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CONDITIONAL FOURIER–FEYNMAN TRANSFORM AND CONDITIONAL CONVOLUTION PRODUCT ASSOCIATED WITH INFINITE DIMENSIONAL CONDITIONING FUNCTION

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Abstract. In this paper, we use an infinite dimensional conditioning function to define a conditional Fourier–Feynman transform (CFFT) and a conditional convolution product (CCP) on the Wiener space. We establish the existences of the CFFT and the CCP for bounded functions which form a Banach algebra. We then provide fundamental relationships between the CFFTs and the CCPs.

1. Introduction

Let $C_0[0,T]$ denote the Wiener space. The concepts of the CFFT and the CCP for functions on the Wiener space $C_0[0,T]$ were introduced by Park and Skoug in [14]. These transform and convolution are similar in many respects to the Fourier transform (with the corresponding convolution) of functions on Euclidean spaces. Since then many authors have provided various results between the CFFT and the CCP [2, 4, 5, 7, 8, 10]. However the conditioning functions defining the CFFT and the CCP studied in [2, 4, 5] (also several analogies [7,8,10] which are more exquisite) are only finite dimensional (vector-valued) functions.

On the other hand, in [13], Park and Skoug derived an evaluation formula for the conditional Wiener integral given an infinite dimensional conditioning function and established useful formulas to calculate their conditional Wiener integrals. In this paper, using the fundamental result in [13], we define a CFFT and a CCP associated with infinite dimensional conditioning functions on the Wiener space $C_0[0,T]$. We then establish the existences of the CFFT and the CCP for bounded functions which form a Banach algebra [1]. We also provide basic relationships between the CFFTs and the CCPs. The conditioning

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function $X_\infty : C_0[0,T] \to \mathbb{R}^\infty$ used in this paper is given by

$$X_\infty(x) = (\langle e_1, x\rangle, \langle e_2, x\rangle, \ldots),$$

where $\{e_n\}$ is an orthonormal sequence of functions in $L_2[0,T]$ and $\langle e, x\rangle$ denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral $\int_0^T e(t)dx(t)$.

2. Preliminaries and definitions

In this section, we introduce the concepts of the CFFT and the CCP for functions on the complete Wiener measure space $(C_0[0,T], \mathcal{W}(C_0[0,T]), m_w)$, where $\mathcal{W}(C_0[0,T])$ denotes the σ-field of all Wiener measurable subsets. The definitions are based on the concept of the conditional Wiener integral associated with an infinite dimensional vector-valued conditioning function.

Let $\mathbb{V}$ be a real normed linear space with norm $|\cdot|$. Clearly, the linear space $\mathbb{V}$ is a topological vector space with respect to the metric topology induced by $|\cdot|$. Let $\mathcal{B}(\mathbb{V})$ be the σ-field generated by the class of all open subsets of $\mathbb{V}$. Then $\mathcal{B}(\mathbb{V})$ is known as the Borel σ-field on $\mathbb{V}$. Let X be a $\mathbb{V}$-valued measurable function and Y a $\mathbb{C}$-valued integrable function on $C_0[0,T]$. Let $\mathcal{F}(X)$ denote the σ-field generated by X. Then by the definition, the conditional expectation of Y given $\mathcal{F}(X)$, written $E(Y|X)$, is any real valued $\mathcal{F}(X)$-measurable function on $C_0[0,T]$ such that

$$\int_A Y(x)dm_w(x) = \int_A E(Y|X)(x)dm_w(x) \quad \text{for} \quad A \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{V}, \mathcal{B}(\mathbb{V}), P_X)$ such that $E(Y|X) = \psi \circ X$, where P_X is the probability distribution of X defined by $P_X(U) = m_w(X^{-1}(U))$ for $U \in \mathcal{B}(\mathbb{V})$. The function $\psi(\eta)$, $\eta \in \mathbb{V}$, is unique up to Borel null sets in $\mathbb{V}$. Following Tucker [15] and Yeh [16], the function $\psi(\eta)$, written $E(Y|X=\eta)$, is called the conditional Wiener integral of Y given X.

We denote the Wiener integral of a Wiener integrable function F by

$$E[F] \equiv E_x[F(x)] = \int_{C_0[0,T]} F(x)dm_w(x),$$

and for $u \in L_2[0,T]$ and $x \in C_0[0,T]$, we let $\langle u, x\rangle = \int_0^T u(t)dx(t)$ denote the PWZ stochastic integral [11, 12].

Let $\mathcal{H}$ be an infinite dimensional subspace of $L_2[0,T]$ with a countable orthonormal basis $\{e_n\}$. Let $X_\infty : C_0[0,T] \to \mathbb{R}^\infty$ be the function defined by

$$X_\infty(x) = (\gamma_1(x), \gamma_2(x), \ldots), \tag{2.1}$$

where $\gamma_n(x) \equiv \langle e_n, x\rangle$ for each $n \in \mathbb{N}$. We note that the stochastic integrals $\gamma_n(x)$, $n \in \mathbb{N}$, form a set of independent standard Gaussian variables on $C_0[0,T]$. Consider the projection map $\mathcal{P}_\mathcal{H} : L_2[0,T] \to \mathcal{H}$ given by

$$\mathcal{P}_\mathcal{H} h(t) = \sum_{n=1}^\infty (h, e_n)_2 e_n(t),$$

where $(\cdot,\cdot)_2$ denotes the L_2-inner product. Then it follows that $\|\mathcal{P}_{\mathcal{H}}h\|_2 \lneq \|h\|_2$ if $\mathcal{H} = \text{Span}\{e_1, e_2, \dots\} \subsetneq L_2[0,T]$. For $x \in C_0[0,T]$ and $\vec{\xi} = (\xi_1, \xi_2, \dots) \in \mathbb{R}^\infty$, let

$$x_\infty(t) = \langle \mathcal{P}_{\mathcal{H}} I_{[0,t]}, x\rangle = \sum_{n=1}^{\infty} \gamma_n(x)(I_{[0,t]}, e_n)_2 = \sum_{n=1}^{\infty} \gamma_n(x)\beta_n(t)$$

and

$$\vec{\xi}_\infty(t) = \sum_{n=1}^{\infty} \xi_n (I_{[0,t]}, e_n)_2 = \sum_{n=1}^{\infty} \xi_n \beta_n(t),$$

where $I_{[0,t]}$ denotes the indicator function of the interval $[0,t]$ and $\beta_n(t) = \int_0^t e_n(s)ds$, $t \in [0,T]$ for each $n \in \mathbb{N}$.

In [13], Park and Skoug proved the facts that the process $\{x(t) - x_\infty(t), 0 \le t \le T\}$ and the Gaussian random variable $\gamma_n(x)$ are stochastically independent for each $n \in \mathbb{N}$, and that the processes $\{x(t) - x_\infty(t), 0 \le t \le T\}$ and $\{x_\infty(t), 0 \le t \le T\}$ are also stochastically independent. Using these basic results, Park and Skoug established the following evaluation formula to express conditional Wiener integrals in terms of ordinary Wiener integrals.

Theorem 2.1 ([13])**.** *Let $F \in L_1(C_0[0,T])$. Then it follows that*

$$\begin{aligned} E(F|X_\infty = \vec{\xi}) &= E_x\big[F\big(x - x_\infty + \vec{\xi}_\infty\big)\big] \\ &= E_x\bigg[F\bigg(x - \sum_{n=1}^{\infty} \gamma_n(x)\beta_n + \sum_{n=1}^{\infty} \xi_n\beta_n\bigg)\bigg] \end{aligned} \tag{2.2}$$

for a.e. $\vec{\xi} \in \mathbb{R}^\infty$.

In order to define the CFFT and the CCP, we need the concept of the scale-invariant measurability on the Wiener space. A subset B of $C_0[0,T]$ is called a scale-invariant measurable (SIM) set if $\rho B \in \mathcal{W}(C_0[0,T])$ for all $\rho > 0$, and a SIM set N is called a scale-invariant null set if $m_w(\rho N) = 0$ for all $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (SI-a.e.). A function F is said to be SIM provided F is defined on a SIM set and $F(\rho\,\cdot\,)$ is $\mathcal{W}(C_0[0,T])$-measurable for every $\rho > 0$. For more detailed studies of the scale-invariant measurability, see [6, 9].

The definitions of the CFFT and the CCP are based on the conditional analytic Wiener integral [2, 4, 14]. In this paper, we shall use exclusively the conditioning function X_∞ given by (2.1) to define a CFFT and a CCP on $C_0[0,T]$.

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C}\setminus\{0\} : \text{Re}(\lambda) \ge 0\}$. Let $X_\infty : C_0[0,T] \to \mathbb{R}^\infty$ be given by (2.1) and let F be a $\mathbb{C}$-valued SIM function such that the Wiener integral $E_x[F(\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. For $\lambda > 0$ and $\vec{\xi}$ in $\mathbb{R}^\infty$, let

$$J_F(\lambda; \vec{\xi}) = E(F(\lambda^{-1/2}\,\cdot\,)|X_\infty(\lambda^{-1/2}\,\cdot\,) = \vec{\xi})$$

denote the conditional Wiener integral of $F(\lambda^{-1/2}\cdot)$ given $X_\infty(\lambda^{-1/2}\cdot)$. If for a.e. $\vec{\xi}\in\mathbb{R}^\infty$, there exists a function $J_F^*(\lambda;\vec{\xi})$, analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda;\vec{\xi})=J_F(\lambda;\vec{\xi})$ for all $\lambda>0$, then $J_F^*(\lambda;\cdot)$ is defined to be the conditional analytic Wiener integral of F over $C_0[0,T]$ given X_∞ with parameter λ. For $\lambda\in\mathbb{C}_+$, we write

$$E^{\mathrm{an}w_\lambda}(F|X_\infty=\vec{\xi})=J_F^*(\lambda;\vec{\xi}).$$

If for a fixed real $q\in\mathbb{R}\setminus\{0\}$, the limit

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E^{\mathrm{an}w_\lambda}(F|X_\infty=\vec{\xi})$$

exists for a.e. $\vec{\xi}\in\mathbb{R}^\infty$, then we will denote the value of this limit by $E^{\mathrm{an}f_q}(F|X_\infty=\vec{\xi})$, and we call it the conditional analytic Feynman integral of F over $C_0[0,T]$ given X_∞ with parameter q.

Let F be a $\mathbb{C}$-valued SIM function on $C_0[0,T]$ such that the Wiener integral $E[F(y+\lambda^{-1/2}\cdot)]\equiv E_x[F(y+\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda>0$. Then one can easily see from (2.2) that for all $\lambda>0$,

$$\begin{aligned}
&E(F(\lambda^{-1/2}\cdot)|X_\infty(\lambda^{-1/2}\cdot)=\vec{\xi})\\
&\equiv E(F(\lambda^{-1/2}\cdot)|\gamma_n(\lambda^{-1/2}\cdot)=\xi_n,\ n=1,2,\ldots)\\
&=E_x\bigg[F\bigg(\lambda^{-1/2}x-\lambda^{-1/2}\sum_{n=1}^\infty\gamma_n(x)\beta_n+\sum_{n=1}^\infty\xi_n\beta_n\bigg)\bigg].
\end{aligned}\tag{2.3}$$

Thus we have that

$$E^{\mathrm{an}w_\lambda}(F|X_\infty=\vec{\xi})=E_x^{\mathrm{an}w_\lambda}\bigg[F\bigg(x-\sum_{n=1}^\infty\gamma_n(x)\beta_n+\sum_{n=1}^\infty\xi_n\beta_n\bigg)\bigg],$$

and

$$E^{\mathrm{an}f_q}(F|X_\infty=\vec{\xi})=E_x^{\mathrm{an}f_q}\bigg[F\bigg(x-\sum_{n=1}^\infty\gamma_n(x)\beta_n+\sum_{n=1}^\infty\xi_n\beta_n\bigg)\bigg],\tag{2.4}$$

where $E_x^{\mathrm{an}w_\lambda}[F(x)]$ and $E_x^{\mathrm{an}f_q}[F(x)]$ denote the analytic Wiener and the analytic Feynman integrals of functions F on $C_0[0,T]$, see [1].

We are now ready to state the definitions of the CFFT and the CCP of functions on $C_0[0,T]$.

Definition 2.2. Let $F:C_0[0,T]\to\mathbb{C}$ be a SIM function on $C_0[0,T]$ such that the Wiener integral $E[F(y+\lambda^{-1/2}\cdot)]$ exists as a finite number for all $\lambda>0$. Let $X_\infty: C_0[0,T]\to\mathbb{R}^\infty$ be given by (2.1). For $\lambda\in\mathbb{C}_+$ and $y\in C_0[0,T]$, let $T_\lambda(F|X_\infty)(y,\vec{\xi})$ denote the conditional analytic Wiener integral of $F(y+\cdot)$

given X_∞, that is to say,

$$\begin{aligned}T_\lambda(F|X_\infty)(y,\vec{\xi}) &= E^{\mathrm{an}w_\lambda}(F(y+\,\cdot\,)|X_\infty=\vec{\xi})\\ &= E_x^{\mathrm{an}w_\lambda}\bigg[F\bigg(y+x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n+\sum_{n=1}^{\infty}\xi_n\beta_n\bigg)\bigg].\end{aligned}$$

We define the L_1 analytic CFFT $T_q^{(1)}(F|X_\infty)(y,\vec{\xi})$ of F given X_∞ by the formula

$$T_q^{(1)}(F|X_\infty)(y,\vec{\xi}) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_\lambda(F|X_\infty)(y,\vec{\xi}).$$

We also define the CCP of SIM functions F and G given X_∞ by the formula

$$\begin{aligned}&[(F*G)_\lambda|X_\infty](y,\vec{\xi})\\ &= \begin{cases} E^{\mathrm{an}w_\lambda}\bigg(F\Big(\dfrac{y+\cdot}{\sqrt{2}}\Big)G\Big(\dfrac{y-\cdot}{\sqrt{2}}\Big)\bigg|X_\infty=\vec{\xi}\bigg), & \lambda\in\mathbb{C}_+,\\ E^{\mathrm{an}f_q}\bigg(F\Big(\dfrac{y+\cdot}{\sqrt{2}}\Big)G\Big(\dfrac{y-\cdot}{\sqrt{2}}\Big)\bigg|X_\infty=\vec{\xi}\bigg), & \lambda=-iq,\ q\in\mathbb{R}\setminus\{0\}.\end{cases}\end{aligned}$$

3. CFFT and CCP given $\mathbb{R}^\infty$-valued conditioning function

In this section, we will establish the existences of the CFFT and the CCP for bounded functions in the Cameron and Storvick's Banach algebra $\mathcal{S}(L_2[0,T])$.

The Banach algebra $\mathcal{S}(L_2[0,T])$ consists of functions on $C_0[0,T]$ having the form

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u,x\rangle\}df(u) \tag{3.1}$$

for SI-a.e. $x\in C_0[0,T]$, where the associated measure f is an element of the Banach algebra $\mathcal{M}(L_2[0,T])$, the space of $\mathbb{C}$-valued countably additive (and hence finite) Borel measures on $L_2[0,T]$. More precisely, since we shall identify functions which coincide SI-a.e. on $C_0[0,T]$, the space $\mathcal{S}(L_2[0,T])$ can be regarded as the space of all s-equivalence classes of functions of the form (3.1). It was also shown in [1] that the correspondence $f\mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L_2[0,T])$ is a Banach algebra with the norm

$$\|F\| \equiv \|f\| = \int_{L_2[0,T]} d|f|(u).$$

In particular, it was shown in [3] that the Banach algebra $\mathcal{S}(L_2[0,T])$ contains many functions of interest in Feynman integration theory. For a more detailed study of the Banach algebra $\mathcal{S}(L_2[0,T])$, see [1,3].

Using the fact that the PWZ stochastic integral $\langle w,x\rangle$ of a function w in $L_2[0,T]$ is a Gaussian random variable, as a function of x, with mean zero and variance $\|w\|_2^2$, and the change of variable theorem, we have the following lemma.

Lemma 3.1. *For each $w \in L_2[0,T]$ and any $\rho > 0$, it follows that*

$$E_x[\exp\{i\rho\langle w,x\rangle\}] = \exp\big\{-\rho^2\|w\|_2^2\big\}. \tag{3.2}$$

From the bilinearity of the PWZ stochastic integral $\langle\cdot,\cdot\rangle$ and equation (3.2) with w replaced with $w-\sum_{n=1}^{\infty}(w,e_n)_2e_n$, we have the following lemma.

Lemma 3.2. *Let $\{e_n\}$ be the countable orthonormal basis in a subspace $\mathcal{H}$ of $L_2[0,T]$. Then for each $w \in L_2[0,T]$ and any $\rho > 0$, it follows that*

$$E_x\bigg[\exp\bigg\{i\rho\bigg\langle w, x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\bigg\rangle\bigg\}\bigg] = \exp\bigg\{-\frac{\rho^2}{2}\bigg[\|w\|^2-\sum_{n=1}^{\infty}(w,e_n)_2^2\bigg]\bigg\}. \tag{3.3}$$

In particular, it follows that for any $q \in \mathbb{R}\setminus\{0\}$ and any $\rho > 0$,

$$E_x^{\mathrm{anf}_q}\bigg[\exp\bigg\{i\rho\bigg\langle w, x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\bigg\rangle\bigg\}\bigg] = \exp\bigg\{-\frac{i\rho^2}{2q}\bigg[\|w\|^2-\sum_{n=1}^{\infty}(w,e_n)_2^2\bigg]\bigg\}. \tag{3.4}$$

In our first theorem of this section, we establish the existences of the CFFT $T_q^{(1)}(F|X_\infty)$ of functions F in the Banach algebra $\mathcal{S}(L_2[0,T])$.

Theorem 3.3. *Let $F \in \mathcal{S}(L_2[0,T])$ be given by equation (3.1), and let X_∞ be given by equation (2.1). Then for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, it follows that*

$$\begin{aligned} &T_q^{(1)}(F|X_\infty)(y,\vec{\xi}) \\ &= \int_{L_2[0,T]}\exp\bigg\{i\langle u,y\rangle - \frac{i}{2q}\bigg[\|u\|_2^2-\sum_{n=1}^{\infty}(u,e_n)_2^2\bigg] + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2\bigg\}df(u) \end{aligned} \tag{3.5}$$

for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$.

Proof. Using (3.1), (2.3) with F replaced with $F(y+\cdot)$, the Fubini theorem, (3.3) with w and ρ replaced with u and $\lambda^{-1/2}$, it follows that for $(\lambda,\vec{\xi}) \in (0,+\infty)\times\mathbb{R}^\infty$,

$$\begin{aligned} &J_{F(y+\cdot)}(\lambda;\vec{\xi}) \equiv E\big(F(y+\lambda^{-1/2}\,\cdot)\big|X_\infty(\lambda^{-1/2}\,\cdot)=\vec{\xi}\big) \\ &= E_x\bigg[F\bigg(y+\lambda^{-1/2}x-\lambda^{-1/2}\sum_{n=1}^{\infty}\gamma_n(x)\beta_n+\sum_{n=1}^{\infty}\xi_n\beta_n\bigg)\bigg] \\ &= \int_{L_2[0,T]}\exp\bigg\{i\langle u,y\rangle + i\bigg\langle u,\sum_{n=1}^{\infty}\xi_n\beta_n\bigg\rangle\bigg\} \\ &\quad\times E_x\bigg[\exp\bigg\{i\lambda^{-1/2}\bigg\langle u, x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\bigg\rangle\bigg\}\bigg]df(u) \\ &= \int_{L_2[0,T]}\exp\bigg\{i\langle u,y\rangle - \frac{1}{2\lambda}\bigg[\|u\|_2^2-\sum_{n=1}^{\infty}(u,e_n)_2^2\bigg] + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2\bigg\}df(u). \end{aligned}$$

Let

$$\begin{aligned} &J^*_{F(y+\cdot)}(\lambda;\vec{\xi}) \\ (3.6) \quad &= \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle - \frac{1}{2\lambda}\left[\|u\|_2^2 - \sum_{n=1}^{\infty}(u,e_n)_2^2\right] + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2 \right\} df(u) \end{aligned}$$

for $\lambda \in \mathbb{C}_+$. Since $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \mathbb{C}_+$, it follows that

$$\begin{aligned} &\left|J^*_{F(y+\cdot)}(\lambda;\vec{\xi})\right| \\ (3.7) \quad &\leq \int_{L_2[0,T]} \left| \exp\left\{ i\langle u,y\rangle - \frac{1}{2\lambda}\left[\|u\|_2^2 - \sum_{n=1}^{\infty}(u,e_n)_2^2\right] + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2 \right\}\right| d|f|(u) \\ &\leq \int_{L_2[0,T]} d|f|(u) = \|f\| < +\infty. \end{aligned}$$

Hence, applying the dominated convergence theorem, we see that $J^*_F(\lambda;\vec{\xi})$ is a continuous function of $\lambda \in \widetilde{\mathbb{C}}_+$. Since

$$K(\lambda) \equiv \exp\left\{ i\langle u,y\rangle - \frac{1}{2\lambda}\left[\|u\|_2^2 - \sum_{n=1}^{\infty}(u,e_n)_2^2\right] + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2 \right\}$$

is analytic on $\mathbb{C}_+$, using the Fubini theorem, it follows that

$$\int_\Gamma J^*_{F(y+\cdot)}(\lambda;\vec{\xi})d\lambda = \int_{L_2[0,T]}\int_\Gamma K(\lambda)d\lambda df(u) = 0$$

for all rectifiable closed curves Γ lying in $\mathbb{C}_+$. Thus by the Morera theorem, $J^*_{F(y+\cdot)}(\lambda;\vec{\xi})$ is analytic on $\mathbb{C}_+$. Therefore, the conditional analytic Wiener integral

$$T_\lambda(F|X_\infty)(y,\vec{\xi}) = E^{\mathrm{anw}_\lambda}(F(y+\cdot)|X_\infty = \vec{\xi}) = J^*_{F(y+\cdot)}(\lambda;\vec{\xi})$$

exists and is given by the right-hand side of (3.6). Finally, by the dominated convergence theorem (the use of which is justified by (3.7)), the L_1 analytic CFFT $T_q^{(1)}(F|X_\infty = \vec{\xi})$ of F exists and is given by the formula (3.5). □

By the Parseval's identity, we have the following corollary.

Corollary 3.4. *Let $F \in \mathcal{S}(L_2[0,T])$ be given by equation* (3.1)*, and given a complete orthonormal basis $\{e_1, e_2, \ldots\}$ of $L_2[0,T]$, let X_∞ be given by equation* (2.1)*. Then for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, it follows that*

$$T_q^{(1)}(F|X_\infty)(y,\vec{\xi}) = \int_{L_2[0,T]} \exp\left\{ i\langle u,y\rangle + i\sum_{n=1}^{\infty}\xi_n(u,e_n)_2 \right\} df(u)$$

for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$.

From the definition of the conditional Feynman integral and the L_1 analytic CFFT, it follows that $T_q^{(1)}(F|X_\infty)(0,\vec{\xi}) = E^{\mathrm{anf}_q}(F|X_\infty = \vec{\xi})$. We thus have the following corollary.

Corollary 3.5. *Let F and X_∞ be as in Theorem 3.3. Then the conditional analytic Feynman integral $E^{\mathrm{an}f_q}(F|X_\infty=\vec{\xi})$ of F exists for all $q\in\mathbb{R}\setminus\{0\}$ and a.e. $\vec{\xi}\in\mathbb{R}^\infty$, and is given by the formula*

$$E^{\mathrm{an}f_q}(F|X_\infty=\vec{\xi}) = \int_{L_2[0,T]}\exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{n=1}^\infty(u,e_n)_2^2\bigg]+i\sum_{n=1}^\infty\xi_n(u,e_n)_2\bigg\}df(u).$$

Remark 3.6. Given a function F in $\mathcal{S}(L_2[0,T])$ with the corresponding measure $f\in\mathcal{M}(L_2[0,T])$, and given a nonzero real number q and a vector $\vec{\xi}\in\mathbb{R}^\infty$, define a set function $f_{q,\vec{\xi}}:\mathcal{B}(L_2[0,T])\to\mathbb{C}$ by the formula

$$f_{q,\vec{\xi}}(U)=\int_U\exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{n=1}^\infty(u,e_n)_2^2\bigg]+i\sum_{n=1}^\infty\xi_n(u,e_n)_2\bigg\}df(u) \tag{3.8}$$

for each U in $\mathcal{B}(L_2[0,T])$, the Borel σ-field on $L_2[0,T]$. Then $f_{q,\vec{\xi}}$ is obviously a complex measure in $\mathcal{M}(L_2[0,T])$ and $\|f_{q,\vec{\xi}}\|=\|f\|$ for any $q\in\mathbb{R}\setminus\{0\}$ and $\vec{\xi}\in\mathbb{R}^\infty$. Then equation (3.5) can be rewritten by

$$T_q^{(1)}(F|X_\infty)(y,\vec{\xi})=\int_{L_2[0,T]}\exp\{i\langle u,y\rangle\}df_{q,\vec{\xi}}(u) \tag{3.9}$$

for SI-a.e. $y\in C_0[0,T]$, and so the L_1 analytic CFFT $T_q^{(1)}(F|X_\infty)(\,\cdot\,,\vec{\xi})$ of F with parameter q is an element of $\mathcal{S}(L_2[0,T])$ for each $\vec{\xi}\in\mathbb{R}^\infty$.

In view of Theorem 3.3 and Remark 3.6, we easily obtain the following theorem.

Theorem 3.7. *Let F and X_∞ be as in Theorem 3.3. Then,*

(i) *for any q in $\mathbb{R}\setminus\{0\}$, it follows that*

$$T_{-q}^{(1)}(T_q^{(1)}(F|X_\infty)(\,\cdot\,,\vec{\xi})|X_\infty)(y,-\vec{\xi})=F(y)$$

for SI-a.e. $y\in C_0[0,T]$ and a.e. $\vec{\xi}\in\mathbb{R}^\infty$; and

(ii) *for any finite sequence $\{q_1,\dots,q_m\}$ in $\mathbb{R}\setminus\{0\}$ which satisfies the condition*

$$\frac{1}{q_1}+\cdots+\frac{1}{q_k}\neq 0 \ \text{ for each } k\in\{1,\dots,m\}, \tag{3.10}$$

it follows that

$$\begin{aligned}&T_{q_m}^{(1)}\Big(T_{q_{m-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}(F|X_\infty)(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_\infty\Big)(\,\cdot\,,\vec{\xi}^{(m-1)})\Big|X_\infty\Big)(y,\vec{\xi}^{(m)})\\ &=T_{\alpha_m}^{(1)}(F|X_\infty)\bigg(y,\sum_{j=1}^m\vec{\xi}^{(j)}\bigg)\end{aligned} \tag{3.11}$$

for SI-a.e. $y \in C_0[0,T]$ *and a.e.* $(\vec{\xi}^{(1)}, \dots, \vec{\xi}^{(m)})$ *in* $(\mathbb{R}^\infty)^m$*, the product of* m *copies of* $\mathbb{R}^\infty$*, where*

$$\alpha_m = \left(\frac{1}{q_1} + \cdots + \frac{1}{q_m}\right)^{-1}. \tag{3.12}$$

Also, both of the expressions in (3.11) *are given by the expression*

$$\int_{L_2[0,T]} \exp\left\{ i\langle u, y\rangle - \frac{i}{2\alpha_m}\left[\|u\|_2^2 - \sum_{n=1}^\infty (u, e_n)_2^2\right] + i\sum_{j=1}^m \sum_{n=1}^\infty \xi_n^{(j)}(u, e_n)_2 \right\} df(u).$$

In our next theorem, we also establish the existence of the CCP of functions F and G in $\mathcal{S}(L_2[0,T])$.

Theorem 3.8. *Let* F *and* G *be the functions in* $\mathcal{S}(L_2[0,T])$ *with corresponding Borel measures* f *and* g*, respectively, in* $\mathcal{M}(L_2[0,T])$*, and let* X_∞ *be given by equation* (2.1)*. Then for a.e.* $\vec{\xi} \in \mathbb{R}^\infty$*, it follows that*

$$\begin{aligned}&[(F * G)_q | X_\infty](y, \vec{\xi}) \\ &= \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}\left[\|u-v\|_2^2 - \sum_{n=1}^\infty (u-v, e_n)_2^2\right]\right. \\ &\qquad\qquad \left. + \frac{i}{\sqrt{2}}\sum_{n=1}^\infty \xi_n (u-v, e_n)_2 \right\} df(u)dg(v)\end{aligned} \tag{3.13}$$

for all $q \in \mathbb{R}\setminus\{0\}$ *and SI-a.e.* $y \in C_0[0,T]$.

Proof. Using similar methods as those in the proof of Theorem 3.3, it follows equation (3.13) immediately by the definition of the CCP. □

Remark 3.9. Given two functions F and G in $\mathcal{S}(L_2[0,T])$ with the corresponding measures f and g in $\mathcal{M}(L_2[0,T])$, and given a nonzero real q and a vector $\vec{\xi} \in \mathbb{R}^\infty$, define a set function $\varphi_{q,\vec{\xi}} : \mathcal{B}(L_2[0,T] \times L_2[0,T]) \to \mathbb{C}$ by the formula

$$\begin{aligned}\varphi_{q,\vec{\xi}}(V) = \iint_V \exp\left\{ -\frac{i}{4q}\left[\|u-v\|_2^2 - \sum_{n=1}^\infty (u-v, e_n)_2^2\right]\right. \\ \left. + \frac{i}{\sqrt{2}}\sum_{n=1}^\infty \xi_n (u-v, e_n)_2 \right\} df(u)dg(v)\end{aligned} \tag{3.14}$$

for each V in $\mathcal{B}(L_2[0,T] \times L_2[0,T])$, the Borel σ-field on $L_2[0,T] \times L_2[0,T]$. Then $\varphi_{q,\vec{\xi}}$ is a complex measure on $\mathcal{B}(L_2[0,T] \times L_2[0,T])$. Define a function $\phi : L_2[0,T] \times L_2[0,T] \to L_2[0,T]$ by $\phi(u,v) = (u+v)/\sqrt{2}$. Then ϕ is a continuous function, and so it is $\mathcal{B}(L_2[0,T] \times L_2[0,T])$-measurable. Thus the set function $\varphi_{q,\vec{\xi}} \circ \phi^{-1} : L_2[0,T] \to \mathbb{C}$ is in $\mathcal{M}(L_2[0,T])$ obviously. Under these setting, equation (3.13) can be rewritten by

$$[(F * G)_q | X_\infty](y, \vec{\xi}) = \int_{L_2[0,T]} \exp\{i\langle w, y\rangle\} d\varphi_{q,\vec{\xi}} \circ \phi^{-1}(w)$$

for SI-a.e. $y \in C_0[0,T]$. Thus the CCP $[(F*G)_q|X_\infty](\,\cdot\,,\vec{\xi})$ of F and G is an element of $\mathcal{S}(L_2[0,T])$ for each $\vec{\xi} \in \mathbb{R}^\infty$.

4. Relationships between the CFFT and the CCP

In this section, we establish basic relationships between the CFFTs and the CCPs. The following theorem is one of our main assertions; namely that the CFFT of the CCP is the product of the CFFTs.

Theorem 4.1. *Let F, G, and X_∞ be as in Theorem 3.8. Then for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}(4.1)\qquad &T_q^{(1)}\Big([(F*G)_q|X_\infty](\,\cdot\,,\vec{\xi}^{(1)})\Big|X_\infty\Big)(y,\vec{\xi}^{(2)})\\ &= T_q^{(1)}(F|X_\infty)\bigg(\frac{y}{\sqrt{2}},\frac{\vec{\xi}^{(2)}+\vec{\xi}^{(1)}}{\sqrt{2}}\bigg)T_q^{(1)}(G|X_\infty)\bigg(\frac{y}{\sqrt{2}},\frac{\vec{\xi}^{(2)}-\vec{\xi}^{(1)}}{\sqrt{2}}\bigg).\end{aligned}$$

Proof. Using (3.9) with F and f replaced with $[(F*G)_q|X_\infty]$ and $\varphi_{q,\vec{\xi}^{(1)}}\circ\phi^{-1}$ respectively, (3.8) with f replaced with $\varphi_{q,\vec{\xi}^{(1)}}\circ\phi^{-1}$, (3.14), the Fubini theorem, and (3.5) together with simple calculations, it follows that

$$\begin{aligned}
&T_q^{(1)}\Big(\big[(F*G)_q\big|X_\infty\big](\,\cdot\,,\vec{\xi}^{(1)})\Big|X_\infty\Big)(y,\vec{\xi}^{(2)})\\
&= \int_{L_2[0,T]}\exp\{i\langle w,y\rangle\}d(\varphi_{q,\vec{\xi}^{(1)}}\circ\phi^{-1})_{q,\vec{\xi}^{(2)}}(w)\\
&= \int_{L_2[0,T]}\exp\bigg\{i\langle w,y\rangle-\frac{i}{2q}\bigg[\|w\|_2^2-\sum_{n=1}^\infty(w,e_n)_2^2\bigg]\\
&\qquad\qquad +i\sum_{n=1}^\infty\xi_n^{(2)}(w,e_n)_2\bigg\}d\varphi_{q,\vec{\xi}^{(1)}}\circ\phi^{-1}(w)\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]}\exp\bigg\{i\Big\langle\frac{u+v}{\sqrt{2}},y\Big\rangle-\frac{i}{2q}\bigg[\Big\|\frac{u+v}{\sqrt{2}}\Big\|_2^2-\sum_{n=1}^\infty\Big(\frac{u+v}{\sqrt{2}},e_n\Big)_2^2\bigg]\\
&\qquad\qquad +i\sum_{n=1}^\infty\xi_n^{(2)}\Big(\frac{u+v}{\sqrt{2}},e_n\Big)_2\bigg\}d\varphi_{q,\vec{\xi}^{(1)}}(u,v)\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]}\exp\bigg\{i\Big\langle\frac{u+v}{\sqrt{2}},y\Big\rangle\\
&\quad -\frac{i}{2q}\bigg[\Big\|\frac{u+v}{\sqrt{2}}\Big\|_2^2-\sum_{n=1}^\infty\Big(\frac{u+v}{\sqrt{2}},e_n\Big)_2^2\bigg]+i\sum_{n=1}^\infty\xi_n^{(2)}\Big(\frac{u+v}{\sqrt{2}},e_n\Big)_2\\
&\quad -\frac{i}{4q}\bigg[\|u-v\|_2^2-\sum_{n=1}^\infty(u-v,e_n)_2^2\bigg]+\frac{i}{\sqrt{2}}\sum_{n=1}^\infty\xi_n^{(1)}(u-v,e_n)_2\bigg\}df(u)dg(v)\\
&= \int_{L_2[0,T]}\exp\bigg\{i\Big\langle u,\frac{y}{\sqrt{2}}\Big\rangle
\end{aligned}$$

$$- \frac{i}{2q}\left[\|u\|_2^2 - \sum_{n=1}^{\infty}(u,e_n)_2^2\right] + i\sum_{n=1}^{\infty}\frac{\xi_n^{(2)}+\xi_n^{(1)}}{\sqrt{2}}(u,e_n)_2\biggr\}df(u)$$
$$\times \int_{L_2[0,T]} \exp\biggl\{i\biggl\langle v, \frac{y}{\sqrt{2}}\biggr\rangle$$
$$- \frac{i}{2q}\left[\|v\|_2^2 - \sum_{n=1}^{\infty}(v,e_n)_2^2\right] + i\sum_{n=1}^{\infty}\frac{\xi_n^{(2)}-\xi_n^{(1)}}{\sqrt{2}}(v,e_n)_2\biggr\}dg(v)$$
$$= T_q^{(1)}(F|X_\infty)\left(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}+\vec{\xi}^{(1)}}{\sqrt{2}}\right)T_q^{(1)}(G|X_\infty)\left(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}-\vec{\xi}^{(1)}}{\sqrt{2}}\right)$$

as desired. □

In order to provide our second main assertion of this section, we need the following lemma.

Lemma 4.2. *Let F, G, and X_∞ be as in Theorem 3.8. Then for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}&\left[\left(T_q^{(1)}(F|X_\infty)(\,\cdot\,,\vec{\xi}^{(1)}) * T_q^{(1)}(G|X_\infty)(\,\cdot\,,\vec{\xi}^{(2)})\right)_{-q}\Big|X_\infty\right](y,\vec{\xi}^{(3)})\\ &= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\biggl\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}\left[\|u+v\|_2^2 - \sum_{n=1}^{\infty}(u+v,e_n)_2^2\right]\\ &\quad + i\sum_{n=1}^{\infty}\left(\xi_n^{(1)} + \frac{\xi_n^{(3)}}{\sqrt{2}}\right)(u,e_n)_2 + i\sum_{n=1}^{\infty}\left(\xi_n^{(2)} - \frac{\xi_n^{(3)}}{\sqrt{2}}\right)(v,e_n)_2\biggr\}df(u)dg(v)\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&T_q^{(1)}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\Big|X_\infty\right)(y,\vec{\xi})\\ &= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\biggl\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}\left[\|u+v\|_2^2 - \sum_{n=1}^{\infty}(u+v,e_n)_2^2\right]\\ &\quad + i\sum_{n=1}^{\infty}\xi_n(u+v,e_n)_2\biggr\}df(u)dg(v).\end{aligned} \tag{4.3}$$

Proof. In view of Remark 3.6, we observe that

$$T_q^{(1)}(F|X_\infty)(y,\vec{\xi}^{(1)}) = \int_{L_2[0,T]} \exp\{i\langle u,y\rangle\}df_{q,\vec{\xi}^{(1)}}(u)$$

and

$$T_q^{(1)}(G|X_\infty)(y,\vec{\xi}^{(2)}) = \int_{L_2[0,T]} \exp\{i\langle v,y\rangle\}dg_{q,\vec{\xi}^{(2)}}(v), \tag{4.4}$$

where $f_{q,\vec{\xi}^{(1)}}$ is the complex measure in $\mathcal{M}(L_2[0,T])$ given by (3.8) with $\vec{\xi}$ replaced with $\vec{\xi}^{(1)}$, and $g_{q,\vec{\xi}^{(2)}}$ is the complex measure in $\mathcal{M}(L_2[0,T])$ given by the formula:

$$g_{q,\vec{\xi}^{(2)}}(U) = \int_U \exp\bigg\{-\frac{i}{2q}\bigg[\|v\|_2^2 - \sum_{n=1}^\infty (v,e_n)_2^2\bigg] + i\sum_{n=1}^\infty \xi_n^{(2)}(v,e_n)_2\bigg\}dg(v)$$

for each $U \in \mathcal{B}(L_2[0,T])$. Then using (3.13) with F, G, $\vec{\xi}$, f and g replaced with $T_q^{(1)}(F|X_\infty)(\cdot,\vec{\xi}^{(1)})$, $T_q^{(1)}(G|X_\infty)(\cdot,\vec{\xi}^{(2)})$, $\vec{\xi}^{(3)}$, $f_{q,\vec{\xi}^{(1)}}$ and $g_{q,\vec{\xi}^{(2)}}$, respectively, (3.9) with $\vec{\xi}$ replaced with $\vec{\xi}^{(1)}$, and (4.4), it follows equation (4.2) immediately.

Next, using the definition of the L_1 analytic CFFT, (2.4) with F replaced with $F((y+\cdot)/\sqrt{2})G((y+\cdot)/\sqrt{2})$, and the Fubini theorem, it follows that

$$\begin{aligned}
&T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_\infty\bigg)(y,\vec{\xi})\\
&= E^{\mathrm{an}f_q}\bigg[F\Big(\frac{y+\cdot}{\sqrt{2}}\Big)G\Big(\frac{y+\cdot}{\sqrt{2}}\Big)\bigg|X_\infty = \vec{\xi}\bigg]\\
&= E^{\mathrm{an}f_q}\bigg[F\bigg(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\bigg[x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\bigg]\bigg)\\
&\qquad\times G\bigg(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\bigg[x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\bigg]\bigg)\bigg]\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle + i\Big\langle u+v, \sum_{n=1}^\infty \xi_n\beta_n\Big\rangle\bigg\}\\
&\qquad\times E^{\mathrm{an}f_q}\bigg[\exp\bigg\{\frac{i}{\sqrt{2}}\Big\langle u+v, x - \sum_{n=1}^\infty \gamma_n(x)\beta_n\Big\rangle\bigg\}df(u)dg(v).
\end{aligned} \tag{4.5}$$

Applying (3.4) with w and ρ replaced with $u+v$ and $1/\sqrt{2}$ in the last expression of (4.5), it follows equation (4.3) as desired. □

Let $(\mathbb{R}^\infty)^4$ denote the product of four copies of $\mathbb{R}^\infty$. A close examination of the right-hand sides of (4.2) and (4.3) shows that they are equal if $(\vec{\xi},\vec{\xi}^{(1)},\vec{\xi}^{(2)},\vec{\xi}^{(3)}) \in (\mathbb{R}^\infty)^4$ is in the solution set of the system

$$\begin{cases}\vec{\xi} - \sqrt{2}\vec{\xi}^{(1)} - \vec{\xi}^{(3)} = \vec{0},\\ \vec{\xi} - \sqrt{2}\vec{\xi}^{(2)} + \vec{\xi}^{(3)} = \vec{0}.\end{cases} \tag{4.6}$$

Theorem 4.3. *Let F, G, and X_∞ be as in Theorem 3.8 and let $(\vec{\xi}, \vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \vec{\xi}^{(3)})$ satisfy the system* (4.6). *Then for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}
&\Big[\big(T_q(F|X_\infty)(\,\cdot\,,\vec{\xi}^{(1)}) * T_q(G|X_\infty)(\,\cdot\,,\vec{\xi}^{(2)})\big)_{-q}\Big|X_\infty\Big](y,\vec{\xi}^{(3)})\\
&= T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_\infty\bigg)(y,\vec{\xi}).
\end{aligned}$$

5. Further results: Iterated CFFT and CCP

In this section, we give more general relationships between the iterated CFFT and the CCP for functions in $\mathcal{S}(L_2[0,T])$.

Theorem 5.1. *Let F, G, and X_∞ be as in Theorem 3.8. Then for any finite sequence $\{q_1,\dots,q_m\}$ in $\mathbb{R}\setminus\{0\}$ which satisfies the condition* (3.10) *above, it follows that*

$$\begin{aligned} &T^{(1)}_{q_m}\Big(T^{(1)}_{q_{m-1}}\Big(\cdots T^{(1)}_{q_1}\Big([(F*G)_{\alpha_m}|X_\infty](\,\cdot\,,\vec{\eta})\Big|X_\infty\Big)(\,\cdot\,,\vec{\xi}^{(1)})\\ &\qquad\qquad\cdots\Big|X_\infty\Big)(\,\cdot\,,\vec{\xi}^{(m-1)})\Big|X_\infty\Big)(y,\vec{\xi}^{(m)})\\ &= T^{(1)}_{\alpha_m}\Big([(F*G)_{\alpha_m}|X_\infty](\,\cdot\,,\vec{\eta})\Big|X_\infty\Big)\Big(y,\sum_{j=1}^m\vec{\xi}^{(j)}\Big)\\ &= T^{(1)}_{\alpha_m}(F|X_\infty)\Big(\frac{y}{\sqrt2},\frac{1}{\sqrt2}\sum_{j=1}^m\vec{\xi}^{(j)}+\frac{\vec{\eta}}{\sqrt2}\Big)\\ &\quad\times T^{(1)}_{\alpha_m}(G|X_\infty)\Big(\frac{y}{\sqrt2},\frac{1}{\sqrt2}\sum_{j=1}^m\vec{\xi}^{(j)}-\frac{\vec{\eta}}{\sqrt2}\Big) \end{aligned}\tag{5.1}$$

for SI-a.e. $y\in C_0[0,T]$, where α_m is the real number given by (3.12).

Proof. Using (3.11) and (4.1) with q replaced with α_m, it follows equation (5.1) immediately. □

Theorem 5.2. *Let F, G, X_∞ and $\{q_1,\dots,q_m\}$ be as in Theorem 5.1. Let*

$$\Big\{\vec{\theta}^{(1)},\dots,\vec{\theta}^{(m)},\vec{\xi}^{(1)},\dots,\vec{\xi}^{(m)},\vec{\zeta}^{(1)},\dots,\vec{\zeta}^{(m)},\vec{\eta}\Big\}$$

be a subset of $\mathbb{R}^\infty$. If

$$\Big(\vec{\theta}^{(1)},\dots,\vec{\theta}^{(m)},\vec{\xi}^{(1)},\dots,\vec{\xi}^{(m)},\vec{\zeta}^{(1)},\dots,\vec{\zeta}^{(m)},\vec{\eta}\Big)$$

is in the solution set of the system

$$\begin{cases}\sum_{j=1}^m\vec{\theta}^{(j)}-\sqrt2\sum_{j=1}^m\vec{\xi}^{(j)}-\vec{\eta}=\vec{0},\\ \sum_{j=1}^m\vec{\theta}^{(j)}-\sqrt2\sum_{j=1}^m\vec{\zeta}^{(j)}+\vec{\eta}=\vec{0},\end{cases}$$

then it follows that

$$\begin{aligned} &\Big[\Big(T^{(1)}_{q_m}\Big(\cdots T^{(1)}_{q_1}(F|X_\infty)(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_\infty\Big)(\,\cdot\,,\vec{\xi}^{(m)})\\ &\quad * T^{(1)}_{q_m}\Big(\cdots T^{(1)}_{q_1}(G|X_\infty)(\,\cdot\,,\vec{\zeta}^{(1)})\cdots\Big|X_\infty\Big)(\,\cdot\,,\vec{\zeta}^{(m)})\Big)_{-q}\Big|X_\infty\Big](y,\vec{\eta})\\ &= T^{(1)}_{q_m}\Big(\cdots T^{(1)}_{q_1}\Big(F\Big(\frac{\cdot}{\sqrt2}\Big)G\Big(\frac{\cdot}{\sqrt2}\Big)\Big|X_\infty\Big)(\,\cdot\,,\vec{\theta}^{(1)})\cdots\Big|X_\infty\Big)(y,\vec{\theta}^{(m)}) \end{aligned}\tag{5.2}$$

for SI-a.e. $y\in C_0[0,T]$.

Proof. Using (3.11), it follows that

$$\begin{aligned}&\bigg[\bigg(T^{(1)}_{q_m}\bigg(\cdots T^{(1)}_{q_1}(F|X_\infty)(\,\cdot\,,\vec{\xi}^{(1)})\cdots\bigg|X_\infty\bigg)(\,\cdot\,,\vec{\xi}^{(m)})\\&\quad * T^{(1)}_{q_m}\bigg(\cdots T^{(1)}_{q_1}(G|X_\infty)(\,\cdot\,,\vec{\zeta}^{(1)})\cdots\bigg|X_\infty\bigg)(\,\cdot\,,\vec{\zeta}^{(m)})\bigg)_{-q}\bigg|X_\infty\bigg](y,\vec{\eta})\\&=\bigg[\bigg(T^{(1)}_{\alpha_m}(F|X_\infty)\bigg(\,\cdot\,,\sum_{j=1}^m\vec{\xi}^{(j)}\bigg) * T^{(1)}_{\alpha_m}(G|X_\infty)\bigg(\,\cdot\,,\sum_{j=1}^m\vec{\zeta}^{(j)}\bigg)\bigg)_{-q}\bigg|X_\infty\bigg](y,\vec{\eta})\end{aligned}$$

and

$$\begin{aligned}&T^{(1)}_{q_m}\bigg(\cdots T^{(1)}_{q_1}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_\infty\bigg)(\,\cdot\,,\vec{\theta}^{(1)})\cdots\bigg|X_\infty\bigg)(y,\vec{\theta}^{(m)})\\&=T^{(1)}_{\alpha_m}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg)\bigg|X_\infty\bigg)\bigg(y,\sum_{j=1}^m\vec{\theta}^{(j)}\bigg),\end{aligned}$$

where α_m is a nonzero real number given by (3.12). Then in view of Theorem 4.3, it follows equation (5.2). □

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