Bull. Korean Math. Soc. **60** (2023), No. 5, pp. 1321–1334
https://doi.org/10.4134/BKMS.b220692
pISSN: 1015-8634 / eISSN: 2234-3016

STRUCTURE OF IDEMPOTENTS IN POLYNOMIAL RINGS AND MATRIX RINGS

Juan Huang, Tai Keun Kwak, Yang Lee, and Zhelin Piao

Abstract. An idempotent e of a ring R is called *right* (resp., *left*) *semicentral* if $er = ere$ (resp., $re = ere$) for any $r \in R$, and an idempotent e of $R\backslash\{0,1\}$ will be called *right* (resp., *left*) *quasicentral* provided that for any $r \in R$, there exists an idempotent $f = f(e,r) \in R\backslash\{0,1\}$ such that $er = erf$ (resp., $re = fre$). We show the whole shapes of idempotents and right (left) semicentral idempotents of upper triangular matrix rings and polynomial rings. We next prove that every nontrivial idempotent of the n by n full matrix ring over a principal ideal domain is right and left quasicentral and, applying this result, we can find many right (left) quasicentral idempotents but not right (left) semicentral.

1. Introduction

The concept of idempotent arises in a number of places in abstract algebra (for example, in ring theory, in the theory of projectors, and in the theory of closure operators). In particular, in ring theory, many kinds of rings are characterized by idempotents, for example, Boolean rings, semisimple Artinian rings, von Neumann regular rings, Baer rings, Rickart rings and Abelian rings. In the studies of such topics we encounter with many types of idempotents.

In this article, we show the whole shapes of idempotents and right (left) semicentral idempotents of upper triangular matrix rings and polynomial rings and then study the structure of rings with idempotents which satisfy a generalized condition of right semicentralness (namely, a right (left) *quasicentral* idempotent), showing that every nonzero nonidentity idempotent of the n by n full matrix ring over a principal ideal domain is both right and left quasicentral.

Throughout this article, every ring is an associative ring with identity unless otherwise stated. Let R be a ring. Let $I(R) = \{e \in R \mid e^2 = e\}$ and $I(R)' = I(R)\backslash\{0,1\}$. We use $Z(R)$, $N^*(R)$ and $N(R)$ to denote the center, the upper

Received October 4, 2022; Accepted June 9, 2023.
2020 *Mathematics Subject Classification.* Primary 16S50, 16U40.
Key words and phrases. Idempotent, right (left) semicentral idempotent, right (left) quasicentral idempotent, right (left) quasi-Abelian ring, matrix ring, polynomial ring.
The fourth named author was supported by the Science and Technology Research Project of Education Department of Jilin Province, China(JJKH20210563KJ).

nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements of R, respectively. It is well-known that $N^*(R) \subseteq N(R)$. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{i+1,j+1}$ for all $1 \leq i \leq n-2$ with $i < j\}$. I_n means the identity matrix in $Mat_n(R)$ and use E_{ij} for the matrix with (i,j)-entry 1 and zeros elsewhere. $R[x]$ denotes the polynomial ring with an indeterminate x over R. We use $\mathbb{Z}$ ($\mathbb{Z}_n$) to denote the ring of integers (modulo n).

2. Structures of matrices and polynomials which are idempotents

In this section we study the structure of idempotents and right semicentral idempotents in upper triangular matrix rings and polynomial rings. We first investigate the shape of idempotents in upper triangular matrix rings.

Lemma 2.1. *Let R be a ring and $n \geq 2$.*

(1) *If $(a_{ij}) \in I(T_n(R))'$, then we have the following*:

(i) $(e_{11}, \ldots, e_{nn}) \in I(R)^n \backslash \{(0, \ldots, 0), (1, \ldots, 1)\}$;

(ii) $e_{ii}a_{i,i+1}e_{i+1,i+1} = 0$ *for all* $i = 1, \ldots, n-1$;

(iii) $e_{ii}(a_{i,i+1}a_{i+1,i+2}) = (a_{i,i+1}a_{i+1,i+2})e_{i+2,i+2} = -e_{ii}a_{i,i+2}e_{i+2,i+2}$ *for all* $i = 1, \ldots, n-2$;

(iv)
$$e_{ii}\left(\sum_{s=1}^{k-2} a_{i,i+s}a_{i+s,i+k-1}\right) = \left(\sum_{s=1}^{k-2} a_{i,i+s}a_{i+s,i+k-1}\right)e_{i+k-1,i+k-1}$$
$$= -e_{ii}a_{i,i+k-1}e_{i+k-1,i+k-1}$$
for all $1 \leq i \leq n-(k-1)$ *and* $4 \leq k \leq n$, *where* $e_{ii} = a_{ii}$.

(2) *If $(a_{ij}) \in I(D_n(R))'$, then we have the following*:

(i) $a_{ii} \in I(R)'$, *e say*;

(ii) $ea_{i,i+1}e = 0$ *for all* $i = 1, \ldots, n-1$;

(iii) $e(a_{i,i+1}a_{i+1,i+2}) = (a_{i,i+1}a_{i+1,i+2})e = -ea_{i,i+2}e$ *for all* $i = 1, \ldots, n-2$;

(iv) $e\left(\sum_{s=1}^{k-2} a_{i,i+s}a_{i+s,i+k-1}\right) = \left(\sum_{s=1}^{k-2} a_{i,i+s}a_{i+s,i+k-1}\right)e = -ea_{i,i+k-1}e$ *for all* $1 \leq i \leq n-(k-1)$ *and* $4 \leq k \leq n$.

(3) *If $(a_{ij}) \in I(V_n(R))'$, then we have the following*:

(i) $a_{ii} \in I(R)'$, *e say*;

(ii) $ea_{12}e = 0$ *and* $ea_{12}^2 = a_{12}^2e = -ea_{13}e$;

(iii) $e\left(\sum_{s=1}^{k-2} a_{1,1+s}a_{1+s,k}\right) = \left(\sum_{s=1}^{k-2} a_{1,1+s}a_{1+s,k}\right)e = -ea_{1,k}e$ *for all* $4 \leq k \leq n$.

Proof. (1) Let $(a_{ij}) \in I(T_n(R))'$. Then, clearly,

(i) $(e_{11}, \ldots, e_{nn}) \in I(R)^n \backslash \{(0, \ldots, 0), (1, \ldots, 1)\}$.

Since $\left(\begin{smallmatrix} e_{ii} & a_{i,i+1} \\ 0 & e_{i+1,i+1} \end{smallmatrix}\right) \in I(T_2(R))$ for all $1 \leq i \leq n-1$, we get $a_{i,i+1} = e_{ii}a_{i,i+1} + a_{i,i+1}e_{i+1,i+1}$ and so we obtain

(ii) $e_{ii}a_{i,i+1}e_{i+1,i+1} = 0$ for all $i = 1, \dots, n-1$, by multiplying this equality by e_{ii} on the left.

Since $\begin{pmatrix} e_{ii} & a_{i,i+1} & a_{i,i+2} \\ 0 & e_{i+1,i+1} & a_{i+1,i+2} \\ 0 & 0 & e_{i+2,i+2} \end{pmatrix} \in I(T_3(R))$ for all $1 \le i \le n-2$, we get $a_{i,i+2} = e_{ii}a_{i,i+2} + a_{i,i+1}a_{i+1,i+2} + a_{i,i+2}e_{i+2,i+2}$ and so we obtain

(iii) $e_{ii}(a_{i,i+1}a_{i+1,i+2}) = (a_{i,i+1}a_{i+1,i+2})e_{i+2,i+2} = -e_{ii}a_{i,i+2}e_{i+2,i+2}$ for all $i = 1, \dots, n-2$, by multiplying this equality by e_{ii} (resp., $e_{i+2,i+2}$) on the left (resp., right).

Since $\begin{pmatrix} e_{ii} & a_{i,i+1} & a_{i,i+2} & a_{i,i+3} \\ 0 & e_{i+1,i+1} & a_{i+1,i+2} & a_{i+1,i+3} \\ 0 & 0 & e_{i+2,i+2} & a_{i+2,i+3} \\ 0 & 0 & 0 & e_{i+3,i+3} \end{pmatrix} \in I(T_4(R))$ for all $1 \le i \le n-3$, we get $a_{i,i+3} = e_{ii}a_{i,i+3} + a_{i,i+1}a_{i+1,i+3} + a_{i,i+2}a_{i+2,i+3} + a_{i,i+3}e_{i+3,i+3}$ and so we obtain

$$\begin{aligned} e_{ii}(a_{i,i+1}a_{i+1,i+3} + a_{i,i+2}a_{i+2,i+3}) &= (a_{i,i+1}a_{i+1,i+3} + a_{i,i+2}a_{i+2,i+3})e_{i+3,i+3} \\ &= -e_{ii}a_{i,i+3}e_{i+3,i+3} \end{aligned}$$

for all $i = 1, \dots, n-3$, by multiplying this equality by e_{ii} (resp., $e_{i+3,i+3}$) on the left (resp., right).

We proceed in this manner. Let $i = 1, \dots, n-(k-1)$ with $4 \le k \le n$ and $(b_{st}) \in T_k(R)$ such that

$$b_{st} = a_{st} \text{ for all } i \le s, t \le i+k-1.$$

Then, since $(b_{st}) \in I(T_k(R))$, we get

$$a_{i,i+h} = e_{ii}a_{i,i+h} + a_{i,i+1}a_{i+1,i+h} + \cdots + a_{i,i+h-1}a_{i+h-1,i+h} + a_{i,i+h}e_{i+h,i+h},$$

where $h = k-1$. So we obtain

(iv)

$$\begin{aligned} &e_{ii}(a_{i,i+1}a_{i+1,i+h} + \cdots + a_{i,i+h-1}a_{i+h-1,i+h}) \\ &= (a_{i,i+1}a_{i+1,i+h} + \cdots + a_{i,i+h-1}a_{i+h-1,i+h})e_{i+h,i+h}, \end{aligned}$$

by multiplying this equality by e_{ii} (resp., $e_{i+h,i+h}$) on the left (resp., right).

(2) and (3) are clear from (1) and (2), respectively. □

Let R be a ring and $e \in I(R)$. By [1], e is called *right* (resp., *left*) *semicentral* in R if $eR = eRe$ (resp., $Re = eRe$). The following are easily checked: (i) e is right (resp., left) semicentral in R if and only if $ea = eae$ (resp., $ae = eae$) for all $a \in R$; (ii) e is right semicentral if and only if $1-e$ is left semicentral; and (iii) e is central in R if and only if e is both right and left semicentral in R. For example, in $T_2(\mathbb{Z}_2)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is right semicentral but not left semicentral, and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is left semicentral but not right semicentral.

Theorem 2.2. *Let R be a ring and $(a_{ij}) \in I(D_n(R))$ $(n \ge 2)$.*

(1) *$a_{ii} \in I(R)$.*

(2) *If a_{ii}, e say, is right semicentral in R, then we have the following*:

(i) *$ea_{ij} = 0$ for all i, j with $i \ne j$;*

(ii) *$a_{ij} = a_{ij}e$ for all i, j;*

(iii) $(a_{ij})(b_{ij}) = (a_{ij})(eI_n)(b_{ij}) = (a_{ij})(b_{ij})(eI_n)$ *for all* $(b_{ij}) \in D_n(R)$.

(3) a_{ii} *is right semicentral in* R *if and only if* (a_{ij}) *is right semicentral in* $D_n(R)$.

Proof. Write $E = D_n(R)$ and $e = a_{ii}$. (1) is clear.

(2) Suppose that e is right semicentral in R. Since $A = (a_{ij}) \in I(E)$, $ea_{i,i+1}e = 0$ for all $i = 1, \ldots, n-1$ by Lemma 2.1(2-ii). Since e is right semicentral, we get $ea_{i,i+1} = ea_{i,i+1}e = 0$, from which we see that

$$a_{i,i+1} = ea_{i,i+1} + a_{i,i+1}e = a_{i,i+1}e \text{ for all } i = 1, \ldots, n-1. \tag{0}$$

Since $a_{i,i+2} = ea_{i,i+2} + a_{i,i+1}a_{i+1,i+2} + a_{i,i+2}e$ and $ea_{i,i+1} = 0$, we get

$$ea_{i,i+2} = ea_{i,i+2} + (ea_{i,i+1})a_{i+1,i+2} + ea_{i,i+2}e = ea_{i,i+2} + ea_{i,i+2}e,$$

and this implies $0 = ea_{i,i+2}e = ea_{i,i+2}$. Inductively we assume that $ea_{i,k} = 0$ for all $k = 1, \ldots, n-1$. Then since $a_{in} = ea_{in} + a_{i,i+1}a_{i+1,n} + \cdots + a_{i,n-1}a_{n-1,n} + a_{i,n}e$, we get

$$ea_{in} = ea_{in} + (ea_{i,i+1})a_{i+1,n} + \cdots + (ea_{i,n-1})a_{n-1,n} + ea_{in}e = ea_{in} + ea_{in}e$$

and this implies $0 = ea_{in}e = ea_{in}$ for all $i = 1, \ldots, n-1$. Therefore we now have

$$ea_{ij} = 0 \text{ for all } i, j \text{ with } i \neq j. \tag{i}$$

Note that $a_{i,i+2} = ea_{i,i+2} + a_{i,i+1}a_{i+1,i+2} + a_{i,i+2}e = ea_{i,i+2} + (a_{i,i+1}e)a_{i+1,i+2} + a_{i,i+2}e = a_{i,i+2}e$ by (0) and (i). Now, inductively, we obtain

$$\begin{aligned} a_{i,i+h} &= ea_{i,i+h} + a_{i,i+1}a_{i+1,i+h} + \cdots + a_{i,i+h-1}a_{i+h-1,i+h} + a_{i,i+h}e \\ &= ea_{i,i+h} + (a_{i,i+1}e)a_{i+1,i+h} + \cdots + (a_{i,i+h-1}e)a_{i+h-1,i+h} + a_{i,i+h}e \\ &= a_{i,i+h}e \end{aligned}$$

by (0) and (i), assuming that $a_{i,i+s} = a_{i,i+s}e$ for all $s = 1, \ldots, h-1$. Therefore

$$a_{ij} = a_{ij}e \text{ for all } i, j. \tag{ii}$$

Lastly, for any $B = (b_{ij}) \in E$,

$$\begin{aligned} AB = (a_{ij}e)(b_{ij}) &= (a_{ij})(eI_n)(b_{ij}) = (a_{ij})(eb_{ij}) \\ &= (a_{ij})(eb_{ij}e) = (a_{ij})(eb_{ij})(eI_n) \\ &= (a_{ij})(eI_n)(b_{ij})(eI_n) = AB(eI_n) \end{aligned}$$

by (i), (ii), and the hypothesis that e is right semicentral in R.

(3) Let e be right semicentral in R. Then we have that for all $B \in D_n(R)$,

$$AB = AB(eI_n) = AB(ea_{ij}) = AB(eI_n)(a_{ij}) = A(eI_n)B(a_{ij}) = ABA$$

by (i) and (iii) of (2).

Conversely if A is right semicentral in E, then e is also right semicentral in R, through the computation that $A(rI_n) = A(rI_n)A$ for all $r \in R$. □

A ring is usually called *Abelian* if every idempotent is central. By Theorem 2.2(2-i,ii), if $(a_{ij}) \in I(D_n(R))$ with $e = a_{ii} \in Z(R)$, then $a_{ij} = a_{ij}e = ea_{ij} = 0$ for all i, j with $i \neq j$. Thus $(a_{ij}) = eI_n$ and so we obtain the following.

Corollary 2.3 ([6, Lemma 2])**.** *Let R be an Abelian ring. Then $D_n(R)$ is an Abelian ring in which every idempotent is of the form eI_n with $e \in I(R)$.*

Theorem 2.2(3) is not valid for $T_n(R)$ $(n \geq 2)$ as we see in the result (1) of the following remark.

Remark 2.4. (1) Let R_0 be any ring and $R = R_0^3$. Consider a matrix $A = \begin{pmatrix} (1,1,0) & (1,0,0) \\ 0 & (0,1,1) \end{pmatrix}$ in $T_2(R)$. Then $A \in I(T_2(R))$ and note that $(1,1,0)$ and $(0,1,1)$ are central idempotents in R. Let $B = \begin{pmatrix} 0 & 0 \\ 0 & (1,0,0) \end{pmatrix} \in T_2(R)$. Then $AB = (1,0,0)E_{12}$ but $AB \neq 0 = ABA$. Hence A is not right semicentral.

(2) For any ring R, $T_n(R)$ contains right (left) semicentral nonzero nonidentity idempotents. Let $e \in I(R)$ (resp., $f \in I(R)$) be right (resp., left) semicentral in R. Consider $A = e(E_{1n} + E_{2n} + \cdots + E_{nn}) \in T_n(R)$. Then $A \in I(T_n(R))'$ and, for all $B = (b_{ij}) \in T_n(R)$, we have

$$\begin{aligned} AB &= eb_{nn}(E_{1n} + E_{2n} + \cdots + E_{nn}) = eb_{nn}e(E_{1n} + E_{2n} + \cdots + E_{nn}) \\ &= eb_{nn}(E_{1n} + E_{2n} + \cdots + E_{nn})e(E_{1n} + E_{2n} + \cdots + E_{nn}) = ABA \end{aligned}$$

since e is right semicentral. Thus A is right semicentral in $T_n(R)$. Next consider $A' = f(E_{11} + E_{12} + \cdots + E_{1n}) \in T_n(R)$. Then A' is similarly shown to be left semicentral in $T_n(R)$.

For any ring R, it is easy to check that $V_n(R)$ is isomorphic to $R[x]/(x^n)$, where $(x^n) = R[x]x^n$.

Theorem 2.5. (1) *Let R be a ring and $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$. If $f(x) \in I(R[x])'$, then we have the following*:

(i) $a_0 \in I(R)'$ *(e say)*, $ea_1e = 0$ *and* $ea_1^2 = a_1^2 e = -ea_2e$;

(ii) $e\left(\sum_{s+t=3}^{k} a_s a_t\right) = \left(\sum_{s+t=3}^{k} a_s a_t\right)e = -ea_k e$ *for all* $3 \leq k \leq m$.

(2) *In (1), $a_0 = e$ is right semicentral in R if and only if $f(x)$ is right semicentral in $R[x]$.*

(3) *Let R be a ring and $e \in I(R)$. If e is right semicentral but not central in R, then there exists $h(x) = \sum_{k=0}^{n} c_k x^k \in I(R[x])$ with $c_0 = e$ such that $h(x)$ is right semicentral in $R[x]$ but $h(x) \notin R$.*

Proof. (1) Note first that $\sum_{i=0}^{m} a_i x^i \in I(R[x])'$ implies

$$\sum_{i=0}^{m} \bar{a}_i \bar{x}^i \in I(R[x]/(x^{m+1}))',$$

whence $(e_{ij}) \in I(V_{m+1}(R))' \subset I(D_{m+1}(R))'$ because $\sum_{i=0}^{m} a_i x^i \in I(R[x])'$ is equivalent to $(e_{ij}) \in I(V_{m+1}(R))' \subset I(D_{m+1}(R))'$, where $e_{ii} = a_0$ and $e_{1,1+s} = a_s$ for all $s = 1, \ldots, m$. So we have the result by Lemma 2.1(3).

(2) Let a_0 be right semicentral in R. Then we have $ea_s = 0$ for all $s = 1, \ldots, m$ and $a_i = a_i e$ for all i, by (1) and Theorem 2.2(2), from which we see that for all $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$,

$$\begin{aligned} f(x)g(x) &= (\sum_{i=0}^{m} a_i e x^i)(\sum_{j=0}^{n} b_j x^j) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} e b_j x^j) \\ &= (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} e b_j e x^j) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} e b_j x^j)e \\ &= (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} e b_j x^j)(\sum_{i=0}^{m} e a_i x^i) \\ &= (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} e b_j e x^j)(\sum_{i=0}^{m} a_i x^i) = f(x)g(x)f(x) \end{aligned}$$

since e is right semicentral. Thus $f(x)$ is right semicentral.

Conversely if $f(x)$ is right semicentral in $R[x]$, then a_0 is also right semicentral in R, through the computation that $f(x)r = f(x)rf(x)$ for all $r \in R$.

(3) Suppose the e is right semicentral in R but not central in R. Then $ea \neq ae$ for some $a \in R$. Note that $ea = eae$ and $(1-e)ae \neq 0$. For any $\ell \geq 1$, set $h_\ell(x) = \sum_{k=0}^{\ell} c_k x^k$ such that $c_0 = e$ and $c_k = (1-e)ae$ for all $1 \leq k \leq \ell$. Then $h_\ell(x) \in I(R[x])'$ and, moreover, $h_\ell(x)$ is right semicentral in $R[x]$ by (2) since e is right semicentral in R. But $h_\ell(x) \notin R$. $\square$

Note. Properties of the case of left semicentral idempotents can be also obtained by symmetry in Theorem 2.2 and Theorem 2.5.

Recall that, for a ring R, the following conditions are equivalent: (1) R is Abelian; (2) $R[x]$ is Abelian; (3) Every idempotent of $R[x]$ is in R ([2, Proposition 2.4]). The following elaborates upon Theorem 2.5(2).

Remark 2.6. Let R be a ring and $f(x) = \sum_{i=0}^{m} a_i x^i \in I(R[x])$. Then $a_0 \in I(R)$ clearly. Here if $a_0 \in Z(R)$, then $f(x) = a_0$ by the proof of [7, Lemma 8]. Thus one may naturally ask whether this result also holds for the case of a_0 being right semicentral but not central. However the answer is not affirmative as follows. Let $R = T_2(\mathbb{Z}_2)$. Then $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is right semicentral but not left semicentral in R as above. Following the construction in the proof of Theorem 2.5(3), let $f(x) = a_0 + (1-a_0)ba_0x \in I(R[x])'$, where $a_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $b = E_{12}$. Then $f(x)$ is right semicentral in $R[x]$ by Theorem 2.5(2), but $f(x) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) x \notin R$.

Let A be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [4], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times_{dor} S$ to denote the Dorroh extension of A by S. Note $1_D = (0, 1)$. In the following we deal with the case of $1 \in A$.

Proposition 2.7. *Let R be a ring of characteristic 2 that is an algebra over a commutative ring S. Then we have the following for $D = R \times_{dor} S$.*

(1) (i) *$I(D) = I(R) \times I(S)$ and $I(R) = \{e + s \mid (e, s) \in I(D)\}$.*

(ii) *$I(D)' = \{(e_1, 0), (e_2, 1) \mid e_1, e_2 \in I(R)\backslash\{0\}\}$ when $I(S)' = \emptyset$; and $I(D)' = \{(e_1, 0), (e_2, 1), (e_3, s) \mid e_1, e_2 \in I(R)\backslash\{0\}, e_3 \in I(R), s \in I(S)'\}$ when $I(S)' \neq \emptyset$.*

(2) *Let $e \in I(R)$. Then e is left semicentral in R if and only if $(e, 1)$ is a right semicentral idempotent in D.*

Proof. (1) (i) Note that $s \in S$ is identified with $s \cdot 1 \in R$, where $1 = 1_R$, the identity of R, and so $R = \{r + s \mid (r, s) \in D\}$. Now, if $(r, s) \in I(D)$, then $(r, s)^2 = (r, s)$ implies $r^2 = r, s^2 = s$ because the characteristic of R is 2, and hence $r \in I(R)$ and $s \in I(S)$, showing that $I(D) \subseteq I(R) \times I(S)$. Conversely, if $(e, t) \in I(R) \times I(S)$, then $(e, t)^2 = (e, t)$ and so $I(R) \times I(S) \subseteq I(D)$. Thus $I(D) = I(R) \times I(S)$.

Next, if $e \in I(R)$, then $(e, 0) \in I(D)$, noting $e = e+0$ with $0 \in S$. Conversely, if $(e, s) \in I(D)$, then $e \in I(R)$ and $s \in I(S)$ as above, from which we see $e + s \in I(R)$.

(ii) is obtained from the fact that $1_D = (0, 1)$.

(2) By (1-i), $(e, 1) \in I(D)$. For $(a, b) \in D$, we have $(e, 1)(a, b) = ((1 + e)a + eb, b)$. Suppose that e is left semicentral. Then $(1 - e)re = 0$ for all $r \in R$, from which we see that

$$\begin{aligned}(e,1)(a,b)(e,1) &= ((1-e)a + eb, b)(e,1)\\ &= ((1-e)ae + eb + eb + (1-e)a + eb, b)\\ &= ((1+e)a + eb, b) = (e,1)(a,b),\end{aligned}$$

using the hypothesis that the characteristic of R is 2.

Suppose that $(e, 1)$ is right semicentral in D. Then

$$(e,1)(r,1)(e,1) = (e,1)(r,1)$$

for all $r \in R$. So we have $ere+re+er+r+e = er+r+e$ since the characteristic of R is 2, entailing and so $ere - re = ere + re = 0$. □

An element u of a ring R is called *regular* if $ur = 0$ and $su = 0$ for some $r, s \in R$ imply $r = s = 0$, i.e., u is not a zero divisor in R.

Proposition 2.8. *Let S be a multiplicatively closed subset of a ring R. Suppose that S consists of central regular elements of R and let $e = u^{-1}a \in I(S^{-1}R)$, where $a \in R$ and $u \in S$. Then we have the following assertions.*

(1) *$a = ea = ae = eae$.*

(2) *e is right semicentral in $S^{-1}R$ if and only if $aru = ara$ for any $r \in R$.*

Proof. (1) From $u^{-1}a = (u^{-1}a)^2 = u^{-2}a^2$, we obtain $a = u^{-1}a^2 = (u^{-1}a)a = a(u^{-1}a) = ea = ae$, and $a = eae$ follows.

(2) Let e be right semicentral in $S^{-1}R$. Then, for any $v^{-1}r \in S^{-1}R$, $e(v^{-1}r) = e(v^{-1}r)e$ and this yields $v^{-1}aru^{-1} = v^{-1}arau^{-2}$, so that $aru = ara$.

Conversely assume the necessity. Then $aru = ara$ implies $av^{-1}r = u^{-1}av^{-1}ra$ for any $v \in S$, from which see that $e(v^{-1}r) = u^{-1}a(v^{-1}r) = u^{-1}a(v^{-1}r)u^{-1}a = e(v^{-1}r)e$. $\square$

In Proposition 2.8, one may ask whether $I(S^{-1}R) = I(R)$. But the answer is negative by the following.

Example 2.9. Let $A = \mathbb{Z}\langle x, y\rangle$, the free algebra generated by the noncommuting indeterminates x, y over $\mathbb{Z}$. Next let I be the ideal of A generated by $yx - xy - 1$ and $R_0 = A/I$. Set $R = T_2(R_0)$ and consider the multiplicatively closed subset $S = \left\{\left(\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right) \in R \,|\, 0 \neq f \in \mathbb{Z}\right\}$ of R. Then S consists of central regular elements in R. Let $a = \left(\begin{smallmatrix} n & g \\ 0 & 0 \end{smallmatrix}\right) \in R$ and $u = \left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right) \in R$, where $n \in \mathbb{Z}\backslash\{0, 1, -1\}$ and $g \in R_0\backslash\mathbb{Z}$. Then

$$u^{-1}a = \begin{pmatrix} n^{-1} & 0 \\ 0 & n^{-1} \end{pmatrix}\begin{pmatrix} n & g \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & n^{-1}g \\ 0 & 0 \end{pmatrix} \in I(S^{-1}R)\backslash I(R),$$

from which we see $I(R) \subsetneq I(S^{-1}R)$.

We end this section by arguing about a property of the difference of two right semicentral idempotents.

Remark 2.10. Let e, f be right semicentral idempotents of a ring R. Then $ef, fe \in I(R)$ and $(e - f)^2 \in I(R)$. For, $(ef)^2 = (efe)f = (ef)f = ef$, $(fe)^2 = (fef)e = (fe)e = fe$, and $(e - f)^4 = (e - ef - fe + f)^2 = e - ef - efe + ef - efe + efef + efe - ef - fe + fef + fefe - fef + fe - fef - fe + f = e - ef - ef + ef - ef + ef + ef - ef - fe + fe + fe - fe + fe - fe - fe + f = e - ef - fe + f = (e - f)^2$. In this situation, if $(e - f)^2 = 0$, then $e = ef$ and $f = fe$ through a simple computation. One may ask whether $e - f \in I(R)$. But the answer is negative. Consider $R = T_2(A)$ over any ring A. It is easily checked that $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are right semicentral idempotents in R. But $e - f = E_{12} \notin I(R)$.

3. Right quasicentral idempotents and right quasi-Abelian rings

In this section we study the structure of rings with idempotents which satisfy a generalized condition of semicentralness.

Lemma 3.1. *A ring is Abelian if and only if every idempotent is right semicentral if and only if every idempotent is left semicentral.*

Proof. Let R be a ring and $e^2 = e, r \in R$. Suppose that every idempotent of R is right semicentral. Then $er = ere$ and $(1 - e)r = (1 - e)r(1 - e)$ for every $e \in I(R)$. Then, for all $r \in R$, $0 = (1 - e)r - (1 - e)r(1 - e) = r - er - r + er + re - ere = re - ere = re - er$; hence R is Abelian. The proof for the case of left semicentral is similar. Other directions are evident. $\square$

Let I be an ideal of a ring R, and suppose that every idempotent of R/I can be lifted to a right semicentral idempotent of R. Then every idempotent of R/I is right semicentral and thus R/I is Abelian by Lemma 3.1. Moreover, motivated by this lemma, we give the following definition.

Definition 3.2. Let R be a ring and $e \in I(R)'$. Then e is called *right quasicentral* (resp., *left quasicentral*) provided that for any $r \in R$, there exists $f = f(e, r) \in I(R)'$ such that $er = erf$ (resp., $re = fre$), and e is called *quasicentral* if e is both right and left quasicentral.

It is evident that for a nonzero nonidentity idempotent e, if e is right (resp., left) semicentral, then e is right (resp., left) quasicentral. But there exist right quasicentral idempotents but not right semicentral.

Remark 3.3. (1) $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is quasicentral in $Mat_2(\mathbb{Z})$ by help of Example 3.5 to follow; but

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq 0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is neither right nor left semicentral.

(2) Let R be a ring and $(a_{ij}) \in I(T_2(R))'$ be such that $a_{11} \in I(R)'$ (resp., $a_{22} \in I(R)'$) is right (resp., left) quasicentral in R. Moreover $(a_{ij})(b_{ij}) = (a_{ij})(b_{ij})(eE_{22})$ (resp., $(b_{ij})(a_{ij}) = (fE_{11})(b_{ij})(a_{ij})$) for all $(b_{ij}) \in \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$, where $e = e(a_{11}, b_{12}), f = f(a_{22}, b_{12}) \in I(R)'$ satisfying $a_{11}b_{12} = a_{11}b_{12}e$ and $b_{12}a_{22} = fb_{12}a_{22}$. For

$$\begin{aligned}(a_{ij})(b_{ij}) &= \begin{pmatrix} 0 & a_{11}b_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{11}b_{12}e \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{11}b_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = (a_{ij})(b_{ij})(eE_{22})\end{aligned}$$

and

$$\begin{aligned}(b_{ij})(a_{ij}) &= \begin{pmatrix} 0 & b_{12}a_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & fb_{12}a_{22} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_{12}a_{22} \\ 0 & 0 \end{pmatrix} = (fE_{11})(b_{ij})(a_{ij}),\end{aligned}$$

noting that eE_{22} and fE_{11} are idempotents. Thus (a_{ij}) is quasicentral in $T_2(R)$. But if a_{11} (resp., a_{22}) is not right (resp., left) semicentral, then (a_{ij}) is not right (resp., left) semicentral.

The following is a main result of our article.

Theorem 3.4. *Every nontrivial idempotent of $Mat_n(P)$ over a principal ideal domain P is quasicentral for $n \geq 2$.*

Proof. Let $R = Mat_n(P)$ and $E \in I(R)'$. Then $EA \notin U(R)$ for any $A \in R$, where $U(R)$ is the group of all units in R. Whence, letting $EA \neq 0$, there exist $X, Y \in U(R)$ such that

$$EA = XDY, \text{ where } D = \sum_{i=1}^{k} \alpha_i E_{ii} \text{ with } k < n \text{ and } 0 \neq \alpha_i \in P,$$

by Smith normal form theorem in linear algebra. Let $F = Y^{-1}CY$, where $C = \sum_{i=1}^{k} E_{ii}$. Then $F \in I(R)'$ and $EAF = XDYF = XDY = EA$. Next, by symmetry, it can be also shown that $AE = F'AE$ for some $F' \in I(R)'$.

Therefore every nontrivial idempotent of $Mat_n(P)$ is quasicentral. □

In the following we provide an actual manner to find nontrivial idempotents in $Mat_2(P)$, based on Theorem 3.4.

Example 3.5. Let P be a principal ideal domain but not a field and $R = Mat_n(P)$. By [8, Lemma 2.3(2)], $I(R)'$ is the union of the following two sets:

$$\left\{E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & f \\ 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 \\ g & 0 \end{pmatrix}, \right.$$
$$\left. E_5 = \begin{pmatrix} 0 & f \\ 0 & 1 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 \\ g & 1 \end{pmatrix} \mid f \neq 0, g \neq 0 \right\}$$

and

$$\left\{E_7 = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \mid a \notin \{0,1\} \text{ and } a(1-a) = bc \right\}.$$

We will show that for any E_i and $A \in R$, there exists $F \in I(R)'$ such that $E_iA = E_iAF$. Let K be the quotient field of P. Since E_iA is not a unit, the row vectors of E_iA are linearly dependent over K.

Case 1. If E_iA is of the form $\left(\begin{smallmatrix} y_1 & 0 \\ y_2 & 0 \end{smallmatrix}\right) \in R$, then $E_iA = E_iAE_1$. If E_iA is of the form $\left(\begin{smallmatrix} 0 & y_3 \\ 0 & y_4 \end{smallmatrix}\right) \in R$, then $E_iA = E_iAE_2$.

So it suffices to consider the following cases.

Case 2. $E_iA = \left(\begin{smallmatrix} s & t \\ 0 & 0 \end{smallmatrix}\right)$ with $s \neq 0$ and $t \neq 0$.

(Subcase 1) If $t \mid s$ ($s = tg$ say), then $E_iA = E_iAE_6$. If $s \mid t$ ($t = sf$ say), then $E_iA = E_iAE_3$.

(Subcase 2) $t \nmid s$ and $s \nmid t$: We claim that there exists $F = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) \in I(R)'$, with $a \notin \{0,1\}$ and $a(1-a) = bc$, such that $E_iA = E_iAF$.

We can let $s^{-1}t = s_1^{-1}t_1$ in K such that $\gcd(s_1, t_1) = 1$, letting $s = s_1k$ and $t = t_1k$ with $s_1, t_1 \notin \{0,1\}$ and $k = \gcd(s,t) > 1$. So we can find $m, n \in P$ such that $s_1n + t_1m = 1$, from which we see

$$t_1^{-1}s_1(t_1n) + s_1^{-1}t_1(s_1m) = s_1n + t_1m = 1.$$

Next set

$$a = s_1n, \; b = s^{-1}ta = s_1^{-1}t_1a = t_1n \in P,$$

and

$$c = t^{-1}s(1-a) = t_1^{-1}s_1(1 - s_1n) = t_1^{-1}s_1t_1m = s_1m \in P.$$

Then $F = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) = \left(\begin{smallmatrix} s_1n & t_1n \\ s_1m & 1-s_1n \end{smallmatrix}\right) \in I(Mat_2(P))'$ with $s_1n \notin \{0,1\}$, and we have

$$E_iA = \begin{pmatrix} s & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} sa+tc & sb+t(1-a) \\ 0 & 0 \end{pmatrix} = E_iAF$$

since $sa+tc = sa+t(t^{-1}s(1-a)) = s$ and $sb+t(1-a) = s(s^{-1}ta)+t(1-a) = t$.

This argument is also applicable to Subcase 1 in which $s_1 = 1$ or $t_1 = 1$. If $s_1 = 1$, then, for any $p \in P$ with $t_1p \notin \{0,1\}$, we have $s_1(1-t_1p)+t_1p = 1$ such that $s_1(1-t_1p) \notin \{0,1\}$. If $t_1 = 1$, then, for any $q \in P$ with $s_1q \notin \{0,1\}$, we have $s_1q + t_1(1-s_1q) = 1$ such that $t_1(1-s_1q) \notin \{0,1\}$. Consequently we can take $s_1n \notin \{0,1\}$ in any case.

Case 3. $E_iA = \left(\begin{smallmatrix} 0 & 0 \\ u & v \end{smallmatrix}\right)$ with $u \neq 0$ and $v \neq 0$.

(Subcase 1) If $u \mid v$ or $v \mid u$, then $E_iA = E_iAE_3$ or $E_iA = E_iAE_6$ similarly.

(Subcase 2) $u \nmid v$ and $v \nmid u$: We apply the argument of (2) to this case. We can let $v^{-1}u = v_1^{-1}u_1$ such that $\gcd(u_1, v_1) = 1$, and so we can find $m, n \in P$ such that $u_1n + v_1m = 1$, from which we see

$$v_1^{-1}u_1(v_1n) + u_1^{-1}v_1(u_1m) = u_1n + v_1m = 1.$$

Next set

$$a = u_1n,\ b = u^{-1}va = u_1^{-1}v_1a = v_1n \in P,$$

and

$$c = v^{-1}u(1-a) = v_1^{-1}u_1(1-u_1n) = v_1^{-1}u_1v_1m = u_1m \in P.$$

Then $F = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_1n & v_1n \\ u_1m & 1-u_1n \end{smallmatrix}\right) \in I(Mat_2(P))'$ with $u_1n \notin \{0,1\}$, and we have

$$E_iA = \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ua+vc & ub+v(1-a) \end{pmatrix} = E_iAF$$

since $ua+vc = ua+v(v^{-1}u(1-a)) = u$ and $ub+v(1-a) = u(u^{-1}va)+v(1-a) = v$.

This argument is also applicable to Subcase 1 in which $u_1 = 1$ or $v_1 = 1$, by a similar manner to one of Case 2.

Case 4. $E_iA = \left(\begin{smallmatrix} \alpha & \beta \\ w\alpha & w\beta \end{smallmatrix}\right)$ with $\alpha, \beta \in P\backslash\{0\}$ and $0 \neq w \in K$. We can find $F = \left(\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right) \in I(R)'$ such that $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right)$, by the argument of Case 2, entailing $\alpha = \alpha p_1 + \beta p_3$ and $\beta = \alpha p_2 + \beta p_4$. From this, we obtain

$$\begin{aligned} E_iAF = \begin{pmatrix} \alpha & \beta \\ w\alpha & w\beta \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} &= \begin{pmatrix} \alpha p_1 + \beta p_3 & \alpha p_2 + \beta p_4 \\ w(\alpha p_1 + \beta p_3) & w(\alpha p_2 + \beta p_4) \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ w\alpha & w\beta \end{pmatrix} = E_iA. \end{aligned}$$

From the results above, we now conclude that every nontrivial idempotent of $Mat_2(P)$ is right quasicentral. The case of left quasicentral idempotents can be also obtained by symmetrically arguments.

Next we argue about a property of right (left) semicentral idempotents of semiprime rings, by which we can find quasicentral idempotents but not right (left) semicentral.

Remark 3.6. In a semiprime ring, every right (or left) semicentral idempotent is easily shown to be central. By help of this fact, we can find many quasicentral idempotents which are not semicentral. Consider $R = Mat_n(P)$ $(n \geq 2)$ over any principal ideal domain P. Then every noncentral idempotent e of R is quasicentral by Theorem 3.4. Assuming that e is right (or left) semicentral, we have that e is central since R is (semi)prime, a contradiction. Therefore e is neither right nor left semicentral.

Following [2], a ring R is called *right* (resp., *left*) *quasi-Abelian* provided that either $I(R)'$ is empty, or else for any $(e, a) \in I(R)' \times R$ (resp., $(a, e) \in R \times I(R)'$) there exists $(b, f) \in R \times I(R)'$ (resp., $(f, b) \in I(R)' \times R$) such that $ea = bf$ (resp., $ae = fb$). R is called *quasi-Abelian* if it is both right and left quasi-Abelian. Note that a non-Abelian ring R (i.e., $\emptyset \neq I(R)' \nsubseteq Z(R)$) is right (resp., left) quasi-Abelian if and only if every $e \in I(R)'$ is right (resp., left) quasicentral, by the argument above. Hence, the quasicentralness is not left-right symmetric by [2, Example 1.5]. Right (left) quasi-Abelian rings are directly finite by [2, Theorem 1.9(1)] and there exists a domain R over which $Mat_2(R)$ is not directly finite by [10, Theorem 1.0]. But if R is a principal ideal domain, then $Mat_n(R)$ $(n \geq 2)$ is quasi-Abelian by Theorem 3.4, providing of quasi-Abelian ring which is not Abelian. Furthermore, Theorem 3.4 is compared with the fact that for a domain R, $T_n(R)$ is right quasi-Abelian for all $n \geq 2$ if and only if R is a division ring ([2, Theorem 2.1]). Thus $T_2(\mathbb{Z})$ is not right quasi-Abelian, but $Mat_n(\mathbb{Z})$ $(n \geq 2)$ is quasi-Abelian by Theorem 3.4, from which we see that the class of right quasi-Abelian rings is not closed under subrings.

Finally, we consider the quasi-Abelian property in other cases that occur frequently in noncommutative ring theory. Note that a ring R is Abelian if and only if $D_n(R)$ is Abelian for any $n \geq 2$ if and only if $D_n(R)$ is Abelian for some $n \geq 2$, by [6, Lemma 2]. We consider a condition under which this result can be extended to the case of right quasi-Abelian.

Proposition 3.7. *Let R be a ring such that $I(R)'$ is orthogonal. Then $D_n(R)$ is right quasi-Abelian for some $n \geq 2$ if and only if R is Abelian.*

Proof. Let $D_n(R)$ be right quasi-Abelian for some $n \geq 2$. Assume on the contrary that there exist $e \in I(R)'$ and $r \in R$ such that $er(1-e) \neq 0$, a say. Let $E = (e + a)I_n$. Then $E \in I(D_n(R))'$. Let $A = E_{1n} \in D_n(R)$. Since $D_n(R)$ is right quasi-Abelian, $EA = EAF$ for some $F = (f_{ij}) \in I(D_n(R))'$. Let $f = f_{ii}$. Then clearly $f \in I(R)'$. From $EA = EAF$, we get $e + er(1-e) = ef + er(1-e)f$; hence $f \neq e$ and $f \neq 1 - e$. Since $I(R)'$ is orthogonal, $ef = 0$ and $(1 - e)f = 0$, entailing $e + a = 0$, a contradiction. Thus R is Abelian. The converse is clear by [6, Lemma 2]. $\square$

We next observe some methods of constructing (right quasi-)Abelian rings through the factorization. We start with the fact that the class of right quasi-Abelian rings is not closed under factor rings. A ring R is called *regular* if for any $a \in R$ there exists $b \in R$ such that $a = aba$ (see [5]). Directly finite regular rings are quasi-Abelian by [2, Theorem 1.9(2)], but there exists a directly finite regular ring R with a right and left primitive ideal P such that R/P is not directly finite (hence not right quasi-Abelian) by [5, Example 5.11].

An ideal I of a ring R is said to be *idempotent-lifting* if idempotents in R/I can be lifted to R. The Jacobson radical of a ring R is denoted by $J(R)$.

Proposition 3.8. (1) *Let R be a right quasi-Abelian ring and I be an ideal of R such that $I \subseteq J(R)$. If I is idempotent-lifting, then R/I is right quasi-Abelian.*

(2) *Let R be a right quasi-Abelian ring. If I is a nil ideal of R, then R/I is right quasi-Abelian.*

(3) *Let R be a ring and $n \geq 2$. If $D_n(R)$ ($n \geq 2$) is right quasi-Abelian, then so is R.*

(4) *Let R be a commutative ring and M be a maximal ideal of R. Then $Mat_n(R/M)$ is quasi-Abelian for any $n \geq 2$.*

Proof. (1) This is clear from [2, Theorem 2.2(1)]. (2) is clear from [9, Proposition 3.6.1] and (1). (3) is proved by (2) and the fact that the factor ring $D_n(R)/I$, by the nil ideal $I = \{(a_{ij}) \in D_n(R) \mid a_{ii} = 0\}$ of $D_n(R)$, is isomorphic to R. (4) is shown by [2, Theorem 1.9(4)] since R/M is a field. □

Note that the right quasi-Abelian property does not go up to polynomial rings by [2, Proposition 2.5(1)]. Following [3], a ring R is called *NR* if $N(R)$ forms a subring. A ring R is NR if and only if $N(R)$ is additively closed ([11, Theorem 2.1]).

Proposition 3.9. (1) *Let R be a commutative ring and M be a maximal ideal of R. Then $Mat_n(R/M)[x]$ is quasi-Abelian for $n \geq 2$.*

(2) *Let R be an NR ring and I be an ideal of R with $N^*(R) \subseteq I$. If I is idempotent-lifting, then $R[x]/I[x]$ is an Abelian ring.*

Proof. (1) Note first that $(R/M)[x]$ is a principal ideal domain, hence

$$Mat_n(R/M)[x] (\cong Mat_n((R/M)[x]))$$

is quasi-Abelian for $n \geq 2$ by Theorem 3.4.

(2) Since R is NR, $R/N^*(R)$ is an Abelian ring by [11, Proposition 3.1]. Write $\bar{R} = R/I$. Let $\bar{f} \in I(\bar{R})$. Since I is idempotent-lifting, $\bar{f} = \bar{e}$ for some $e \in I(R)$. Since $R/N^*(R)$ is Abelian, we have $er - re \in N^*(R)$ for all $r \in R$, entailing $er - re \in I$ by hypothesis. Thus $\bar{f}\bar{r} = \bar{e}\bar{r} = \bar{r}\bar{e} = \bar{r}\bar{f}$ in $\bar{R}$, so that $\bar{R}$ is Abelian. Therefore $R[x]/I[x]$ ($\cong (R/I)[x]$) is Abelian by [7, Lemma 8]. □

References

[1] G. F. Birkenmeier, *Idempotents and completely semiprime ideals*, Comm. Algebra **11** (1983), no. 6, 567–580. https://doi.org/10.1080/00927878308822865

[2] E.-K. Cho, T. K. Kwak, Y. Lee, Z. Piao, and Y. S. Seo, *A structure of noncentral idempotents*, Bull. Korean Math. Soc. **55** (2018), no. 1, 25–40. https://doi.org/10.4134/BKMS.b160801

[3] Y. Chun, Y. C. Jeon, S. Kang, K. N. Lee, and Y. Lee, *A concept unifying the Armendariz and NI conditions*, Bull. Korean Math. Soc. **48** (2011), no. 1, 115–127. https://doi.org/10.4134/BKMS.2011.48.1.115

[4] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), no. 2, 85–88. https://doi.org/10.1090/S0002-9904-1932-05333-2

[5] K. R. Goodearl, *Von Neumann Regular Rings*, Monographs and Studies in Mathematics, 4, Pitman, Boston, MA, 1979.

[6] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52. https://doi.org/10.1016/S0022-4049(01)00149-9

[7] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488. https://doi.org/10.1006/jabr.1999.8017

[8] T. K. Kwak, S. I. Lee, and Y. Lee, *Quasi-normality of idempotents on nilpotents*, Hacet. J. Math. Stat. **48** (2019), no. 6, 1744–1760. https://doi.org/10.15672/HJMS.2018.634

[9] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Co., Waltham, MA, 1966.

[10] J. C. Shepherdson, *Inverses and zero divisors in matrix rings*, Proc. London Math. Soc. (3) **1** (1951), 71–85. https://doi.org/10.1112/plms/s3-1.1.71

[11] J. Šter, *Rings in which nilpotents form a subring*, Carpathian J. Math. **32** (2016), no. 2, 251–258. https://doi.org/10.48550/arXiv.1510.07523

Juan Huang
Department of Mathematics
Pusan National University
Busan 46241, Korea
Email address: hj1117@163.com

Tai Keun Kwak
Department of Data Science
Daejin University
Pocheon 11159, Korea
Email address: tkkwak@daejin.ac.kr

Yang Lee
Department of Mathematics
Yanbian University
Yanji 133002, P. R. China
and
Institute for Applied Mathematics and Optics
Hanbat National University
Daejeon 34158, Korea
Email address: ylee@pusan.ac.kr

Zhelin Piao
Department of Mathematics
Yanbian University
Yanji 133002, P. R. China
Email address: zlpiao@ybu.edu.cn