $p(\cdot)$ satisfies the log-Hölder continuity conditions*

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}, \;\; x, y \in \mathbb{R}^n, \;\; |x - y| < 1/2, \tag{12}$$

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x|)}, \;\; x, y \in \mathbb{R}^n, \;\; |y| \geq |x|. \tag{13}$$

Then, the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 4.4 was proved by Cruz–Uribe, Fiorenza and Neugebauer [8]. If a given function $p(\cdot)$ satisfies the conditions of Theorem 4.4, then we say $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, that is, if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Lemma 4.5 ([11], see also [42])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following assertions are equivalent*:

(i) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,
(ii) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$,
(iii) $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$ *for some* $1 < r < p_-$,
(iv) $(p(\cdot)/r)' \in \mathcal{B}(\mathbb{R}^n)$ *for some* $1 < r < p_-$.

For $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space (see [7, p. 73]) with associated space $L^{p'(\cdot)}(\mathbb{R}^n)$. Moreover it is known from [22] that

$$\frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C$$

for all cubes Q in $\mathbb{R}^n$.

Corollary 4.6. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Remark 4.7. Corollary 4.6 was first proved by second author and J. Wu in [42, Theorem 3.1].

Corollary 4.8. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M, b]$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.
(iii) *There exists a constant $c > 0$ such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \leq c < \infty.$$

Remark 4.9. Corollary 4.8 was first proved by second author and J. Wu in [41, Theorem 1.2]. However, the authors have shown that this theorem is true under the condition $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Obviously, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is weaker than $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. So, Corollary 4.8 extends the Zhang's result in [41, Theorem 1.2].

Corollary 4.10. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M^\#, b]$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.
(iii) *There exists a constant $c > 0$ such that*

$$\sup_Q \frac{\|(b - 2M^\#(b\chi_Q))\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \leq c < \infty.$$

Remark 4.11. Corollary 4.10 was first proved by second author and J. Wu in [41, Theorem 1.3]. For similar reasons as in Remark 4.9, Corollary 4.10 extends the Zhang's result in [41, Theorem 1.3].

4.3. Orlicz spaces

A function Φ is called a Young function if it is continuous, nonnegative, convex and strictly increasing on $[0,\infty)$ with $\Phi(0)=0$ and $\Phi(\infty)=\infty$.

The Orlicz space denoted by $L^\Phi = L^\Phi(\mathbb{R}^n)$ consists of all measurable function $f:\mathbb{R}^n\to\mathbb{R}$ such that for some $\lambda>0$

$$\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx<\infty.$$

When Φ is a Young function, $L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to norm

$$\|f\|_\Phi := \inf\left\{\lambda>0:\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}.$$

Given a Young function Φ, the complementary function $\tilde{\Phi}$ is defined by

$$\widetilde{\Phi}(t)=\sup\{ts-\Phi(s)\ ;s\ge 0\},\ \ t\in\mathbb{R}.$$

The complementary function $\widetilde{\Phi}$ is also a Young function when Φ is a Young function.

A Young function Φ is said to satisfy Δ_2 condition (we shall write $\Phi\in\Delta_2$) if there exist constants $k\ge 0$ and $T\ge 0$ such that $\Phi(2t)\le\ k\,\Phi(t)$ for all $t\ge T$.

A Young function Φ is said to satisfy ∇_2 condition, denoted $\Phi\in\nabla_2$, if for some $c>1$, $\Phi(ct)\ge\ 2c\,\Phi(t)$ for all $t>0$.

Also $\Phi\in\Delta_2$ if and only if $\widetilde{\Phi}\in\nabla_2$. For a complete information of Orlicz spaces, see for instance [3], [26], [33].

Theorem 4.12 ([25]). *M is bounded on $L^\Phi(\mathbb{R}^n)$ if and only if $\Phi\in\nabla_2$.*

Note that the Orlicz spaces $L^\Phi(\mathbb{R}^n)$ is a Banach function space with the associated space $L^{\widetilde{\Phi}}(\mathbb{R}^n)$ (see [7, p. 71]).

Corollary 4.13. *Assume that M is bounded on both $L^\Phi(\mathbb{R}^n)$ and $L^{\widetilde{\Phi}}(\mathbb{R}^n)$. Then M_b is bounded from $L^\Phi(\mathbb{R}^n)$ to itself if and only if $b\in\mathrm{BMO}(\mathbb{R}^n)$.*

Remark 4.14. The boundedness of M on both $L^\Phi(\mathbb{R}^n)$ and $L^{\widetilde{\Phi}}(\mathbb{R}^n)$ is equivalent to $\Phi\in\Delta_2\cap\nabla_2$. It is known that $\Phi\in\Delta_2\cap\nabla_2$ if and only if $1<a_\Phi\le b_\Phi<\infty$ (see [26]). It was proved in Corollary 2.3 in [13] that if $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $1<a_\Phi\le b_\Phi<\infty$, then M_b is bounded on $L^\Phi(\mathbb{R}^n)$. Corollary 4.13 also shows that the $b\in\mathrm{BMO}(\mathbb{R}^n)$ is necessary and sufficient condition for the boundedness of M_b on $L^\Phi(\mathbb{R}^n)$.

Corollary 4.15. *Assume that M is bounded on both $L^\Phi(\mathbb{R}^n)$ and $L^{\widetilde{\Phi}}(\mathbb{R}^n)$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M, b]$ *is bounded on* $L^\Phi(\mathbb{R}^n)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_\Phi}{\|\chi_Q\|_\Phi} \le c < \infty.$$

Corollary 4.16. *Assume that* M *is bounded on both* $L^\Phi(\mathbb{R}^n)$ *and* $L^{\widetilde{\Phi}}(\mathbb{R}^n)$. *Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M^\#, b]$ *is bounded on* $L^\Phi(\mathbb{R}^n)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - 2M^\#(b\chi_Q))\chi_Q\|_\Phi}{\|\chi_Q\|_\Phi} \le c < \infty.$$

4.4. Musielak–Orlicz (generalized Orlicz) spaces

Let $\varphi : [0,\infty) \to [0,\infty]$ be an increasing function such that $\varphi(0) = \lim_{t\to 0} \varphi(t) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$. Such a function φ is called a Φ-*prefunction.* Furthermore, we say that φ is

(1) a *weak* Φ-*function*, denoted by $\varphi \in \Phi_w$, if, additionally, $t \to \frac{\varphi(t)}{t}$ is almost increasing on $(0,\infty)$;

(2) a Φ-*function*, denoted by $\varphi \in \Phi$, if, additionally, it is left continuous and convex;

(3) a *strong* Φ-*function*, denoted by $\varphi \in \Phi_s$, if, additionally, it is continuous on $\overline{\mathbb{R}}$ and convex.

While studies on Φ-functions are frequently encountered, sometimes weak and sometimes strong Φ-functions are used for convenience. In doing so, the following result is used:

Lemma 4.17 ([18], Proposition 2.3)**.** *Every weak* Φ-*function is equivalent to a strong* Φ-*function.*

To define the generalized Orlicz spaces, we extend the definition of Φ-functions to depend on the location in space.

Definition 3. The set $\Phi(\Omega)$ of *generalized* Φ-*functions* consists of those functions $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ such that
(1) $\varphi(y, \cdot) \in \Phi$ for every $y \in \Omega$;
(2) $\varphi(\cdot, t) \in \mathcal{M}(\Omega)$ for every $t \ge 0$.
The families $\Phi_s(\Omega)$ and $\Phi_w(\Omega)$ are defined analogously.

Weak, strong, and generalized Φ-functions will all be written as Φ-functions for simplicity. We can now define generalized Orlicz spaces. We can take weak Φ-functions in our definitions by Lemma 4.17, though in the references above these definitions are made for Φ-functions.

Definition 4. Let $\varphi \in \Phi_w(\mathbb{R}^n)$ and define semimodular $\rho_{\varphi(\cdot)}$ for any measurable function f by

$$\rho_{\varphi(\cdot)}(f) := \int_{\mathbb{R}^n} \Phi(x, |f(x)|)dx.$$

The Musielak–Orlicz spaces also called generalized Orlicz spaces $L^{\varphi(\cdot)}(\mathbb{R}^n)$ is defined as the set

$$L^{\varphi(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \lim_{\lambda\to 0} \rho_{\varphi(\cdot)}(\lambda f) = 0 \right\}$$

equipped with the norm (Luxemburg–Nakano norm)

$$\|f\|_{\varphi(\cdot)(\mathbb{R}^n)} := \|f\|_{\varphi(\cdot)} := \inf \left\{ \lambda > 0 : \rho_{\varphi(\cdot)} \left(\frac{|f(x)|}{\lambda} \right) \leq 1 \right\}.$$

For more complete information, see for instance [18] and [9].

Note that the Musielak–Orlicz spaces $L^{\varphi(\cdot)}(\mathbb{R}^n)$ is a Banach function space (see [9, Lemma 3.2]) with the associated space $(L^{\varphi(\cdot)})'(\mathbb{R}^n)$ (see [17, Theorem 1.1]).

We now give a family of hypotheses that are closely related to the boundedness of the maximal operator on generalized Orlicz spaces.

Definition 5. Given $\varphi \in \Phi_w(\mathbb{R}^n)$ and $0 < p < \infty$, we define the following conditions:

(A0) $\varphi^{-1}(x,1) \approx 1$ uniformly in $x \in \Omega$.

(A1) There exists $\beta \in (0,1)$ such that $\beta\varphi^{-1}(x,t) \leq \varphi^{-1}(y,t)$ for every $t \in \left[1, \frac{1}{|x-y|^n}\right]$ and every $x, y \in \Omega$ with $|x-y| \leq 1$.

(A2) $L^{\varphi(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = L^{\varphi_\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\varphi_\infty(t) := \limsup_{|x|\to\infty} \varphi(x,t)$ and $\varphi_\infty \in \Phi_w$.

$(\mathrm{Inc})_p$ $s \mapsto s^{-p}\varphi(x,s)$ is increasing for all $x \in \Omega$.

$(\mathrm{Dec})_p$ $s \mapsto s^{-p}\varphi(x,s)$ is decreasing for all $x \in \Omega$.

$(\mathrm{aInc})_p$ $s \mapsto s^{-p}\varphi(x,s)$ is almost increasing uniformly for all $x \in \Omega$.

$(\mathrm{aDec})_p$ $s \mapsto s^{-p}\varphi(x,s)$ is almost decreasing uniformly for all $x \in \Omega$.

We say that φ satisfies (aInc) if it satisfies $(\mathrm{aInc})_p$ for some $p > 1$ and (aDec) if it satisfies $(\mathrm{aDec})_p$ for some $p < \infty$.

Theorem 4.18 ([19], Theorem 4.6)**.** *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ satisfy the conditions* (A0)–(A2) *and* (aInc). *Then the maximal operator M is bounded from $L^{\varphi(\cdot)}(\mathbb{R}^n)$ to $L^{\varphi(\cdot)}(\mathbb{R}^n)$.*

Corollary 4.19. *Assume that M is bounded on both $L^{\varphi(\cdot)}(\mathbb{R}^n)$ and $(L^{\varphi(\cdot)})'(\mathbb{R}^n)$. Then M_b is bounded from $L^{\varphi(\cdot)}(\mathbb{R}^n)$ to itself if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Corollary 4.20. *Assume that M is bounded on both $L^{\varphi(\cdot)}(\mathbb{R}^n)$ and $(L^{\varphi(\cdot)})'(\mathbb{R}^n)$. Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M, b]$ *is bounded on* $L^{\varphi(\cdot)}(\mathbb{R}^n)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_{\varphi(\cdot)}}{\|\chi_Q\|_{\varphi(\cdot)}} \leq c < \infty.$$

Corollary 4.21. *Assume that* M *is bounded on both* $L^{\varphi(\cdot)}(\mathbb{R}^n)$ *and* $(L^{\varphi(\cdot)})'(\mathbb{R}^n)$. *Then the following assertions are equivalent*:

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) $[M^\#, b]$ *is bounded on* $L^{\varphi(\cdot)}(\mathbb{R}^n)$.
(iii) *There exists a constant* $c > 0$ *such that*

$$\sup_Q \frac{\|(b - 2M^\#(b\chi_Q))\chi_Q\|_{\varphi(\cdot)}}{\|\chi_Q\|_{\varphi(\cdot)}} \leq c < \infty.$$

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