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WEAK POTENCY AND CYCLIC SUBGROUP SEPARABILITY OF CERTAIN FREE PRODUCTS AND TREE PRODUCTS

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Abstract. In this note, we shall show that the generalized free products of subgroup separable groups amalgamating a subgroup which itself is a finite extension of a finitely generated normal subgroup of both the factor groups are weakly potent and cyclic subgroup separable. Then we apply our result to generalized free products of finite extensions of finitely generated torsion-free nilpotent groups. Finally, we shall show that their tree products are cyclic subgroup separable.

1. Introduction

A group G is called weakly potent if for any element x of infinite order in G, we can find a positive integer r with the property that for each positive integer n, there exists a normal subgroup M_n of finite index in G such that xM_n has order exactly rn in the finite group G/M_n. A group G is cyclic subgroup separable if G is H-separable for every cyclic subgroup H of G.

Weak potency is a strong form of residual finiteness in the sense that a finitely generated torsion-free weakly potent group is residually finite. A cyclic subgroup separable group is residually finite. A group is termed Hopfian if it is not isomorphic to the quotient group by any nontrivial normal subgroup, in short, it is not isomorphic to any of its proper quotients. We note that Baumslag [6] constructed a generalized free product of two finitely generated free nilpotent groups, amalgamating a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$, which is non-Hopfian, whence not residually finite (see [15]). On the positive side, Baumslag [7] proved that generalized free product of two finitely generated torsion-free nilpotent groups, amalgamating a cyclic subgroup is residually finite.

The concept of weak potency was introduced by Evans [8] with the name regular quotient in order to show the residual finiteness and cyclic subgroup

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separability of certain generalised free products. In the same paper he showed that free groups and finitely generated torsion-free nilpotent groups are weakly potent. The term weak potency was properly and independently defined by Tang [17] and he proved that finite extensions of free groups and finitely generated torsion-free nilpotent groups are weakly potent. Wong and Wong [23] further improved the result of Tang [17] by showing that every finite extensions of finitely generated nilpotent groups are weakly potent. Kim and Tang [14] and Tang [17] used weak potency to determine the conjugacy separability of certain generalised free products of conjugacy separable groups. Since then, weak potency has been used in establishing the residual finiteness and conjugacy separability in various generalized free products, tree products, polygonal products and one-relator groups (see [1, 3, 14, 17, 20, 21, 23]).

On the other hand, separability properties are interesting in both group theory and topology (see [9, 18]). In [22], Wong and Wong have shown that the tree products of subgroup separable groups (particularly, the graph products of polycyclic-by-finite groups) amalgamating finitely generated normal subgroups are π_c. Zhou et al. [26] have shown the conjugacy separability of generalized free products of polycyclic-by-finite groups amalgamating a finite extension of central subgroup.

Recently, Zhou and Kim [24] proved that certain generalized free products of abelian subgroup separable groups, amalgamating an infinite cyclic subgroup, are abelian subgroup separable. Applying this, it was shown that any tree product of free groups or finitely generated nilpotent groups, amalgamating infinite cyclic subgroups, is abelian subgroup separable. They also have prove that certain HNN extensions is abelian subgroup separable [25]. Their study of abelian subgroup separability was inspired by [10].

In this paper, we consider the weak potency and cyclic subgroup separability of generated free products of subgroup separable groups amalgamating a subgroup which itself is a finite extension of a finitely generated normal subgroup of both the factor groups. Then we apply our result to generalized free products of finite extensions of finitely generated torsion-free nilpotent groups. We shall show that their tree products are cyclic subgroup separable.

The outline of this paper is as follows: We state the definitions and essential lemmas in Section 2. In Section 3, we established a stronger criterion for weak potency of generalized free products. Then, in Section 4, we apply the new criterion to generalized free products of finite extensions of finitely generated torsion-free nilpotent groups amalgamating a finitely generated normal subgroup or amalgamating a finite extension of a finitely generated normal subgroup which is either torsion-free nilpotent or abelian. Finally, in Section 5, we prove that these types of generalized free products and tree products are cyclic subgroup separable.

The notations used in this note are standard. In addition, the following notations will be used for any group G:

(i) $N \leq_f G$ (resp. $N \lhd_f G$) means N is a subgroup (resp. normal subgroup) of finite index in G.
(ii) $N \operatorname{char}_f G$ means N is a characteristic subgroup of finite index in G.
(iii) $G = A *_H B$ denotes the generalized free product G of A and B amalgamating the subgroup H.
(iv) If G is a generalized free product, then $\|g\|$ denotes the usual reduced length of g in G.

2. Preliminaries

We now state some definitions as well as some essential lemmas.

Definition 2.1. Let G be a group and H a subgroup of G.

(i) A group G is called *H-separable* for the subgroup H if for each $x \in G \backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$ (or $N \cap xH = \emptyset$).
(ii) G is termed *subgroup separable* if G is H-separable for every finitely generated subgroup H.
(iii) G is termed *cyclic subgroup separable* (briefly, π_c) if G is $\langle x \rangle$-separable for each $x \in G$.
(iv) G is termed *residually finite* if G is $\{1\}$-separable.

From the above definition, a subgroup separable group is π_c and a π_c group is residually finite. We note that a group G is also called *subgroup separable* if every finitely generated subgroup of G is closed in the *profinite topology*, the topology whose open basis consists of the cosets of finite index subgroups of G.

Definition 2.2 ([17])**.** Let G be a group and x be an element of infinite order in G. Then G is called *weakly $\langle x \rangle$–potent*, briefly $\langle x \rangle$–wpot, if we can find a positive integer r with the property that for each positive integer n, there exists $M_n \lhd_f G$ such that xM_n has order exactly rn in the finite group G/M_n; G is called *weakly potent* if G is $\langle x \rangle$–wpot for each element x of infinite order in G.

We note here that the subgroup M_n in Definition 2.2 depends on n. Hence when there is no confusion, we shall write M instead of M_n.

Free groups, polycyclic groups, finitely generated nilpotent groups and their finite extensions are known to be weakly potent for elements of infinite order and subgroup separable (see [8, 17, 21]). On the other hand, there are infinite groups with elements of finite order that are weakly potent but not residually finite. For example, we have $G = Z(p^\infty) \times \langle h \rangle$, where $Z(p^\infty)$ is the Prüfer group. Then G is weakly potent for elements of infinite order but G is not residually finite.

The following lemma can be derived easily from definition of the separability of a normal subgroup.

Lemma 2.3. *Let G be a group and $N \lhd G$. If G is N–separable, then G/N is residually finite.*

Lemma 2.4. *Let $G = A *_H B$, where A and B are finite. Then G is weakly potent and residually finite.*

Proof. We note that G is free-by-finite [11]. Hence G is weakly potent by [17, Lemma 2.2] and G is residually finite by [7, Theorem 2]. □

Lemma 2.5 ([2])**.** *Let $G = A *_H B$, where A and B are subgroup separable* (*or π_c or residually finite*)*. If H is finite, then G is subgroup separable* (*or π_c or residually finite respectively*)*.*

Lemma 2.6 ([4])**.** *Let $G = A *_H B$, where A and B are weakly potent. If H is finite, then G is weakly potent.*

We now state one criterion for the residual finiteness and another criterion for cyclic subgroup separability (π_c) of generalized free products.

Theorem 2.7 ([19])**.** *Let $G = A *_H B$. Suppose that*

(a) *A and B are residually finite and H-separable*;
(b) *for each $R \lhd_f H$, there exist $M_A \lhd_f A$ and $M_B \lhd_f B$ such that $M_A \cap H = M_B \cap H \subseteq R$.*

Then G is residually finite.

Theorem 2.8 ([13])**.** *Let $G = A *_H B$. Suppose that*

(a) *A and B are π_c and H-separable*;
(b) *for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$.*

Then G is π_c.

3. A new criterion for weak potency

In this section, we prove a new criterion for weak potency. In [20], Wong and Tang proved a criterion for the weak potency of generalized free product. Then, Asri, Wong and Wong [5] established the following stronger version below and we shall use it to establish a new criterion in Theorem 3.7.

Theorem 3.1 ([5])**.** *Let $G = A *_H B$. Suppose that*

(a) *A and B are H-separable*;
(b) *for each $R \lhd_f H$, there exist $M_A \lhd_f A$ and $M_B \lhd_f B$ such that $M_A \cap H = M_B \cap H \subseteq R$; and*
(c) *for any $x \in A$* (*or $x \in B$*) *of infinite order, we can find a positive integer r, such that for each positive integer n, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H$ and $N_A \cap \langle x \rangle = \langle x^{rn} \rangle$* (*or $N_B \cap \langle x \rangle = \langle x^{rn} \rangle$ if $x \in B$*)*.*

Then G is weakly potent.

For ease of exposition, from now on we define condition (c) in Theorem 3.1 as Condition ($*$). Thus, if A and B satisfy condition (c) of Theorem 3.1, then A and B are said to satisfy Condition ($*$).

3.1. Lemmas needed

Lemma 3.2. *Let A be a group and H be a subgroup of A. Suppose C is a normal subgroup of A such that $C \leq_f H$. If A is C-separable, then there exists $N \lhd_f A$ such that $N \cap H = C$.*

Proof. We form $\overline{A} = A/C$. Since A is C-separable, then $\overline{A}$ is residually finite (see Lemma 2.3). Since $\overline{H}$ is finite, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \overline{H} = \overline{1}$. Let N be the preimage of $\overline{N}$ in A. Then $N \lhd_f A$ and $N \cap H = C$. □

Next we show how we can construct a characteristic subgroup from a subgroup of finite index in a finitely generated group.

Let H be a finitely generated group and $R \lhd_f H$. If R is a characteristic subgroup of H, then we set $f_H(R) = R$. Suppose R is not a characteristic subgroup of H. Let $[H : R] = m$, where m is a positive integer. Since H is finitely generated, the number of subgroups of index m in H is finite. Let N be the intersection of all these subgroups of index m in H. Then $N \operatorname{char}_f H$ and $N \subseteq R$. We set $f_H(R) = N$ (see [21, Lemma 3.1]). The above statement can be summarized as follows:

Lemma 3.3. *Let H be a finitely generated group. If $R \lhd_f H$, then there exists $f_H(R) \subseteq R$ such that $f_H(R) \operatorname{char}_f H$ and $f_H(R)$ is finitely generated. Furthermore, if $R \operatorname{char}_f H$, then $f_H(R) = R$*

Lemma 3.4. *Let A be a subgroup separable group and H be a subgroup of A. Suppose C is a finitely generated normal subgroup of A such that $C \leq_f H$. Then for each $S \lhd_f C$, there exists $N \lhd_f A$ such that $N \cap H = f_C(S)$.*

Proof. Let $S \lhd_f C$ be given. Since C is finitely generated, then by Lemma 3.3, there exists $f_C(S) \subseteq S$ such that $f_C(S) \operatorname{char}_f C$ and $f_C(S)$ is finitely generated. Hence $f_C(S)$ is a finitely generated normal subgroup of A such that $f_C(S) \leq_f H$. Since A is subgroup separable, then A is $f_C(S)$-separable. Therefore by Lemma 3.2, there exists $N \lhd_f A$ such that $N \cap H = f_C(S)$. □

Lemma 3.5. *Let $G = A *_H B$, where A and B are subgroup separable. Suppose C is a finitely generated normal subgroup of A and B such that $C \leq_f H$. Then for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$.*

Proof. Let $R \lhd_f H$ be given. If $R = C$, then R is a finitely generated normal subgroup of A and B such that $R \leq_f H$. Since A and B are subgroup separable, A and B are R-separable. Hence by Lemma 3.2, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H = R$.

Suppose that $R \neq C$. Since $R \lhd_f H$ and $C \leq_f H$, then $R \cap C \lhd_f C$. Then by Lemma 3.4, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H = f_C(R \cap C) \subseteq R$. □

Lemma 3.6. *Let $G = A *_H B$. If G is weakly potent, then A and B satisfy Condition $(*)$.*

Proof. Let $x \in A$ be of infinite order (similarly, we can show for if $x \in B$). Since G is weakly potent, we can find a positive integer r, such that for each positive integer n there exists $N \lhd_f G$ such that $N \cap \langle x \rangle = \langle x^{rn} \rangle$. Let $N_A = N \cap A$ and $N_B = N \cap B$. Then $N_A \lhd_f A$, $N_B \lhd_f B$ and $N_A \cap \langle x \rangle = N \cap A \cap \langle x \rangle = N \cap \langle x \rangle = \langle x^{rn} \rangle$. Furthermore, we have $N_A \cap H = (N \cap A) \cap H = N \cap (A \cap H) = N \cap H = N \cap (B \cap H) = (N \cap B) \cap H = N_B \cap H$. □

3.2. The new criterion

From Theorem 3.1 with Lemmas 3.5 and 3.6, we have the following new criterion.

Theorem 3.7. *Let $G = A *_H B$, where A and B are subgroup separable groups. Suppose C is a finitely generated normal subgroup of A and B such that $C \leq_f H$. Then G is weakly potent if and only if A and B satisfy Condition $(*)$.*

Proof. Suppose G is weakly potent. Then the result follows from Lemma 3.6.

We now prove the converse by using Theorem 3.1. First, we show that A and B are H-separable. Since A and B are subgroup separable and C is a finitely generated subgroup of A and B, then A and B are C-separable. Let $a \in A \setminus H$. Since A is C-separable, then $\overline{A} = A/C$ is residually finite (see Lemma 2.3). Note that $\overline{H} = H/C$ is finite and $\overline{a} = aC \notin \overline{H}$. Then there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N} \cap \overline{a}\overline{H} = \emptyset$. Let N be the preimage of $\overline{N}$ in A. Then $a \notin NH$. Hence A is H-separable. Similarly, B is H-separable.

Next note that by assumption, there exists a finitely generated normal subgroup C of A and B such that $C \leq_f H$. Hence by Lemma 3.5, for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$. Finally, by assumption, both A and B satisfy Condition $(*)$. Therefore, by Theorem 3.1, G is weakly potent. □

4. Weak potency for generalized free products

In this section, we shall apply our criterion Theorem 3.7 to show that the generalized free product of finite extensions of finitely generated torsion-free nilpotent groups amalgamating finitely generated normal subgroup (Theorem 4.7) or amalgamating a subgroup which itself is a finite extension of a finitely generated normal and either a torsion-free nilpotent or an abelian subgroup of both factor groups are weakly potent (Theorem 4.8).

4.1. Characteristic weak potency of nilpotent groups

We begin by state an important property of finitely generated torsion-free nilpotent group, that is, the concept of characteristic weak potency which will play a central role in proving our results. This concept is a stronger form of weak potency and it was first introduced by Evans [8] with the name completely regular quotient.

Definition 4.1 ([8])**.** A group G is said to be *characteristically weakly potent* if, for every element $x \in G$ of infinite order, we can find a positive integer r such that for each positive integer n, there exists a characteristic subgroup N of finite index in G such that xN has order exactly rn in G/N.

Clearly every characteristically weakly potent group is weakly potent.

Lemma 4.2 ([8, 16])**.** *Every finitely generated torsion-free nilpotent group is characteristically weakly potent.*

For convenience we gather here the various residual properties of finitely generated torsion-free nilpotent groups and their finite extensions.

Lemma 4.3 ([8, 17, 21–23])**.** *Let G be a finite extension of a finitely generated nilpotent group. Then G is weakly potent and subgroup separable* (*and hence residually finite*)*.*

4.2. Application of the new criterion

We begin by proving several essential lemmas.

Lemma 4.4. *Let $G = A *_H B$. Suppose C is a finitely generated normal and either a torsion-free nilpotent or an abelian subgroup of A and B such that $C \leq_f H$. Furthermore, suppose A/S and B/S are weakly potent and residually finite for any finitely generated normal subgroup S of A and B such that $S \leq_f C$. Then A and B satisfy Condition* $(*)$*.*

Proof. Let $x \in A$ be of infinite order (we can similarly show for if $x \in B$).

Case 1. Suppose that $C \cap \langle x \rangle = 1$. Then $|xC| = \infty$. Denote $\overline{x} = xC$ in the group $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/C$, $\overline{B} = B/C$ and $\overline{H} = H/C$. Then $|\overline{x}| = \infty$ in $\overline{G}$. By assumption, $\overline{A}$ and $\overline{B}$ are weakly potent. Since $\overline{H}$ is finite, by Lemma 2.6, $\overline{G}$ is weakly potent. Thus, we can find a positive integer r such that for each positive integer n, there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{x}\overline{N}$ has order exactly rn in the finite group $\overline{G}/\overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $N \cap \langle x \rangle = \langle x^{rn} \rangle$. Now let $N_A = N \cap A$ and $N_B = N \cap B$. Then $N_A \lhd_f A$ and $N_B \lhd_f B$. Furthermore we have $N_A \cap \langle x \rangle = (N \cap A) \cap \langle x \rangle = N \cap (A \cap \langle x \rangle) = N \cap \langle x \rangle = \langle x^{rn} \rangle$ and $N_A \cap H = (N \cap A) \cap H = N \cap (A \cap H) = N \cap H = N \cap (B \cap H) = (N \cap B) \cap H = N_B \cap H$.

Case 2. Suppose that $C \cap \langle x \rangle = \langle x^s \rangle$ for some positive integer s. Clearly $|x^s| = \infty$. Since C is a finitely generated normal and torsion-free nilpotent, then C is characteristically weakly potent by Lemma 4.2 (if C is abelian, we use

Lemma 4.2 of [23] instead and the rest of the proof are essentially the same). Hence we can find a positive integer r' such that for each positive integer n, there exists $N_C \operatorname{char}_f C$ such that $N_C \cap \langle x^s \rangle = \langle x^{r'sn} \rangle$. This implies that $N_C \cap \langle x \rangle = (N_C \cap C) \cap \langle x \rangle = N_C \cap (C \cap \langle x \rangle) = N_C \cap \langle x^s \rangle = \langle x^{r'sn} \rangle$. Since $N_C \operatorname{char}_f C$ we have $N_C \lhd_f H$ and $N_C \lhd A$, $N_C \lhd B$. Denote $\overline{x} = xN_C$ in the group $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$ where $\overline{A} = A/N_C, \overline{B} = B/N_C$ and $\overline{H} = H/N_C$. Then $|\overline{x}| = r'sn$ in $\overline{G}$. By assumption, $\overline{A}$ and $\overline{B}$ are residually finite. Since $\overline{H}$ is finite, by Lemma 2.5, $\overline{G}$ is residually finite. Thus there exists $\overline{N} \lhd_f \overline{G}$ such that $\overline{x}, \overline{x}^2, \ldots, \overline{x}^{r'sn-1} \notin \overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_f G$ and $N \cap \langle x \rangle = \langle x^{r'sn} \rangle$. Now set $N_A = N \cap A$ and $N_B = N \cap B$. Then we have $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap \langle x \rangle = \langle x^{r'sn} \rangle$ and $N_A \cap H = N_B \cap H$. The lemma follows by noting that we can take $r = r's$. □

Lemma 4.5. *Let A be a finite extension of a finitely generated torsion-free nilpotent group and H is a finitely generated normal subgroup of A. Then A contains a finitely generated normal and torsion-free nilpotent subgroup C such that $C \leq_f H$.*

Proof. Let A be a finite extension of a finitely generated torsion-free nilpotent group, say W. Then $W \lhd_f A$. This implies that $W \cap H$ is a finitely generated normal and torsion-free nilpotent subgroup of A. Furthermore, we have $W \cap H \lhd_f H$. Let $C = W \cap H$. Then C is a finitely generated normal and torsion-free nilpotent subgroup of A such that $C \leq_f H$. □

Recall that finite extensions of finitely generated nilpotent groups are weakly potent and subgroup separable (see Lemma 4.3). Now let A be a finite extension of a finitely generated nilpotent group W. Then $W \lhd_f A$. If $C \lhd A$, then $WC/C \cong W/(W \cap C)$ is finitely generated nilpotent and $WC/C \lhd_f A/C$. This implies that A/C is a finite extension of the finitely generated nilpotent group WC/C. Hence A/C is weakly potent and subgroup separable (and hence residually finite). We summarized this fact as follows:

Lemma 4.6. *If A is a finite extension of a finitely generated nilpotent group, say W and $C \lhd A$, then A/C is a finite extension of the finitely generated nilpotent group WC/C and A/C is weakly potent and subgroup separable.*

Now we are ready to apply Theorem 3.7 to prove the following results.

Theorem 4.7. *Let $G = A *_H B$, where A and B are finite extensions of finitely generated torsion-free nilpotent groups and H is a finitely generated normal subgroup of A and B. Then G is weakly potent.*

Proof. We prove that G is weakly potent by using Theorem 3.7. Note that A and B are subgroup separable and hence residually finite (see Lemma 4.3).

Next we show that there exists a finitely generated normal and torsion-free nilpotent subgroup C of A and B such that $C \leq_f H$. Since H is finitely generated normal, by Lemma 4.5, A contains a finitely generated normal and torsion-free nilpotent subgroup $R_1 \leq_f H$. Similarly, B contains a finitely generated normal and torsion-free nilpotent subgroup $R_2 \leq_f H$. Let $R = R_1 \cap R_2$. Then R is a finitely generated normal and torsion-free nilpotent subgroup of H such that $R \leq_f H$. Since H is finitely generated, by Lemma 3.3, there exists $f_H(R) \subseteq R$ such that $f_H(R)\ \mathrm{char}_f\, H$ and $f_H(R)$ is finitely generated. Let $C = f_H(R)$. Then C is a finitely generated normal and torsion-free nilpotent subgroup of A and B such that $C \leq_f H$.

Now let S be any finitely generated normal subgroup of A and B such that $S \leq_f C$. Since A and B are finite extensions of finitely generated torsion-free nilpotent groups, then by Lemma 4.6, A/S and B/S are weakly potent and residually finite. Therefore by Lemma 4.4, A and B satisfy Condition $(*)$. This implies that all the conditions of Theorem 3.7 are satisfied and hence G is weakly potent. $\square$

Theorem 4.8. *Let $G = A *_H B$, where A and B are finite extensions of finitely generated nilpotent groups. Suppose C is a finitely generated normal and either a torsion-free nilpotent or an abelian subgroup of A and B such that $C \leq_f H$. Then G is weakly potent.*

Proof. By Lemma 4.6, A/S and B/S are weakly potent and residually finite for any finitely generated normal subgroup S of A and B such that $S \leq_f C$. So, the theorem follows from Lemma 4.4 and Theorem 3.7. $\square$

5. Cyclic subgroup separability for generalized free products

In this section, we first show that the generalised free products of subgroup separable groups amalgamating a subgroup which itself a finite extension of a finitely generated normal subgroup of the factor groups are π_c. Then we apply our result to generalized free products of finite extensions of finitely generated torsion-free nilpotent groups. Finally, we extend our result by proving their tree products are cyclic subgroup separable.

5.1. Cyclic subgroup separability

Theorem 5.1. *Let $G = A *_H B$, where A and B are subgroup separable groups. Suppose C is a finitely generated normal subgroup of A and B such that $C \leq_f H$. Then G is π_c.*

Proof. We prove this theorem by using Theorem 2.8. Since A and B are subgroup separable and C is a finitely generated subgroup of A and B, then A and B are π_c and C-separable. Since C is finitely generated and $C \leq_f H$, H is finitely generated. Therefore A and B are H-separable. So condition (a) of Theorem 2.8 is satisfied.

By assumption there exists a finitely generated normal subgroup C of A and B such that $C \leq_f H$. Then by Lemma 3.5, for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$. So condition (b) of Theorem 2.8 is satisfied. Therefore, G is π_c by Theorem 2.8. $\square$

The following corollary is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let $G = A *_H B$, where A and B are finite extensions of finitely generated nilpotent groups and C is a finitely generated normal subgroup of A and B such that $C \leq_f H$. Then G is π_c.*

5.2. Cyclic subgroup separability for tree products

Next, we extend our previous results (i.e., Theorem 5.1 and Corollary 5.2) to tree products. We begin with the following description of tree products.

Definition 5.3 ([22])**.** Let Γ be a tree with vertex set V and edge set E. To each vertex $v \in V$, we assign a group G_v. To each edge $e = (u, v) \in E$ (here $u, v \in V$), we assign a group H_e together with monomorphisms α_u and β_v embedding H_e into the two vertex groups at the end of edge e, G_u and G_v respectively. The tree product T of Γ with vertex groups $\{G_v\}_{v\in V}$ amalgamating the edge subgroups $\{H_e\}_{e\in E}$ is defined to be the group generated by the generators and relations of the vertex groups and additional generators $\alpha_u(g_e) = \beta_v(g_e)$ for each $g_e \in H_e$ and all $e \in E$ (here, u and v are the vertices joined by the edge e).

Definition 5.4. Let $u, v \in V$. If $(u, v) \in E$, i.e., u is adjacent to v in Γ. Then we shall write $u \sim v$ and the edge subgroup shall be written as H_{uv}.

If $(u, v) \in E$, i.e., if $u \sim v$, then H_{uv} is a subgroup of G_u and G_v. Since (u, v) and (v, u) represent the same edge in Γ, we have $H_{uv} = H_{vu}$.

Definition 5.5. Let I be a finite set. A family of subgroups $\{A_b\}_{b\in I}$ of a group G is said to satisfy the *intersection property* if for each $b_0 \in I$, there exists an ordering $(b_1, b_2, \ldots, b_n)$, where $I \setminus \{b_0\} = \{b_1, b_2, \ldots, b_n\}$ such that

$$A_{b_0} \cap A_{b_1} A_{b_2} \cdots A_{b_n} = \{1\}.$$

We note here that $A_{b_1} A_{b_2} \cdots A_{b_n}$ is just a subset of G. Also, if $b'_1, b'_2, \ldots, b'_n$ is a permutation of $b_1, b_2, \ldots, b_n$, then it is not necessary that $A_{b_1} A_{b_2} \cdots A_{b_n} = A_{b'_1} A_{b'_2} \cdots A_{b'_n}$. However, if each $A_{b_i} \lhd G$, then $A_{b_1} A_{b_2} \cdots A_{b_n} = A_{b'_1} A_{b'_2} \cdots A_{b'_n}$ and $A_{b_1} A_{b_2} \cdots A_{b_n}$ is a subgroup of G.

Lemma 5.6. *Let $I = \{b_1, b_2, \ldots, b_n\}$ and $\{A_b\}_{b\in I}$ be a family of subgroups of a group G satisfying the intersection property. Suppose $\{S_b\}_{b\in I}$ is a family of normal subgroups of G such that $S_b \subseteq A_b$ for all $b \in I$. Then, for all $a \in I$,*

$$S_{b_1} S_{b_2} \cdots S_{b_n} \cap A_a = S_a.$$

Proof. Clearly, $S_a \subseteq S_{b_1} S_{b_2} \cdots S_{b_n} \cap A_a$. It remains to show that $S_{b_1} S_{b_2} \cdots S_{b_n} \cap A_a \subseteq S_a$. Let $y \in S_{b_1} S_{b_2} \cdots S_{b_n} \cap A_a$.

Since $\{S_b\}_{b\in I}$ is a family of normal subgroups of G, we have

$$S_{b_1} S_{b_2} \cdots S_{b_n} = S_{b_{i_1}} S_{b_{i_2}} \cdots S_{b_{i_n}},$$

where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$. Let $a = b_j$. Then,

$$S_{b_1} S_{b_2} \cdots S_{b_n} = S_{b_j} S_{b_1} \cdots S_{b_{j-1}} S_{b_{j+1}} \cdots S_{b_n}.$$

So,

$$y = c_j c_1 \cdots c_{j-1} c_{j+1} \cdots c_n = h,$$

where $c_i \in S_{b_i}$ and $h \in A_{b_j}$. This implies that

$$c_j^{-1} h \in A_{b_j} \cap S_{b_1} \cdots S_{b_{j-1}} S_{b_{j+1}} \cdots S_{b_n}.$$

By the intersection property, there is an ordering $(d_1, d_2, \ldots, d_{n-1})$, where $I \setminus \{b_j\} = \{d_1, d_2, \ldots, d_{n-1}\}$ such that

$$A_{b_j} \cap A_{d_1} A_{d_2} \cdots A_{d_n} = \{1\}.$$

Since $S_{b_1} \cdots S_{b_{j-1}} S_{b_{j+1}} \cdots S_{b_n} = S_{d_1} S_{d_2} \cdots S_{d_n}$, we have

$$c_j^{-1} h \in A_{b_j} \cap A_{d_1} A_{d_2} \cdots A_{d_n} = \{1\}.$$

Hence, $y = h = c_j \in S_{b_j}$ and $S_{b_1} S_{b_2} \cdots S_{b_n} \cap A_{b_j} \subseteq S_{b_j}$. □

Definition 5.7. For each $u \in V$, let

$$I(u) = \{v \in V \mid v \sim u\}.$$

We say that the edge subgroups satisfying the *intersection property* if for each edge $(u, v) \in E$, the family of subgroups $\{H_{uv}\}_{v \in I(u)}$ of G_u satisfies the intersection property.

Let Γ be a tree with vertex set V and edge set E. Assume that $|V| = n$. Then we can find a vertex v_n, say, such that it is joined to a unique vertex, say v_{n-1}. Indeed, a tree always has such a vertex (usually called as *external vertex*). By removing the external vertex v_n and the edge (v_{n-1}, v_n) from Γ, we obtain a tree Γ' with vertex set $V' = V \setminus \{v_n\}$ and edge set $E' = E \setminus \{(v_{n-1}, v_n)\}$.

Let T be the tree product of Γ with vertex groups $\{G_u\}_{u \in V}$ amalgamating the edge subgroups $\{H_e\}_{e \in E}$ and let T' be the tree product of Γ' with vertex groups $\{G_u\}_{u \in V'}$ amalgamating the edge subgroups $\{H_e\}_{e \in E'}$. Then

$$T = T' *_H G_{v_n}, \tag{1}$$

where $H = H_{v_{n-1} v_n} = H_{v_n v_{n-1}}$.

Lemma 5.8. *If the edge subgroups of Γ satisfy the intersection property, then the edge subgroups of Γ' also satisfy the intersection property.*

Proof. Let $I_\Gamma(u) = \{v \in V \mid v \sim u \text{ in } \Gamma\}$ and $I_{\Gamma'}(u) = \{v \in V' \mid v \sim u \text{ in } \Gamma'\}$. Note that $I_\Gamma(u) = I_{\Gamma'}(u)$ for all $u \in V' \setminus \{v_{n-1}\}$ and $I_\Gamma(v_{n-1}) = I_{\Gamma'}(v_{n-1}) \cup \{v_n\}$.

Let $(u, v) \in E'$. Suppose $u \neq v_{n-1}$. Then there is an ordering $(b_1, b_2, \ldots, b_m)$, where $I_\Gamma(u) \setminus \{v\} = \{b_1, b_2, \ldots, b_m\}$ such that

$$H_{uv} \cap H_{ub_1} H_{ub_2} \cdots H_{ub_m} = \{1\}.$$

Since $I_{\Gamma'}(u) = I_\Gamma(u)$, we are done.

Suppose $u = v_{n-1}$. Then there is an ordering $(c_1, c_2, \ldots, c_{m'})$, where $I_\Gamma(u) \setminus \{v\} = \{c_1, c_2, \ldots, c_m'\}$ such that

$$H_{uv} \cap H_{uc_1} H_{uc_2} \cdots H_{uc_{m'}} = \{1\}.$$

Now, $v_n = c_i$ for some i and

$$H_{uc_1} \cdots H_{uc_{i-1}} H_{uc_{i+1}} \cdots H_{uc_{m'}} \subseteq H_{uc_1} H_{uc_2} \cdots H_{uc_{m'}}.$$

Therefore $I_{\Gamma'}(u) \setminus \{v\} = \{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m'\}$ and

$$H_{uv} \cap H_{uc_1} \cdots H_{uc_{i-1}} H_{uc_{i+1}} \cdots H_{uc_{m'}} = \{1\}.$$

Hence, the edge subgroups of Γ' also satisfy the intersection property. $\square$

For the rest of this section, we shall assume that T is the tree product with vertex groups $\{G_u\}_{u \in V}$ amalgamating the edge subgroups $\{H_{uv}\}_{u \sim v}$ and the intersection property holds.

We first prove the following case where the vertex groups are residually finite and the edge subgroups are finite.

Theorem 5.9. *Let T be a tree product where the vertex groups are residually finite and the edge subgroups are finite. Then T is residually finite.*

Proof. We prove this theorem by induction on n. The case for $n = 2$ follows from Lemma 2.5. Let $n \geq 3$. We assume the theorem holds for any tree with vertices fewer than n. Write the tree product $T = T' *_H G_{v_n}$ as in (1).

We now prove that T is residually finite by using Lemma 2.5 again. By the inductive hypothesis, T' is residually finite and by assumption, G_{v_n} is residually finite. Since H is finite, then T is residually finite by Lemma 2.5. $\square$

Lemma 5.10. *Let T be a tree product where the vertex groups are subgroup separable and the edge subgroups satisfy the intersection property. Suppose for each edge $(u, v) \in E$, there is a finitely generated normal subgroup C_{uv} of G_u and G_v such that $C_{uv} \leq_f H_{uv}$. Here we assume that $C_{uv} = C_{vu}$. Let $u_0 \in V$ be fixed and $I(u_0) = \{v_1, v_2, \ldots, v_n\}$. Suppose P is a subgroup of the vertex group G_{u_0} such that $\{P, H_{v_1}, H_{v_2}, \ldots, H_{v_n}\}$ satisfies the intersection property. If there exists a finitely generated normal subgroup C_p of G_{u_0} such that $C_p \leq_f P$, then for any $S_P \lhd_f C_P$, there exists $N \lhd_f T$ such that $N \cap P = f_{C_P}(S_P)$.*

Proof. For each $u \in V \setminus \{u_0\}$, let

$$S_u = C_{uw_1} C_{uw_2} \cdots C_{uw_{s_u}},$$

where $I(u) = \{w_1, w_2, \ldots, w_{s_u}\}$ and

$$S_{u_0} = f_{C_P}(S_P) C_{u_0v_1} C_{u_0v_2} \cdots C_{u_0v_n}.$$

Note that S_u is a finitely generated normal subgroup of G_u for all $u \in V$. By Lemma 5.6, for all $u \in V$, and for each edge $(u, v) \in E$

$$S_u \cap H_{uv} = C_{uv} = C_{vu} = S_v \cap H_{vu}.$$

Therefore, we can form the tree product $\overline{T}$ with vertex groups $\overline{G}_u = G_u/S_u$ amalgamating the edge subgroups $\overline{H}_{uv} = H_{uv}S_u/S_u = H_{vu}S_v/S_v = \overline{H}_{vu}$, where $(u,v) \in E$. Note that $\overline{T}$ is an epimorphic image of T under the natural homomorphism, i.e., $\overline{T}$ is a quotient group of T. By Lemma 2.3, $\overline{G}_u$ is residually finite for all $u \in V$. Since all the vertex groups of $\overline{T}$ are residually finite and all the edge subgroups of $\overline{T}$ are finite, by Theorem 5.9, $\overline{T}$ is residually finite.

By Lemma 5.6,

$$S_{u_0} \cap P = f_{C_P}(S_P).$$

So, $\overline{P} = PS_{u_0}/S_{u_0}$ is finite. Therefore, there exists $\overline{N} \lhd_f \overline{T}$ such that $\overline{N} \cap \overline{P} = \overline{1}$. Let N be the preimage of $\overline{N}$ in T. We shall show that N is the required subgroup.

Clearly, $f_{C_P}(S_P) \subseteq N \cap P$. It remains to show that $N \cap P \subseteq f_{C_P}(S_P)$. Let $y \in N \cap P$. Then, $\overline{y} \in \overline{N} \cap \overline{P} = \overline{1}$. This implies that $\overline{y} = 1$ in $\overline{G}_{u_0} = G_{u_0}/S_{u_0}$. Thus, $y \in S_{u_0}$. So, $y \in S_{u_0} \cap P = f_{C_P}(S_P)$. Hence, $y \in f_{C_P}(S_P)$ and $N \cap P \subseteq f_{C_P}(S_P)$. □

Lemma 5.11 ([12]). *Let $G = A *_H B$, where A and B are H-separable. Suppose for each $R \lhd_f H$, there exist $N_A \lhd_f A$, $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$. Let K be any subgroup of B. If B is K-separable, then G is K-separable.*

Lemma 5.12. *Let $G = A *_H B$, where A and B are subgroup separable. Suppose C is a finitely generated normal subgroup of A and B such that $C \leq_f H$. Let K be any subgroup of B. If B is K-separable, then G is K-separable.*

Proof. We show that this lemma satisfies all the conditions in Lemma 5.11. Clearly A and B are C-separable. Since C is finitely generated and $C \leq_f H$, H is finitely generated. Therefore A and B are H-separable. Furthermore, by Lemma 3.5, for each $R \lhd_f H$, there exist $N_A \lhd_f A$ and $N_B \lhd_f B$ such that $N_A \cap H = N_B \cap H \subseteq R$. The lemma now follows from Lemma 5.11. □

Lemma 5.13. *Let T be a tree product where the vertex groups are subgroup separable and the edge subgroups satisfy the intersection property. Suppose for each edge $(u,v) \in E$, there is a finitely generated normal subgroup C_{uv} of G_u and G_v such that $C_{uv} \leq_f H_{uv}$. Here we assume that $C_{uv} = C_{vu}$. Let $u_0 \in V$ be fixed and P be a subgroup of G_{u_0} such that G_{u_0} is P-separable. Then T is P-separable.*

Proof. We prove this lemma by induction on n. The case for $n = 2$ follows from Lemma 5.12. Let $n \geq 3$. We assume the theorem holds for any tree with vertices fewer than n. Write the tree product $T = T' *_H G_{v_n}$ as in (1), where $H = H_{v_{n-1}v_n} = H_{v_n v_{n-1}}$. Let $C = C_{v_{n-1}v_n} = C_{v_n v_{n-1}}$. We note here that C is a finitely generated normal subgroup of G_{v_n} and $G_{v_{n-1}}$ and $C \leq_f H$.

Next we show that the tree product $T = T' *_H G_{v_n}$ satisfies all the conditions in Lemma 5.11. By Lemma 5.8, the edge subgroups of T' satisfy the intersection property. Since C is finitely generated and $C \leq_f H$, H is finitely generated. Therefore G_{v_n} and $G_{v_{n-1}}$ are H-separable. By induction, T' is H-separable.

Now let $R \lhd_f H$ be given. Since $R \lhd_f H$ and $C \leq H$, then $R \cap C \lhd_f C$. For $H = H_{v_{n-1}v_n}$ and $C = C_{v_{n-1}v_n}$, by Lemma 5.10, we can find $N_{T'} \lhd_f T'$ such that $N_{T'} \cap H = f_C(R \cap C)$. For $H = H_{v_n v_{n-1}}$ and $C = C_{v_n v_{n-1}}$, by Lemma 3.4, we can find $N_n \lhd_f G_{v_n}$ such that $N_n \cap H = f_C(R \cap C)$. Hence the tree product $T = T' *_H G_{v_n}$ satisfies all the conditions of Lemma 5.11.

Suppose that $u_0 = v_n$. Since G_{v_n} is P-separable, by Lemma 5.11, T is P-separable. Suppose that $u_0 \neq n$. By induction, T' is P-separable and thus by Lemma 5.11 again, T is P-separable. □

Now we are ready to prove our results.

Theorem 5.14. *Let T be a tree product where the vertex groups are subgroup separable and the edge subgroups satisfy the intersection property. Suppose for each edge $(u, v) \in E$, there is a finitely generated normal subgroup C_{uv} of G_u and G_v such that $C_{uv} \leq_f H_{uv}$. Here we assume that $C_{uv} = C_{vu}$. Then T is π_c.*

Proof. We prove this theorem by induction on n. The case for $n = 2$ follows from Theorem 5.1. Suppose $n \geq 3$. We assume that the theorem holds for any tree with vertices less than n. Write the tree product $T = T' *_H G_{v_n}$ as in (1), where $H = H_{v_{n-1}v_n} = H_{v_n v_{n-1}}$. Let $C = C_{v_{n-1}v_n} = C_{v_n v_{n-1}}$. We note here that C is a finitely generated normal subgroup of G_{v_n} and $G_{v_{n-1}}$ and $C \leq_f H$.

We complete the proof by using Theorem 2.8. By Lemma 5.8, the edge subgroups of T' satisfy the intersection property. By induction, T' is π_c and by assumption, G_{v_n} is π_c. Since C is finitely generated and $C \leq_f H$, H is finitely generated. Therefore G_{v_n} and $G_{v_{n-1}}$ are H-separable. By Lemma 5.13, T' is H-separable. So condition (a) of Theorem 2.8 is satisfied.

Let $R \lhd_f H$ be given. Since $R \lhd_f H$ and $C \leq H$, then $R \cap C \lhd_f C$. For $H = H_{v_{n-1}v_n}$ and $C = C_{v_{n-1}v_n}$, by Lemma 5.10, we can find $N_{T'} \lhd_f T'$ such that $N_{T'} \cap H = f_C(R \cap C)$. For $H = H_{v_n v_{n-1}}$ and $C = C_{v_n v_{n-1}}$, by Lemma 3.5, we can find $N_n \lhd_f G_{v_n}$ such that $N_n \cap H = f_C(R \cap C)$. So condition (b) of Theorem 2.8 is satisfied. Therefore, T is π_c by Theorem 2.8. □

From Theorem 5.14, we can get the following two corollaries.

Corollary 5.15 ([22]). *Let T be a tree product where the vertex groups are finite extensions of finitely generated torsion-free nilpotent groups and the edge subgroups are finitely generated normal subgroups in their respective vertex groups. If the edge subgroups satisfy the intersection property, then T is π_c.*

Corollary 5.16. *Let T be a tree product where the vertex groups are finite extensions of finitely generated nilpotent groups and the edge subgroups satisfy the intersection property. Suppose for each edge $(u, v) \in E$, there is a finitely generated normal subgroup C_{uv} of G_u and G_v such that $C_{uv} \leq_f H_{uv}$. Here we assume that $C_{uv} = C_{vu}$. Then T is π_c.*

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