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DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS ON BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

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Abstract. We characterize the boundedness and compactness of differences of weighted composition operators acting from weighted Bergman spaces A^p_ω to Lebesgue spaces $L^q(d\mu)$ for all $0 < p, q < \infty$, where ω is a radial weight on the unit disk admitting a two-sided doubling condition.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. Let ω be a radial weight on $\mathbb{D}$, that is, a nonnegative integrable function on $\mathbb{D}$ satisfying $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$, the Lebesgue space L^p_ω consists of measurable functions f satisfying

$$\|f\|^p_{L^p_\omega} := \int_{\mathbb{D}} |f|^p \omega dA < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The weighted Bergman space A^p_ω is defined by

$$A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$$

with inherited (quasi-)norm. As usual, the standard weighted Bergman spaces are denoted by A^p_α, i.e., the Bergman spaces induced by the weights $(\alpha+1)(1-|\cdot|^2)^\alpha$, $\alpha > -1$. For a positive Borel measure μ on $\mathbb{D}$, we use $L^p(d\mu)$ to denote the Lebesgue spaces induced by μ.

For a radial weight ω, write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)ds$ for all $z \in \mathbb{D}$. In this paper we always assume $\widehat{\omega}(z) > 0$, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$ for each $0 < p < \infty$. A weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) \geq 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $0 \leq r < 1$, then we

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write $\omega \in \check{\mathcal{D}}$. Weights ω belonging to $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ are called doubling. These classes of weights emerge from fundamental questions in operator theory; see [19]. Recently, function theory and operator theory on Bergman spaces induced by doubling weights have drawn many attentions; see [15–18, 20].

Let $\mathcal{S}(\mathbb{D})$ be the set of all analytic self-maps of $\mathbb{D}$. Given $\varphi \in \mathcal{S}(\mathbb{D})$, the composition operator C_φ with symbol φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in \mathcal{H}(\mathbb{D}).$$

A weighted composition operator is a product of a composition operator and a multiplication operator. More explicitly, for $\varphi \in \mathcal{S}(\mathbb{D})$ and a Borel function u on $\mathbb{D}$, the weighted composition operator uC_φ is defined by

$$uC_\varphi(f) = u(f \circ \varphi), \quad f \in \mathcal{H}(\mathbb{D}).$$

The properties of (weighted) composition operators on various settings have been studied extensively during the past few decades; see for example [2, 5–7, 9, 10, 24, 25].

Motivated by the study of path-connected components in the space of composition operators (see [26]), study on differences of (weighted) composition operators has been of growing interest for the last three decades. Moorhouse [14] characterized the compactness of $C_\varphi - C_\psi$ on the standard weighted Bergman space A^2_α by some Julia–Carathéodory type conditions. Saukko [22, 23] generalized Moorhouse's results by characterizing the compactness of $C_\varphi - C_\psi$ acting from A^p_α to A^q_β for $0 < p, q < \infty$. Acharyya and Wu [1] first obtained a compactness characterization for the weighted composition operator differences $uC_\varphi - vC_\psi : A^p_\alpha \to A^q_\beta$ in the case $p \leq q$, where the weights u, v are analytic functions satisfying some growth conditions. Recently, Liu, Rättyä and Wu [12, 13] generalized the results in [1, 23] to the doubling weight setting. On the other hand, with the lack of the change-of-variable formula for the difference of composition operators, Koo and Wang [11] introduced the notion of joint Carleson measure to obtain a Carleson measure characterization for the differences $C_\varphi - C_\psi$ acting on standard weighted Bergman spaces. Very recently, using the joint Carleson measure again, Choe, Choi, Koo and Yang [3, 4] completely characterized the boundedness and compactness of differences $uC_\varphi - vC_\psi$ acting from standard weighted Bergman spaces into the corresponding Lebesgue spaces, where the weights u, v are allowed to be non-analytic and unbounded.

In this paper, we follow the line of research to study the boundedness and compactness of differences of weighted composition operators acting on Bergman spaces induced by doubling weights via the joint Carleson measure. We generalize the main results of [3, 4] to the doubling weight setting.

To state our results, we introduce some notation. We reserve symbol functions $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and weights u, v to be considered throughout the paper. We put

$$\rho(z) = d(\varphi(z), \psi(z)), \quad z \in \mathbb{D},$$

where $d(\cdot,\cdot)$ is the pseudohyperbolic distance on $\mathbb{D}$; see Section 2.1. Given a positive Borel measure ν on $\mathbb{D}$, the pullback measure $\nu\circ\varphi^{-1}$ is defined by $(\nu\circ\varphi^{-1})(E)=\nu[\varphi^{-1}(E)]$ for all Borel sets $E\subset\mathbb{D}$. Suppose now that μ is a positive Borel measure on $\mathbb{D}$, $0<q,\beta<\infty$ and $r\in(0,1)$. The pullback measures $\lambda=\lambda^{\mu,q}_{\varphi,u;\psi,v}$, $\sigma^\beta=\sigma^{\mu,q;\beta}_{\varphi,u;\psi,v}$ and $\sigma_r=\sigma^{\mu,q}_{\varphi,u;\psi,v;r}$ are defined by

$$\lambda=(|\rho u|^q d\mu)\circ\varphi^{-1}+(|\rho v|^q d\mu)\circ\psi^{-1},$$

$$\sigma^\beta=[(1-\rho)^\beta|u-v|^q d\mu]\circ\varphi^{-1}+[(1-\rho)^\beta|u-v|^q d\mu]\circ\psi^{-1},$$

and

$$\sigma_r=(\chi_{G_r}|u-v|^q d\mu)\circ\varphi^{-1}+(\chi_{G_r}|u-v|^q d\mu)\circ\psi^{-1},$$

respectively, where χ_{G_r} is the characteristic function of the set

$$G_r:=\{z\in\mathbb{D}:\rho(z)<r\}.$$

Recall that for $0<p,q<\infty$ and $\omega\in\mathcal{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a q-Carleson measure for A^p_ω if the embedding operator $I_d:A^p_\omega\to L^q(d\mu)$ is bounded; μ is called a vanishing q-Carleson measure for A^p_ω if $I_d:A^p_\omega\to L^q(d\mu)$ is compact. Our main results read as follows.

Theorem 1.1. *Let $\omega\in\mathcal{D}$, $0<p\le q<\infty$, $0<r<1$, and $\beta>0$ be large enough. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi,\psi\in\mathcal{S}(\mathbb{D})$ and $u,v\in L^q(d\mu)$. Then the following conditions are equivalent*:

(a) *$uC_\varphi-vC_\psi:A^p_\omega\to L^q(d\mu)$ is bounded (resp. compact)*;
(b) *$\lambda+\sigma^\beta$ is a (resp. vanishing) q-Carleson measure for A^p_ω*;
(c) *$\lambda+\sigma_r$ is a (resp. vanishing) q-Carleson measure for A^p_ω.*

Theorem 1.2. *Let $\omega\in\mathcal{D}$, $0<q<p<\infty$, $0<r<1$, and $\beta>0$ be large enough. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi,\psi\in\mathcal{S}(\mathbb{D})$ and $u,v\in L^q(d\mu)$. Then the following conditions are equivalent*:

(a) *$uC_\varphi-vC_\psi:A^p_\omega\to L^q(d\mu)$ is bounded*;
(b) *$uC_\varphi-vC_\psi:A^p_\omega\to L^q(d\mu)$ is compact*;
(c) *$\lambda+\sigma^\beta$ is a q-Carleson measure for A^p_ω*;
(d) *$\lambda+\sigma_r$ is a q-Carleson measure for A^p_ω.*

After some preliminary results are given in Section 2, we give some criterions for q-Carleson measures in Section 3. In particular, we prove some new characterizations of q-Carleson measures for A^p_ω in the case $p>q$. In Section 4, we give the proofs of our main results.

Throughout the paper, the notation $A\lesssim B$ or $B\gtrsim A$ means that $A\le CB$ for some inessential constant $C>0$. If $A\lesssim B\lesssim A$, then we write $A\asymp B$.

2. Preliminaries

In this section, we introduce some well-known results that will be used throughout the paper.

2.1. Pseudohyperbolic distance

We first introduce some basic facts of the pseudohyperbolic distance on the unit disk $\mathbb{D}$. Recall that the pseudohyperbolic distance between $z, w \in \mathbb{D}$ is defined by

$$d(z,w) = \left| \frac{z-w}{1-z\overline{w}} \right|.$$

We use $\Delta(a,\delta)$ to denote the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius $\delta \in (0,1)$, which is actually a Euclidean disk with center $\frac{1-\delta^2}{1-\delta^2|a|^2} z$ and radius $\frac{1-|a|^2}{1-\delta^2|a|^2}\delta$. It is well-known that

$$1-|z|^2 \asymp 1-|w|^2 \asymp |1-z\overline{w}|$$

and

$$|1-\overline{a}z| \asymp |1-\overline{a}w|$$

for any $a \in \mathbb{D}$ and $z, w \in \mathbb{D}$ with $d(z,w) < \delta$. The implicit constants depend only on δ. We will use these estimates frequently.

Let $\delta \in (0,1)$ and $\{a_n\}$ be a sequence of distinct points in $\mathbb{D}$. We say that $\{a_n\}$ is δ-separated if $d(a_n, a_m) \geq \delta$ for all n and m with $n \neq m$. We say that $\{a_n\}$ is a δ-lattice if it is $\delta/2$-separated and

$$\mathbb{D} = \bigcup_{n \geq 1} \Delta(a_n, \delta).$$

The existence of δ-lattice for any $\delta \in (0,1)$ follows from [27, Lemma 4.8]. Furthermore, by [27, Lemma 4.7], for any δ-lattice $\{a_n\}$ and $R \in (0,1)$, there exists a positive constant $N = N(\delta, R)$ such that each point $z \in \mathbb{D}$ belongs to at most N of the pseudohyperbolic disks $\Delta(a_n, \delta)$.

2.2. Estimates on weights and test functions

We now introduce some estimates on weights. It is known that if $\omega \in \mathcal{D}$, then there exist $0 < \alpha = \alpha(\omega) \leq \beta = \beta(\omega) < \infty$ and $C = C(\omega) \geq 1$ such that

$$\frac{1}{C}\left(\frac{1-r}{1-t}\right)^{\alpha} \leq \frac{\widehat{\omega}(r)}{\widehat{\omega}(t)} \leq C\left(\frac{1-r}{1-t}\right)^{\beta}, \quad 0 \leq r \leq t < 1. \tag{2.1}$$

In fact, the right hand inequality holds if and only if $\omega \in \widehat{\mathcal{D}}$ (see [15, Lemma 2.1]) and the left hand inequality holds if and only if $\omega \in \check{\mathcal{D}}$ (see the proof of [19, Theorem 15]). Based on (2.1), one can deduce that (see [12]) for any $r \in (0,1)$,

$$\omega(\Delta(z,r)) \lesssim \widehat{\omega}(z)(1-|z|), \quad z \in \mathbb{D}, \tag{2.2}$$

and there exists $r_0 = r_0(\omega) \in (0,1)$ such that

$$\omega(\Delta(z,r_0)) \gtrsim \widehat{\omega}(z)(1-|z|)|z|, \quad z \in \mathbb{D}. \tag{2.3}$$

We also need the twisted weight $\widetilde{\omega}$, defined by

$$\widetilde{\omega}(z) = \frac{\widehat{\omega}(z)}{1-|z|}, \quad z \in \mathbb{D}.$$

By [20, Proposition 5], we know that if $\omega \in \mathcal{D}$, then

$$\|f\|_{A^p_\omega} \asymp \|f\|_{A^p_{\widetilde{\omega}}}, \quad f \in \mathcal{H}(\mathbb{D}). \tag{2.4}$$

We will use these estimates repeatedly in the sequel.

To characterize the boundedness and compactness of $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$, we will use the functions $K_a(z) = 1/(1-\overline{a}z)$, $a, z \in \mathbb{D}$ to be the test functions in the case $p \le q$. It is known (see [15, Lemma 2.1]) that if $\omega \in \widehat{\mathcal{D}}$, then there exists $\gamma(\omega) \ge 0$ such that for any $\gamma > \gamma(\omega)$,

$$\int_{\mathbb{D}} \frac{\omega(z)dA(z)}{|1-\overline{a}z|^{\gamma+1}} \asymp \frac{\widehat{\omega}(a)}{(1-|a|)^\gamma}.$$

Based on the above estimate, we know that for any $t > (\gamma(\omega)+1)/p$,

$$\|K_a^t\|_{A^p_\omega} \asymp \frac{\widehat{\omega}(a)^{\frac{1}{p}}}{(1-|a|)^{t-\frac{1}{p}}}. \tag{2.5}$$

In the case $p > q$, our test functions are given by the following proposition, which can be found in [21, Theorem 1].

Proposition 2.1. *Let $\omega \in \mathcal{D}$, $0 < p < \infty$ and $\{a_n\}$ be a separated sequence in $\mathbb{D}$. If*

$$t > 1 + \frac{\beta(\omega)+\gamma(\omega)+1}{p}$$

and $c = \{c_n\} \in l^p$, then the function F defined by

$$F(z) = \sum_{n=1}^\infty c_n \frac{(1-|a_n|)^{t-\frac{1}{p}}\widehat{\omega}(a_n)^{-\frac{1}{p}}}{(1-\overline{a_n}z)^t}$$

belongs to $\mathcal{H}(\mathbb{D})$, and there exists a constant $C = C(t,\omega,p) > 0$ such that

$$\|F\|_{A^p_\omega} \le C\|c\|_{l^p}.$$

3. Carleson measures

In this section, we give some characterizations of the q-Carleson measures for A^p_ω, which will be needed in the sequel. Given $\omega \in \mathcal{D}$, $0 < \theta < \infty$, $r \in (0,1)$, and a positive Borel measure μ on $\mathbb{D}$, the weighted mean function $\widehat{\mu}_{\omega,r,\theta}$ is defined by

$$\widehat{\mu}_{\omega,r,\theta}(z) = \frac{\mu(\Delta(z,r))}{\omega(\Delta(z,r))^\theta}, \quad z \in \mathbb{D}.$$

We write $\widehat{\mu}_{\omega,r} := \widehat{\mu}_{\omega,r,1}$ for simplicity.

The following theorem gives the characterization of q-Carleson measures for A^p_ω in the case $p \le q$, which can be found in [12, Theorem 2].

Theorem 3.1. *Let $0 < p \le q < \infty$, $\omega \in \mathcal{D}$, and μ be a positive Borel measure on $\mathbb{D}$. Then the following assertions hold*:

(1) *μ is a q-Carleson measure for A^p_ω if and only if for some (or any) $r \in [r_0(\omega), 1)$, $\widehat{\mu}_{\omega,r,\frac{q}{p}} \in L^\infty(\mathbb{D})$;*

(2) *μ is a vanishing q-Carleson measure for A^p_ω if and only if for some (or any) $r \in [r_0(\omega), 1)$, $\widehat{\mu}_{\omega,r,\frac{q}{p}}(z) \to 0$ as $|z| \to 1$.*

In the case $p > q$, we have the following theorem characterizing the q-Carleson measures for A^p_ω via the weighted mean functions $\widehat{\mu}_{\omega,r,\theta}$, which generalizes [13, Theorem 2].

Theorem 3.2. *Let $0 < q < p < \infty$, $\omega \in \mathcal{D}$, and μ be a positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent*:

(a) *μ is a q-Carleson measure for A^p_ω;*

(b) *μ is a vanishing q-Carleson measure for A^p_ω;*

(c) *For some (or any) $r \in [r_0(\omega), 1)$, the function $\widehat{\mu}_{\omega,r}$ belongs to $L^{\frac{p}{p-q}}_{\widetilde{\omega}}$;*

(d) *For some (or any) $r \in [r_0(\omega), 1)$ and any δ-lattice $\{a_n\}$,*

$$\left\{\widehat{\mu}_{\omega,r,\frac{q}{p}}(a_n)\right\} \in l^{\frac{p}{p-q}};$$

(e) *For some (or any) $r \in [r_0(\omega), 1)$ and any δ-lattice $\{a_n\}$,*

$$\left\{\frac{\mu(\Delta(a_n, r))}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}}\right\} \in l^{\frac{p}{p-q}}.$$

Proof. The equivalence of (d) and (e) follows easily from the inequalities (2.2) and (2.3). We finish the proof by proving the sequence of implications

$$\text{(a)} \implies \text{(e)} \implies \text{(c)} \implies \text{(b)}.$$

Assume first that (a) holds. Fix $r \in [r_0(\omega), 1)$ and a δ-lattice $\{a_n\}$. For any $c = \{c_n\} \in l^p$, define

$$F_\tau(z) = \sum_{n=1}^{\infty} c_n \phi_n(\tau) f_n(z), \quad \tau \in [0, 1], \quad z \in \mathbb{D},$$

where $\{\phi_n\}$ is the sequence of Rademacher functions (see [8, Appendix A]), and

$$f_n(z) = \frac{(1-|a_n|)^{t-\frac{1}{p}}}{\widehat{\omega}(a_n)^{1/p}(1-\overline{a_n}z)^t}$$

for some sufficiently large $t > 0$. Then by Proposition 2.1, $F_\tau \in A^p_\omega$ with

$$\|F_\tau\|_{A^p_\omega} \lesssim \|c\|_{l^p}$$

for any $\tau \in [0, 1]$. Since the embedding operator $I_d : A^p_\omega \to L^q(d\mu)$ is bounded, we have for any $\tau \in [0, 1]$,

$$\int_{\mathbb{D}} \left|\sum_{n=1}^{\infty} c_n \phi_n(\tau) f_n(z)\right|^q d\mu(z) \le \|I_d\|^q \|F_\tau\|^q_{A^p_\omega} \lesssim \|I_d\|^q \|c\|^q_{l^p}.$$

Integrating with respect to τ on $[0,1]$ and using Fubini's theorem and Khinchine's inequality (see [8, Appendix A]), we arrive at

$$\int_{\mathbb{D}} \left(\sum_{n=1}^{\infty} |c_n|^2 |f_n(z)|^2 \right)^{q/2} d\mu(z) \lesssim \|I_d\|^q \|c\|_{l^p}^q. \tag{3.1}$$

Since each $z \in \mathbb{D}$ belongs to at most $N = N(\delta, r)$ of the sets $\Delta(a_n, r)$, we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty} |c_n|^2 |f_n(z)|^2 \right)^{q/2} &\gtrsim \sum_{n=1}^{\infty} |c_n|^q |f_n(z)|^q \chi_{\Delta(a_n,r)}(z) \\ &\asymp \sum_{n=1}^{\infty} |c_n|^q (\widehat{\omega}(a_n)(1-|a_n|))^{-q/p} \chi_{\Delta(a_n,r)}(z). \end{aligned}$$

Inserting this into (3.1) yields

$$\sum_{n=1}^{\infty} |c_n|^q \frac{\mu(\Delta(a_n,r))}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}} \lesssim \|I_d\|^q \cdot \|c\|_{l^p}^q.$$

Combining the above inequality with the duality $(l^{p/q})^* = l^{p/(p-q)}$, we obtain that (e) holds.

Assume now (e) holds. Fix $r \in [r_0, 1)$ and $\delta \in (0, 1-r)$, and choose $K \geq 1$ such that $\left(\mathbb{D} \setminus \frac{1}{2}\mathbb{D}\right) \subset \cup_{n \geq K} \Delta(a_n, \delta)$ and $\epsilon_0 \mathbb{D} \cap (\cup_{n \geq K} \Delta(a_n, \delta)) = \emptyset$ for some $\epsilon_0 \in (0, 1/2)$. Then by (2.1) and (2.3), we have

$$\begin{aligned} &\int_{\mathbb{D} \setminus \frac{1}{2}\mathbb{D}} \widehat{\mu}_{\omega,r}(z)^{\frac{p}{p-q}} \widetilde{\omega}(z) dA(z) \\ &\leq \sum_{n \geq K} \int_{\Delta(a_n,\delta)} \widehat{\mu}_{\omega,r}(z)^{\frac{p}{p-q}} \frac{\widehat{\omega}(z)}{1-|z|} dA(z) \\ &\lesssim \sum_{n \geq K} \frac{\widehat{\omega}(a_n)}{1-|a_n|} \int_{\Delta(a_n,\delta)} \left(\frac{\mu(\Delta(z,r))}{\widehat{\omega}(z)(1-|z|)} \right)^{\frac{p}{p-q}} dA(z) \\ &\lesssim \sum_{n \geq K} \left(\frac{\mu(\Delta(a_n, \delta + r))}{(\widehat{\omega}(a_n)(1-|a_n|)^{q/p})} \right)^{\frac{p}{p-q}} < \infty. \end{aligned}$$

On the other hand, since $\omega(\Delta(z,r)) > 0$ for any $z \in \mathbb{D}$, the absolute continuity of integration implies that the function $\widehat{\mu}_{\omega,r}$ is continuous on $\mathbb{D}$. Consequently,

$$\int_{\frac{1}{2}\mathbb{D}} \widehat{\mu}_{\omega,r}(z)^{\frac{p}{p-q}} \widetilde{\omega}(z) dA(z) \lesssim \int_{\frac{1}{2}\mathbb{D}} \frac{\widehat{\omega}(z)}{1-|z|} dA(z) \leq 2\widehat{\omega}(0).$$

Therefore, the condition (c) holds.

We finally assume that (c) holds. For any $f \in \mathcal{H}(\mathbb{D})$, by the subharmonic property, Fubini's theorem and (2.2), we have

$$\|f\|_{L^q(d\mu)}^q = \int_{\mathbb{D}} |f(z)|^q d\mu(z) \tag{3.2}$$

$$\lesssim \int_{\mathbb{D}} \frac{1}{(1-|z|)^2} \int_{\Delta(z,r)} |f(\xi)|^q dA(\xi) d\mu(z)$$
$$\asymp \int_{\mathbb{D}} |f(\xi)|^q \frac{\mu(\Delta(\xi,r))}{(1-|\xi|)^2} dA(\xi)$$
$$\lesssim \int_{\mathbb{D}} |f(\xi)|^q \widehat{\mu}_{\omega,r}(\xi)\widetilde{\omega}(\xi) dA(\xi).$$

Suppose $\{g_n\}$ is a bounded sequence in A^p_ω converging to 0 uniformly on compact subsets of $\mathbb{D}$. It is enough to show $\|g_n\|_{L^q(d\mu)} \to 0$. Since the function $\widehat{\mu}_{\omega,r}$ belongs to $L^{p/(p-q)}_{\widetilde{\omega}}$, for any $\epsilon > 0$, there exists $\eta \in (0,1)$ such that

$$\int_{\mathbb{D}\setminus\eta\mathbb{D}} \widehat{\mu}_{\omega,r}(z)^{\frac{p}{p-q}} \widetilde{\omega}(z) dA(z) < \epsilon^{\frac{p}{p-q}}.$$

Therefore, using (3.2), Hölder's inequality and (2.4), we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} \|g_n\|^q_{L^q(d\mu)} &\lesssim \limsup_{n\to\infty} \left(\int_{\eta\mathbb{D}} + \int_{\mathbb{D}\setminus\eta\mathbb{D}} \right) |g_n(z)|^q \widehat{\mu}_{\omega,r}(z)\widetilde{\omega}(z) dA(z) \\
&\le \limsup_{n\to\infty} \left(\int_{\eta\mathbb{D}} |g_n|^q \widetilde{\omega} dA \right)^{\frac{q}{p}} \left(\int_{\mathbb{D}} \widehat{\mu}_{\omega,r}^{\frac{p}{p-q}} \widetilde{\omega} dA \right)^{\frac{p-q}{p}} \\
&\quad + \sup_{n\ge 1} \|g_n\|^q_{A^p_{\widetilde{\omega}}} \left(\int_{\mathbb{D}\setminus\eta\mathbb{D}} \widehat{\mu}_{\omega,r}^{\frac{p}{p-q}} \widetilde{\omega} dA \right)^{\frac{p-q}{p}} \\
&\lesssim \epsilon.
\end{aligned}$$

The arbitrariness of ϵ gives (b), which completes the proof. □

4. Differences of weighted composition operators

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2. In fact, we will establish more detailed versions of these two theorems. To this end, we decompose the measure λ, σ^β and σ_r (associated with $\varphi, \psi, u, v, q, \mu$) defined in the Introduction into two parts as follows:

$$\begin{aligned}
\lambda &= \lambda_{\varphi,u} + \lambda_{\psi,v}, \\
\sigma^\beta &= \sigma^\beta_\varphi + \sigma^\beta_\psi, \\
\sigma_r &= \sigma_{\varphi,r} + \sigma_{\psi,r},
\end{aligned}$$

where the measures $\lambda_{\varphi,u}$, σ^β_φ, $\sigma_{\varphi,r}$ are defined by

$$\begin{aligned}
\lambda_{\varphi,u} &= (|\rho u|^q d\mu) \circ \varphi^{-1}, \\
\sigma^\beta_\varphi &= \left[(1-\rho)^\beta |u-v|^q d\mu\right] \circ \varphi^{-1}, \\
\sigma_{\varphi,r} &= (\chi_{G_r} |u-v|^q d\mu) \circ \varphi^{-1},
\end{aligned}$$

and the measures $\lambda_{\psi,v}$, σ^β_ψ, $\sigma_{\psi,r}$ are defined similarly. Parameters omitted in these notation should be clear from the context.

We will separate into two cases.

4.1. The case $p \le q$

We need the following lemma, which is a generalization of [4, Lemma 3.3].

Lemma 4.1. *Let $\omega \in \mathcal{D}$, $0 < p \le q < \infty$, $0 < r < 1$ and μ be a positive Borel measure on $\mathbb{D}$. Then for any $f \in A^p_\omega$,*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \lesssim \|f\|_{A^p_\omega}^{q-p} \int_{\mathbb{D}} |f(z)|^p \widehat{\mu}_{\omega,r,\frac{q}{p}}(z)\widetilde{\omega}(z)dA(z).$$

Proof. Fix $f \in A^p_\omega$. By the subharmonic property of $|f|^p$ and (2.4),

$$\begin{aligned}|f(z)|^q &\lesssim \left(\frac{1}{(1-|z|)^2}\int_{\Delta(z,r)}|f|^p dA\right)^{q/p}\\ &\asymp \left(\frac{1}{\widehat{\omega}(z)(1-|z|)}\int_{\Delta(z,r)}|f|^p\widetilde{\omega}dA\right)^{q/p}\\ &\lesssim \frac{1}{(\widehat{\omega}(z)(1-|z|))^{q/p}}\|f\|_{A^p_\omega}^{q-p}\int_{\Delta(z,r)}|f|^p\widetilde{\omega}dA.\end{aligned}$$

Integrating against the measure μ, and using Fubini's theorem and (2.2), we obtain

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|^q d\mu(z) &\lesssim \|f\|_{A^p_\omega}^{q-p}\int_{\mathbb{D}}\frac{1}{(\widehat{\omega}(z)(1-|z|))^{q/p}}\int_{\Delta(z,r)}|f|^p\widetilde{\omega}dAd\mu(z)\\ &\asymp \|f\|_{A^p_\omega}^{q-p}\int_{\mathbb{D}}|f(\xi)|^p\frac{\mu(\Delta(\xi,r))}{(\widehat{\omega}(\xi)(1-|\xi|))^{q/p}}\widetilde{\omega}(\xi)dA(\xi)\\ &\lesssim \|f\|_{A^p_\omega}^{q-p}\int_{\mathbb{D}}|f(\xi)|^p\widehat{\mu}_{\omega,r,\frac{q}{p}}(\xi)\widetilde{\omega}(\xi)dA(\xi),\end{aligned}$$

which completes the proof. □

The following proposition gives sufficient conditions for the boundedness and compactness of the operators $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$.

Proposition 4.2. *Let $\omega \in \mathcal{D}$, $0 < p \le q < \infty$ and $0 < r < 1$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. If $\lambda + \sigma_{\varphi,r}$ or $\lambda + \sigma_{\psi,r}$ is a (resp. vanishing) q-Carleson measure for A^p_ω, then $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded (resp. compact).*

Proof. Fix $R \in (r_0(\omega), 1)$. By symmetry, we only need to consider the measure $\nu := \lambda + \sigma_{\varphi,r}$. Assume first that ν is a q-Carleson measure for A^p_ω. Then by Theorem 3.1, $\widehat{\nu}_{\omega,R,\frac{q}{p}} \in L^\infty(\mathbb{D})$. For any $f \in A^p_\omega$ with $\|f\|_{A^p_\omega} \le 1$, we have

$$\begin{aligned}\|(uC_\varphi - vC_\psi)f\|_{L^q(d\mu)}^q &= \left(\int_{\mathbb{D}\setminus G_r} + \int_{G_r}\right)|u(f\circ\varphi) - v(f\circ\psi)|^q d\mu\\ &=: I_r + II_r.\end{aligned} \tag{4.1}$$

Noting that $\rho(z) \geq r$ if $z \in \mathbb{D} \setminus G_r$, and using Lemma 4.1, we obtain

$$\begin{aligned} I_r &\lesssim \int_{\mathbb{D}\setminus G_r} (|u(f\circ\varphi)|^q + |v(f\circ\psi)|^q)\, d\mu \\ &\leq \frac{1}{r^q} \int_{\mathbb{D}\setminus G_r} (|\rho u(f\circ\varphi)|^q + |\rho v(f\circ\psi)|^q)\, d\mu \\ &\leq \frac{1}{r^q} \int_{\mathbb{D}} |f|^q d\lambda \\ &\leq \frac{1}{r^q} \int_{\mathbb{D}} |f|^q d\nu \\ &\lesssim \int_{\mathbb{D}} |f|^p \widehat{\nu}_{\omega,R,\frac{q}{p}} \widetilde{\omega} dA. \end{aligned}$$

For the term II_r, using Lemma 4.1 again, we have

$$\begin{aligned} II_r &\lesssim \int_{G_r} (|(u-v) f\circ\varphi|^q + |v(f\circ\varphi - f\circ\psi)|^q)\, d\mu \\ &= \int_{\mathbb{D}} |f|^q d\sigma_{\varphi,r} + \int_{G_r} |v(f\circ\varphi - f\circ\psi)|^q d\mu \\ &\lesssim \int_{\mathbb{D}} |f|^p \widehat{\nu}_{\omega,R,\frac{q}{p}} \widetilde{\omega} dA + \underbrace{\int_{G_r} |v(f\circ\varphi - f\circ\psi)|^q d\mu}_{III_r}. \end{aligned}$$

To estimate the term III_r, we use [12, Lemma 1], which asserts that for $z \in G_r$,

$$|f\circ\varphi(z) - f\circ\psi(z)|^q \lesssim \frac{\rho(z)^q}{(\widehat{\omega}(\psi(z))(1-|\psi(z)|))^{q/p}} \int_{\Delta(\psi(z),R)} |f(\xi)|^p \widetilde{\omega}(\xi) dA(\xi).$$

Therefore, by Fubini's theorem, (2.1) and (2.2), we arrive at

$$\begin{aligned} III_r &\lesssim \int_{G_r} \frac{\rho(z)^q |v(z)|^q}{(\widehat{\omega}(\psi(z))(1-|\psi(z)|))^{q/p}} \int_{\Delta(\psi(z),R)} |f(\xi)|^p \widetilde{\omega}(\xi) dA(\xi) d\mu(z) \\ &\leq \int_{\mathbb{D}} |f(\xi)|^p \int_{\psi^{-1}(\Delta(\xi,R))} \frac{\rho(z)^q |v(z)|^q}{(\widehat{\omega}(\psi(z))(1-|\psi(z)|))^{q/p}} d\mu(z) \widetilde{\omega}(\xi) dA(\xi) \\ &\lesssim \int_{\mathbb{D}} |f(\xi)|^p \frac{\nu(\Delta(\xi,R))}{(\widetilde{\omega}(\xi)(1-|\xi|))^{q/p}} \widetilde{\omega}(\xi) dA(\xi) \\ &\lesssim \int_{\mathbb{D}} |f|^p \widehat{\nu}_{\omega,R,\frac{q}{p}} \widetilde{\omega} dA. \end{aligned}$$

Combining the above estimates on I_r, II_r and III_r with (4.1) and (2.4), we obtain

$$\|(uC_\varphi - vC_\psi)f\|^q_{L^q(d\mu)} \lesssim \int_{\mathbb{D}} |f|^p \widehat{\nu}_{\omega,R,\frac{q}{p}} \widetilde{\omega} dA \lesssim \|\widehat{\nu}_{\omega,R,\frac{q}{p}}\|_{L^\infty(\mathbb{D})}. \tag{4.2}$$

Hence $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded.

Assume now that ν is a vanishing q-Carleson measure for A^p_ω. Then for any $\epsilon > 0$, by Theorem 3.1, there exists $\eta \in (0,1)$ such that

$$\widehat{\nu}_{\omega,R,\frac{q}{p}}(z) < \epsilon, \quad \text{whenever } |z| \geq \eta. \tag{4.3}$$

Suppose that $\{f_n\} \subset A^p_\omega$ is a bounded sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then by (4.2), (4.3) and (2.4), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|(uC_\varphi - vC_\psi)f_n\|^q_{L^q(d\mu)} \\
&\lesssim \limsup_{n\to\infty} \|f_n\|^{q-p}_{A^p_\omega} \left(\int_{\eta\mathbb{D}} + \int_{\mathbb{D}\setminus\eta\mathbb{D}} \right) |f_n|^p \widehat{\nu}_{\omega,R,\frac{q}{p}} \widetilde{\omega} dA \\
&\lesssim \epsilon,
\end{aligned}$$

which implies that $\|(uC_\varphi - vC_\psi)f_n\|_{L^q(d\mu)} \to 0$. Hence the operator $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is compact, and the proof is complete. □

We now consider the necessary conditions. To this end, we introduce some notation. Given $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $b \in \mathbb{D}$, let

$$Q_b(z) = \frac{1 - \bar{b}\varphi(z)}{1 - \bar{b}\psi(z)}, \quad z \in \mathbb{D}.$$

Given $R \in (0,1)$, $0 < t, q < \infty$, $a, b \in \mathbb{D}$ and a positive Borel measure μ on $\mathbb{D}$, denote

$$\Theta^{\mu,q}_{R,t}(a,b) = \int_{\varphi^{-1}(\Delta(a,R))} |u - vQ^t_b|^q d\mu.$$

For $a \in \mathbb{D}$ and $M > 0$, write

$$a_M = ae^{-iM(1-|a|)},$$

and let $\Gamma_M(a)$ be the arc

$$\Gamma_M(a) = \{a\zeta : |\zeta| = 1 \text{ and } |\text{Arg } \zeta| \leq M(1-|a|)\}.$$

The following lemma is implicit in the proof of [3, Theorem 4.1].

Lemma 4.3. *Let $0 < q, \beta < \infty$, $0 < \gamma < \frac{\beta}{q}$ and $R \in (0,1)$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. If $M > 0$ is large so that*

$$\text{Arg}\left[1 + \frac{8i}{M(1-R)}\right] < \min\left\{\frac{\pi}{12}, \frac{\pi}{12\gamma}\right\},$$

then there exists $\eta = \eta(M,R) \in (0,1)$ such that for all $a \in \mathbb{D}$ with $|a| > \eta$,

$$\begin{aligned}
&\int_{\varphi^{-1}(\Delta(a,R))} \left(|\rho u|^q + (1-\rho)^\beta |u - v|^q\right) d\mu \\
&\lesssim \Theta^{\mu,q}_{R,\gamma}(a, a_M) + \Theta^{\mu,q}_{R,2\gamma}(a, a_M) + \Theta^{\mu,q}_{R,\gamma}(a, \overline{a_M}) \\
&\quad + \Theta^{\mu,q}_{R,2\gamma}(a, \overline{a_M}) + \Theta^{\mu,q}_{R,\gamma}(a, a) + \Theta^{\mu,q}_{R,\gamma+1}(a, a).
\end{aligned}$$

Proposition 4.4. *Let $\omega \in \mathcal{D}$, $0 < p \le q < \infty$ and $\frac{\beta}{q} > \frac{\gamma(\omega)+1}{p}$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. If $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded (resp. compact), then the measures $\lambda_{\varphi,u} + \sigma^\beta_\varphi$ and $\lambda_{\psi,v} + \sigma^\beta_\psi$ are (resp. vanishing) q-Carleson measures for A^p_ω.*

Proof. Suppose first that $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded. Let $K_b(z) = 1/(1-\bar{b}z)$, $b, z \in \mathbb{D}$. Fix $R \in (r_0(\omega), 1)$ and $\gamma > 0$ such that

$$\frac{\gamma(\omega)+1}{p} < \gamma < \frac{\beta}{q},$$

and choose sufficiently large $M > 0$ so that

$$\operatorname{Arg}\left[1 + \frac{8i}{M(1-R)}\right] < \min\left\{\frac{\pi}{12}, \frac{\pi}{12\gamma}\right\}.$$

Write $\nu = \lambda_{\varphi,u} + \sigma^\beta_\varphi$ and $T = uC_\varphi - vC_\psi$ to save the notation. Since $u, v \in L^q(d\mu)$, ν is a finite measure. Let $a \in \mathbb{D}$. For any $t > (\gamma(\omega)+1)/p$ and $b \in \Gamma_M(a)$, write

$$f_{b,t} = \frac{K^t_b}{\|K^t_b\|_{A^p_\omega}}.$$

Then for all a with $M(1-|a|) < \pi$, by [3, Lemma 3.3] and (2.5), we have

$$\begin{aligned}
\|Tf_{b,t}\|^q_{L^q(d\mu)} &= \frac{1}{\|K^t_b\|^q_{A^p_\omega}} \int_{\mathbb{D}} \left| \frac{u}{(1-\bar{b}\varphi)^t} - \frac{v}{(1-\bar{b}\psi)^t} \right|^q d\mu \\
&\ge \frac{1}{\|K^t_b\|^q_{A^p_\omega}} \int_{\varphi^{-1}(\Delta(a,R))} \frac{|u - vQ^t_b|^q}{|1-\bar{b}\varphi|^{qt}} d\mu \\
&\asymp \frac{1}{\|K^t_b\|^q_{A^p_\omega}(1-|a|)^{tq}} \int_{\varphi^{-1}(\Delta(a,R))} |u - vQ^t_b|^q d\mu \\
&\asymp \frac{1}{(\widehat{\omega}(a)(1-|a|))^{q/p}} \int_{\varphi^{-1}(\Delta(a,R))} |u - vQ^t_b|^q d\mu.
\end{aligned}$$

Combining the above estimate with Lemma 4.3, we know that there exists $\eta = \eta(M,R) \in (0,1)$ such that for all $a \in \mathbb{D}$ with $|a| \ge \eta$,

$$\begin{aligned}
&\frac{\nu(\Delta(a,R))}{(\widehat{\omega}(a)(1-|a|))^{q/p}} \\
&\lesssim \|Tf_{a_M,\gamma}\|^q_{L^q(d\mu)} + \|Tf_{a_M,2\gamma}\|^q_{L^q(d\mu)} + \|Tf_{\overline{a_M},\gamma}\|^q_{L^q(d\mu)} \\
&\quad + \|Tf_{\overline{a_M},2\gamma}\|^q_{L^q(d\mu)} + \|Tf_{a,\gamma}\|^q_{L^q(d\mu)} + \|Tf_{a,\gamma+1}\|^q_{L^q(d\mu)} \\
&\lesssim \|T\|^q.
\end{aligned} \tag{4.4}$$

Therefore, by (2.3),

$$\sup_{|a|\ge\eta} \widehat{\nu}_{\omega,R,\frac{q}{p}}(a) \lesssim \sup_{|a|\ge\eta} \frac{\nu(\Delta(a,R))}{(\widehat{\omega}(a)(1-|a|))^{q/p}} \lesssim \|T\|^q.$$

It is easy to see that $\widehat{\nu}_{\omega,R,\frac{q}{p}}$ is continuous on $\mathbb{D}$. Thus $\widehat{\nu}_{\omega,R,\frac{q}{p}} \in L^\infty(\mathbb{D})$, and by Theorem 3.1, $\nu = \lambda_{\varphi,u} + \sigma_\varphi^\beta$ is a q-Carleson measure for A^p_ω.

If T is compact, then for any $t > \frac{\gamma(\omega)+1}{p}$, $\|Tf_{b,t}\|_{L^q(d\mu)} \to 0$ as $|b| \to 1$. Hence (2.3) and (4.4) yield that $\widehat{\nu}_{\omega,R,\frac{q}{p}}(a) \to 0$ as $|a| \to 1$, which implies that ν is a vanishing q-Carleson measure for A^p_ω by Theorem 3.1.

Finally, by symmetry, the boundedness (resp. compactness) of T also implies that the measure $\lambda_{\psi,v}+\sigma_\psi^\beta$ is a (resp. vanishing) q-Carleson measure for A^p_ω. □

We are now ready to give the main theorem of this subsection, which is a more detailed version of Theorem 1.1.

Theorem 4.5. *Let $\omega \in \mathcal{D}$, $0 < p \le q < \infty$, $0 < r < 1$, and $\frac{\beta}{q} > \frac{\gamma(\omega)+1}{p}$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. Then the following conditions are equivalent*:

(a) *$uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded (resp. compact)*;

(b) *$\lambda + \sigma_\varphi^\beta$ and $\lambda + \sigma_\psi^\beta$ are (resp. vanishing) q-Carleson measures for A^p_ω*;

(c) *$\lambda + \sigma_\varphi^\beta$ or $\lambda + \sigma_\psi^\beta$ is a (resp. vanishing) q-Carleson measure for A^p_ω*;

(d) *$\lambda + \sigma_{\varphi,r}$ and $\lambda + \sigma_{\psi,r}$ are (resp. vanishing) q-Carleson measures for A^p_ω*;

(e) *$\lambda + \sigma_{\varphi,r}$ or $\lambda + \sigma_{\psi,r}$ is a (resp. vanishing) q-Carleson measure for A^p_ω.*

Proof. The implication (a) $\Longrightarrow$ (b) follows from Proposition 4.4, and the implication (e) $\Longrightarrow$ (a) follows from Proposition 4.2. The implications (b) $\Longrightarrow$ (c), and (d) $\Longrightarrow$ (e) are trivial. Since $1 - \rho(z) > 1 - r$ for $z \in G_r$, the implications (b) $\Longrightarrow$ (d), and (c) $\Longrightarrow$ (e) are clear for any $\beta > 0$. The proof is complete. □

4.2. The case $q < p$

We first give some sufficient conditions for the compactness of the operators $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$.

Proposition 4.6. *Let $\omega \in \mathcal{D}$, $0 < q < p < \infty$ and $0 < r < 1$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. If $\lambda + \sigma_{\varphi,r}$ or $\lambda + \sigma_{\psi,r}$ is a q-Carleson measure for A^p_ω, then $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is compact.*

Proof. Fix $R \in (r_0(\omega), 1)$. Assume that $\nu := \lambda + \sigma_{\varphi,r}$ is a q-Carleson measure for A^p_ω. Then by Theorem 3.2, $\widehat{\nu}_{\omega,R} \in L^{p/(p-q)}_{\widetilde{\omega}}$. For any $f \in A^p_\omega$, using the first inequality in (4.2) with $p = q$, Hölder's inequality and (2.4), we obtain

$$\begin{aligned} \|(uC_\varphi - vC_\psi)f\|^q_{L^q(d\mu)} &\lesssim \int_{\mathbb{D}} |f|^q \widehat{\nu}_{\omega,R} \widetilde{\omega}\, dA \\ &\le \|\widehat{\nu}_{\omega,R}\|_{L^{\frac{p}{p-q}}_{\widetilde{\omega}}} \|f\|^q_{A^p_{\widetilde{\omega}}} \\ &\asymp \|\widehat{\nu}_{\omega,R}\|_{L^{\frac{p}{p-q}}_{\widetilde{\omega}}} \|f\|^q_{A^p_\omega}, \end{aligned}$$

which implies that $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded. To prove the compactness, suppose that $\{f_n\} \subset A^p_\omega$ is a bounded sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$. It is sufficient to show

$$\|(uC_\varphi - vC_\psi)f_n\|_{L^q(d\mu)} \to 0.$$

Fix $\epsilon > 0$. We may choose $\eta \in (0,1)$ such that

$$\int_{\mathbb{D}\setminus\eta\mathbb{D}} \widehat{\nu}_{\omega,R}^{\frac{p}{p-q}} \widetilde{\omega} dA < \epsilon^{\frac{p}{p-q}}.$$

Hence as before, we establish that

$$\begin{aligned}\limsup_{n\to\infty} \|(uC_\varphi - vC_\psi)f_n\|^q_{L^q(d\mu)} &\lesssim \limsup_{n\to\infty} \left(\int_{\eta\mathbb{D}} + \int_{\mathbb{D}\setminus\eta\mathbb{D}}\right) |f_n|^q \widehat{\nu}_{\omega,R}\widetilde{\omega} dA \\ &\le \limsup_{n\to\infty} \left(\int_{\eta\mathbb{D}} |f_n|^p \widetilde{\omega} dA\right)^{\frac{q}{p}} \|\widehat{\nu}_{\omega,R}\|_{L^{\frac{p}{p-q}}_{\widetilde{\omega}}} \\ &\quad + \sup_{n\ge 1} \|f_n\|^q_{A^p_{\widetilde{\omega}}} \left(\int_{\mathbb{D}\setminus\eta\mathbb{D}} \widehat{\nu}_{\omega,R}^{\frac{p}{p-q}} \widetilde{\omega} dA\right)^{\frac{p-q}{p}} \\ &\lesssim \epsilon.\end{aligned}$$

The arbitrariness of $\epsilon > 0$ gives the desired compactness.

Similarly, if $\lambda + \sigma_{\psi,r}$ is a q-Carleson measure for A^p_ω, then the operator $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is also compact. □

To obtain the necessary conditions for the boundedness of the operators $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ in the case $q < p$, we need the following decomposition lemma.

Lemma 4.7. *Let $R \in (0,1)$, K be a positive integer and $\{a_n\}$ be a sequence of distinct points in $\mathbb{D}$. If any collection of more than K of the pseudohyperbolic disks $\Delta(a_n, R)$ contains no point in common, then $\{a_n\}$ is a union of $K+1$ separated sequences.*

Proof. Put $a_{1,1} = a_1, a_{2,1} = a_2, \ldots, a_{K+1,1} = a_{K+1}$. For a_{K+2}, by the assumption, there exists $k \in \{1,2,\ldots,K+1\}$ such that $a_{K+2} \notin \Delta(a_k, R)$, i.e., $d(a_{k,1}, a_{K+2}) = d(a_k, a_{K+2}) \ge R$. Put $a_{k,2} = a_{K+2}$. Inductively, assume that M is a positive integer and $\{a_n\}_{n=1}^M = \cup_{k=1}^{K+1}\{a_{k,n}\}_{n=1}^{N_k}$ such that for any $k \in \{1,2,\ldots,K+1\}$ and $1 \le i,j \le N_k$, $i \ne j$,

$$d(a_{k,i}, a_{k,j}) \ge R.$$

For a_{M+1}, we claim that there exists $k_0 \in \{1,2,\ldots,K+1\}$ such that

$$a_{M+1} \notin \bigcup_{n=1}^{N_{k_0}} \Delta(a_{k_0,n}, R).$$

If not, assume that $a_{M+1} \in \cap_{k=1}^{K+1} \cup_{n=1}^{N_k} \Delta(a_{k,n}, R)$. Then for any $k \in \{1, 2, \ldots, K+1\}$, there exists $n_k \in \{1, 2, \ldots, N_k\}$ such that $a_{M+1} \in \Delta(a_{k,n_k}, R)$. Consequently,

$$a_{K+1} \in \bigcap_{k=1}^{K+1} \Delta(a_{k,n_k}, R),$$

which contradicts the assumption. Thus the claim holds. Then put $a_{k_0, N_{k_0}+1} = a_{M+1}$. Therefore, we have

$$\{a_n\} = \bigcup_{k=1}^{K+1} \{a_{k,n}\}_{n=1}^{\infty},$$

and each $\{a_{k,n}\}_{n=1}^{\infty}$ is an R-separated sequence. □

The following proposition gives some necessary conditions for the boundedness of $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$.

Proposition 4.8. *Let $\omega \in \mathcal{D}$, $0 < q < p < \infty$ and $\frac{\beta}{q} > 1 + \frac{\beta(\omega)+\gamma(\omega)+1}{p}$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. If $uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded, then the measures $\lambda_{\varphi,u} + \sigma^\beta_\varphi$ and $\lambda_{\psi,v} + \sigma^\beta_\psi$ are q-Carleson measures for A^p_ω.*

Proof. Fix $\gamma, M > 0$ so that

$$1 + \frac{\beta(\omega) + \gamma(\omega) + 1}{p} < \gamma < \frac{\beta}{q}$$

and

$$\operatorname{Arg}\left[1 + \frac{8i}{M} \frac{1}{1 - \frac{1}{3(1+8M)}}\right] < \frac{\pi}{12\gamma}.$$

Let $R = \frac{1}{3(1+8M)}$ and $\{a_n\}$ be a δ-lattice, where $0 < \delta < \min\{R, 1 - r_0(\omega)\}$. Choose large $L > 0$ such that $M(1 - |a_n|) < \pi$ whenever $n \geq L$. We first prove that for any $t > 1 + \frac{\beta(\omega)+\gamma(\omega)+1}{p}$,

$$\left\| \left\{ \frac{\Theta^{\mu,q}_{R,t}(a_n, b_n)}{(\widehat{\omega}(a_n)(1 - |a_n|))^{\frac{q}{p}}} \right\}_{n \geq L} \right\|_{l^{\frac{p}{p-q}}} \lesssim \|uC_\varphi - vC_\psi\|^q, \tag{4.5}$$

where $\{b_n\}_{n\geq L}$ is a sequence given by one of $\{a_n\}_{n\geq L}$, $\{a_{n,M}\}_{n\geq L}$ and $\{\overline{a_{n,M}}\}_{n\geq L}$ (write $a_{n,M} = (a_n)_M$). To this end, define

$$F_\tau(z) = \sum_{n\geq L} c_n \phi_n(\tau) \frac{(1 - |a_n|)^{t - \frac{1}{p}} \widehat{\omega}(a_n)^{-\frac{1}{p}}}{(1 - \overline{b_n} z)^t},$$

where $t > 1 + \frac{\beta(\omega)+\gamma(\omega)+1}{p}$, $c = \{c_n\} \in l^p$ and $\{\phi_n\}$ is the sequence of Rademacher functions (see [8, Appendix A]). By [4, Lemma 4.4] and Lemma 4.7, there exists a positive integer K, such that $\{b_n\}$ is a union of K separated sequences. Consequently, Proposition 2.1 implies that for any $\tau \in [0, 1]$,

$F_\tau \in A^p_\omega$ with $\|F_\tau\|_{A^p_\omega} \lesssim \|c\|_{l^p}$. The boundedness of $T := uC_\varphi - vC_\psi$ then yields

$$\|TF_\tau\|^q_{L^q(d\mu)} = \int_{\mathbb{D}} \left| \sum_{n\geq L} c_n \phi_n(\tau) \frac{(1-|a_n|)^{t-\frac{1}{p}} \widehat{\omega}(a_n)^{-\frac{1}{p}}}{(1-\overline{b_n}\varphi)^t} (u - vQ^t_{b_n}) \right|^q d\mu$$
$$\lesssim \|T\|^q \|c\|^q_{l^p}.$$

Integrating with respect to τ on $[0,1]$ and using Fubini's theorem and Khinchine's inequality, we obtain

$$\int_{\mathbb{D}} \underbrace{\left(\sum_{n\geq L} \left| c_n \frac{(1-|a_n|)^{t-\frac{1}{p}} \widehat{\omega}(a_n)^{-\frac{1}{p}}}{(1-\overline{b_n}\varphi)^t} (u - vQ^t_{b_n}) \right|^2 \right)^{\frac{q}{2}}}_{\Phi} d\mu \lesssim \|T\|^q \|c\|^q_{l^p}. \tag{4.6}$$

Since each point $z \in \mathbb{D}$ belongs to at most $N = N(\delta, R)$ of the sets $\Delta(a_n, R)$, by [3, Lemma 3.3], we establish

$$\Phi \gtrsim \sum_{n\geq L} |c_n|^q \left| \frac{(1-|a_n|)^{t-\frac{1}{p}} \widehat{\omega}(a_n)^{-\frac{1}{p}}}{(1-\overline{b_n}\varphi)^t} (u - vQ^t_{b_n}) \right|^q \chi_{\Delta(a_n,R)} \circ \varphi$$
$$\asymp \sum_{n\geq L} |c_n|^q \frac{|u - vQ^t_{b_n}|^q}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}} \chi_{\Delta(a_n,R)} \circ \varphi.$$

Inserting this into (4.6) yields

$$\sum_{n\geq L} |c_n|^q \frac{\Theta^{\mu,q}_{R,t}(a_n, b_n)}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}} \lesssim \int_{\mathbb{D}} \Phi d\mu \lesssim \|T\|^q \|c\|^q_{l^p}.$$

Since $c \in l^p$ is arbitrary and $p > q$, the duality $\left(l^{p/q}\right)^* = l^{p/(p-q)}$ gives the desired estimate (4.5).

To finish the proof, write $\nu = \lambda_{\varphi,u} + \sigma^\beta_\varphi$. For any $n \geq L$, Lemma 4.3 gives that

$$\nu(\Delta(a_n,\delta)) \leq \nu(\Delta(a_n,R))$$
$$\lesssim \Theta^{\mu,q}_{R,\gamma}(a_n, a_{n,M}) + \Theta^{\mu,q}_{R,2\gamma}(a_n, a_{n,M}) + \Theta^{\mu,q}_{R,\gamma}(a_n, \overline{a_{n,M}})$$
$$+ \Theta^{\mu,q}_{R,2\gamma}(a_n, \overline{a_{n,M}}) + \Theta^{\mu,q}_{R,\gamma}(a_n, a_n) + \Theta^{\mu,q}_{R,\gamma+1}(a_n, a_n).$$

Combining this with (4.5), we arrive at

$$\sum_{n\geq L} \left(\frac{\nu(\Delta(a_n,\delta))}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}} \right)^{\frac{p}{p-q}} < \infty. \tag{4.7}$$

Fix $R' \in [r_0(\omega), 1-\delta)$. For any $n \geq 1$, let

$$\Pi_n = \{m \geq 1 : \Delta(a_n, R') \cap \Delta(a_m, \delta) \neq \emptyset\}$$

and

$$\Pi_n' = \{m \geq 1 : \Delta(a_m, R') \cap \Delta(a_n, \delta) \neq \emptyset\}.$$

Then there exists $N = N(\delta, R')$ such that for any $n \geq 1$, $\sharp\Pi_n, \sharp\Pi_n' \leq N$. Therefore, by (2.1) and (4.7), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{\nu(\Delta(a_n,R'))}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}}\right)^{\frac{p}{p-q}} &\leq \sum_{n=1}^{\infty}\left(\frac{\sum_{m\in\Pi_n}\nu(\Delta(a_m,\delta))}{(\widehat{\omega}(a_n)(1-|a_n|))^{q/p}}\right)^{\frac{p}{p-q}}\\
&\lesssim \sum_{n=1}^{\infty}\sum_{m\in\Pi_n}\left(\frac{\nu(\Delta(a_m,\delta))}{(\widehat{\omega}(a_m)(1-|a_m|))^{q/p}}\right)^{\frac{p}{p-q}}\\
&= \sum_{m=1}^{\infty}\sum_{n\in\Pi_m'}\left(\frac{\nu(\Delta(a_m,\delta))}{(\widehat{\omega}(a_m)(1-|a_m|))^{q/p}}\right)^{\frac{p}{p-q}}\\
&< \infty.
\end{aligned}$$

Hence by Theorem 3.2, the measure $\nu = \lambda_{\varphi,u} + \sigma_{\varphi}^{\beta}$ is a q-Carleson measure for A^p_ω. Symmetrically, the same assertion also holds for the measure $\lambda_{\psi,v}+\sigma_{\psi}^{\beta}$. □

We are now in a position to give a more detailed version of Theorem 1.2.

Theorem 4.9. *Let $\omega \in \mathcal{D}$, $0 < q < p < \infty$, $0 < r < 1$, and $\frac{\beta}{q} > 1 + \frac{\beta(\omega)+\gamma(\omega)+1}{p}$. Let μ be a positive Borel measure on $\mathbb{D}$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ and $u, v \in L^q(d\mu)$. Then the following conditions are equivalent*:

(a) *$uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is bounded*;
(b) *$uC_\varphi - vC_\psi : A^p_\omega \to L^q(d\mu)$ is compact*;
(c) *$\lambda + \sigma_\varphi^\beta$ and $\lambda + \sigma_\psi^\beta$ are q-Carleson measures for A^p_ω*;
(d) *$\lambda + \sigma_\varphi^\beta$ or $\lambda + \sigma_\psi^\beta$ is a q-Carleson measure for A^p_ω*;
(e) *$\lambda + \sigma_{\varphi,r}$ and $\lambda + \sigma_{\psi,r}$ are q-Carleson measures for A^p_ω*;
(f) *$\lambda + \sigma_{\varphi,r}$ or $\lambda + \sigma_{\psi,r}$ is a q-Carleson measure for A^p_ω.*

Proof. The implication (f)$\Longrightarrow$(b) follows from Proposition 4.6, and the implication (a)$\Longrightarrow$(c) follows from Proposition 4.8. The rest implications are clear as in the proof of Theorem 4.5. □

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