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SPIN STRUCTURES ON COMPLEX PROJECTIVE SPACES AND CIRCLE ACTIONS

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Abstract. It is known that the complex projective space $\mathbb{CP}^n$ admits a spin structure if and only if n is odd. In this paper, we provide another proof that $\mathbb{CP}^{2m}$ does not admit a spin structure, by using a circle action.

1. Introduction

Spinors were considered in the study of Dirac operators by physicists. Spin structures have played an important role in quantum field theory in mathematical physics, and have been also studied in algebraic topology, differential geometry, K-theory, etc.

It is known that the complex projective space $\mathbb{CP}^n$ admits a spin structure if and only if n is odd. This can be seen as follows. A manifold admits a spin structure if and only if its second Whitney-Stiefel class is zero [2]. The second Whitney-Stiefel class is modulo 2 reduction of the first Chern class. The first Chern class of $\mathbb{CP}^n$ is $(n+1)x$, where x is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

In this paper, we provide another proof that $\mathbb{CP}^{2m}$ does not admit a spin structure if $m>0$, by using a circle action.

Theorem 1.1. *The complex projective space $\mathbb{CP}^{2m}$ does not admit a spin structure, for any positive integer m.*

2. Background

Let M be an orientable manifold with a Riemannian metric g. A *spin structure* on M is an equivariant lift P (called a principal $\mathbf{Spin}(n)$-bundle) of the oriented orthonormal frame bundle Q (called the principal $\mathbf{SO}(n)$-bundle) over M with respect to the double covering $\pi:\mathbf{Spin}(n)\to\mathbf{SO}(n)$. An orientable Riemannian manifold (M,g) with a spin structure is called a *spin manifold*.

Let M be a compact oriented spin manifold. The $\hat{A}$-genus is the genus belonging to the power series $\frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$. Atiyah and Hirzebruch proved that

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if a compact oriented spin manifold M admits a non-trivial smooth action of the circle group S^1, the equivariant index of the Dirac operator on (the spinor bundle of) M is equal to the $\hat{A}$-genus of M and it vanishes.

Theorem 2.1 ([1, Lemma, p. 21]). *Let the circle act on a compact oriented spin manifold. Then* $\hat{A}(M) = 0$.

Let the circle act on a $2n$-dimensional oriented manifold M. Let p be an isolated fixed point. The tangent space T_pM at p has a decomposition

$$T_pM = \bigoplus_{i=1}^{n} L_i$$

into real 2-dimensional irreducibles L_i. For each i, we choose an orientation of L_i so that the circle acts on L_i by multiplication by $g^{w_{p,i}}$ for all $g \in S^1 \subset \mathbb{C}$, for some positive integer $w_{p,i}$. The positive integers $w_{p,i}$ are called *weights* at p. Let $\epsilon(p) = +1$ if the orientation on M agrees with that on the representation space $\bigoplus_{i=1}^{n} L_i$, and $\epsilon(p) = -1$ otherwise. If M admits a spin structure, the following formula holds.

Theorem 2.2 ([1, p. 20]). *Let the circle act on a compact oriented spin manifold M with a discrete fixed point set. Then*

$$0 = \hat{A}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \cdot \prod_{i=1}^{n} \frac{t^{\frac{w_{p,i}}{2}}}{1 - t^{w_{p,i}}}$$

for all indeterminates t.

A standard linear action of the circle on the complex projective space $\mathbb{CP}^n$ has $n+1$ fixed points. In particular, we consider the following action.

Example 2.3. Let the circle act on $\mathbb{CP}^n$ by

$$g \cdot [z_0 : \cdots : z_n] = [z_0 : gz_1 : g^2 z_2 : \cdots : g^n z_n].$$

The action has $n+1$ fixed points $p_0 = [1 : 0 : \cdots : 0]$, $p_1 = [0 : 1 : 0 : \cdots : 0]$, ..., $p_n = [0 : \cdots : 0 : 1]$. At each fixed point $p_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, using local complex coordinates $(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i})$, the circle acts near p_i by

$$\begin{aligned} g \cdot (\tfrac{z_0}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i}) &= (\tfrac{z_0}{g^i z_i}, \ldots, \tfrac{g^{i-1} z_{i-1}}{g^i z_i}, \tfrac{g^{i+1} z_{i+1}}{g^i z_i}, \ldots, \tfrac{g^n z_n}{g^i z_i}) \\ &= (g^{-i} \tfrac{z_0}{z_i}, \ldots, g^{-1} \tfrac{z_{i-1}}{z_i}, g \tfrac{z_{i+1}}{z_i}, \ldots, g^{n-i} \tfrac{z_n}{z_i}) \end{aligned}$$

for all $g \in S^1 \subset \mathbb{C}$. Therefore, as complex S^1-representation, the weights at p_i are $\{j - i\}_{0 \le j \le n, j \ne i}$. Since p_i has i negative weights as complex S^1-representations, as an oriented manifold, $\epsilon(p_i) = (-1)^i$ and the weights at p_i as an oriented manifold are $\{|j - i|\}_{0 \le j \le n, j \ne i}$.

3. Proof

For a compact oriented manifold admitting a circle action, the following lemma provides an obstruction to the existence of a spin structure.

Lemma 3.1. *Let M be a $2n$-dimensional compact oriented manifold. Suppose that M admits a circle action with a non-empty discrete fixed point set with the following property: there is a fixed point q such that $\sum_{i=1}^n w_{q,i} < \sum_{i=1}^n w_{p,i}$ for all fixed points p with $\epsilon(p) \neq \epsilon(q)$. Then M does not admit a spin structure.*

Proof. Assume on the contrary that M admits a spin structure. By Theorem 2.2,

$$\begin{aligned} 0 &= \sum_{p \in M^{S^1}} \epsilon(p) \cdot \prod_{i=1}^n \frac{t^{\frac{w_{p,i}}{2}}}{1 - t^{w_{p,i}}} \\ &= \sum_{p \in M^{S^1}} \epsilon(p) \cdot t^{\frac{\sum_{i=1}^n w_{p,i}}{2}} \cdot \prod_{i=1}^n \left(1 + t^{w_{p,i}} + t^{2w_{p,i}} + \cdots\right) \end{aligned}$$

for all indeterminates t.

In the equation, the fixed point q contributes the term $\epsilon(q) \cdot t^{\frac{\sum_{i=1}^n w_{q,i}}{2}}$, which is the lowest degree term in the equation. Since $\sum_{i=1}^n w_{q,i} < \sum_{i=1}^n w_{p,i}$ for every fixed point p with $\epsilon(p) \neq \epsilon(q)$, the term $\epsilon(q) \cdot t^{\frac{\sum_{i=1}^n w_{q,i}}{2}}$ cannot be canceled out. This leads to a contradiction. □

Using Lemma 3.1, we prove Theorem 1.1.

Proof of Theorem 1.1. Consider the circle action on $\mathbb{CP}^{2m}$ in Example 2.3. For each fixed point p_i, $\epsilon(p_i) = (-1)^i$ and the weights at p_i are $\{|j - i|\}_{0 \leq j \leq 2m, j \neq i}$, $0 \leq i \leq 2m$. For any integer k such that $-m \leq k \leq m$, the sum $\sum_{l=1}^{2m} w_{p_{m+k},l}$ of the weights at p_{m+k} is

$$\begin{aligned} \sum_{l=1}^{2m} w_{p_{m+k},l} &= \sum_{0 \leq l \leq 2m, l \neq m+k} |l - m - k| \\ &= (m+k) + (m+k-1) + \cdots + 2 + 1 + 1 + 2 + \cdots \\ &\quad + (m-k-1) + (m-k) \\ &= \frac{(m+k)(m+k+1)}{2} + \frac{(m-k)(m-k+1)}{2} \\ &= m(m+1) + k^2. \end{aligned}$$

Hence, $\sum_{l=1}^{2m} w_{p_{m+k},l} > \sum_{l=1}^{2m} w_{p_m,l}$ if $k \neq 0$. Therefore, by Lemma 3.1, $\mathbb{CP}^{2m}$ does not admit a spin structure. □

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