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DEFERRED STATISTICAL EQUIVALENCE FOR DOUBLE SEQUENCES OF SETS

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Abstract. The main purpose of this paper is to introduce the concept of asymptotical deferred statistical equivalence in the Wijsman sense for double set sequences. Also, we give some properties of this concept and prove some theorems associated with this concept. Furthermore, we examine the connection between the concepts of asymptotical deferred statistical and Cesàro equivalence in the Wijsman sense for double set sequences.

1. Introduction and Backgrounds

After the definition of convergence in the Pringsheim's sense for double sequences was given in [19], researchers have developed this concept using the concepts of statistical convergence, double lacunary sequence and asymptotical equivalence in [12, 17, 18].

The concept of deferred Cesàro mean for real (or complex) valued sequences, which introduced by Agnew [1], was extended to the double sequences by Dağadur and Sezgek [5]. Also, for double sequences, the concept of deferred statistical convergence was given in [5].

The concept of convergence in the Wijsman sense, that is one of the convergence concepts for set sequences, is taken as basis in this paper (see, [4, 23]). Nuray et al. [13] extended this concept to the double set sequences. Also, the concepts of statistical, lacunary statistical convergence and asymptotical statistical, lacunary statistical equivalence in the Wijsman sense for double set sequences were introduced in [14, 15, 16].

In [21], Ulusu and Gülle studied on new concepts which named deferred Cesàro mean ($D_{\varphi,\phi}$) and deferred statistical convergence in the Wijsman sense for double set sequences. Also, Ulusu [22] defined the concepts of asymptotical deferred Cesàro equivalence in the Wijsman sense for double set sequences.

For the basic properties and facts on these concepts, see [2, 3, 8, 6, 7, 9, 10, 11].

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Now, as for remind, the following basic concepts from [13, 14, 16, 20, 21, 23] needed in the course of the paper.

A double sequence $\theta_2 = \{(k_v, j_u)\}$ is said to be double lacunary sequence if there exists increasing integer sequences (k_v) and (j_u) such that

$$k_0 = 0,\ h_v = k_v - k_{v-1} \to \infty \quad \text{and}$$
$$j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \quad \text{as} \quad v, u \to \infty.$$

For a metric space $(\mathcal{X}, d)$, distance from x to A is denoted by $\rho_x(A)$ where

$$\rho_x(A) := \rho(x, A) = \inf_{a \in A} d(x, a)$$

for any $x \in \mathcal{X}$ and any non-empty $A \subseteq \mathcal{X}$.

For a non-empty set $\mathcal{X}$, a function $f : \mathbb{N} \to P_{\mathcal{X}}$ is defined by $f(n) = A_n \in P_{\mathcal{X}}$ for each $n \in \mathbb{N}$, where $P_{\mathcal{X}}$ denotes the power set of $\mathcal{X}$. The sequence $\{A_n\} = \{A_1, A_2, \ldots\}$, which consists of the range elements of the function f, is called set sequence.

Throughout the study, we will consider $(\mathcal{X}, d)$ as a separable metric space and A, A_{nm}, B_{nm} as any non-empty closed subsets of $\mathcal{X}$.

A double set sequence $\{A_{nm}\}$ is said to be bounded if $\sup_{n,m}\{\rho_x(A_{nm})\} < \infty$ for each $x \in \mathcal{X}$. Also, L^2_∞ denotes the class of all bounded double set sequences.

A double set sequence $\{A_{nm}\}$ is said to be convergent to a set A in the Wijsman sense if

$$\lim_{n,m\to\infty} \rho_x(A_{nm}) = \rho_x(A)$$

for each $x \in \mathcal{X}$.

A double set sequence $\{A_{nm}\}$ is said to be statistically convergent to a set A in the Wijsman sense if for every $\delta > 0$,

$$\lim_{j,i\to\infty} \frac{1}{j\,i} \big|\big\{(n, m) : n \le j,\ m \le i,\ |\rho_x(A_{nm}) - \rho_x(A)| \ge \delta\big\}\big| = 0$$

for each $x \in \mathcal{X}$.

For any non-empty closed subsets $\{A_{nm}\}, \{B_{nm}\} \in \mathcal{X}$ such that $\rho_x(A_{nm}) > 0$ and $\rho_x(B_{nm}) > 0$ for each $x \in \mathcal{X}$ and each $n, m \in \mathbb{N}$, the double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are said to be;

(i) asymptotical equivalent to multiple η in the Wijsman sense if

$$\lim_{n,m\to\infty} \frac{\rho_x(A_{nm})}{\rho_x(B_{nm})} := \lim_{n,m\to\infty} \rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) = \eta$$

for each $x \in \mathcal{X}$ and the notation $A_{nm} \overset{W_2^\eta}{\sim} B_{nm}$ is used for this case.

(ii) asymptotical statistically equivalent to multiple η in the Wijsman sense if for every $\delta > 0$,

$$\lim_{j,i\to\infty} \frac{1}{j\,i}\left|\left\{(n,m) : n \le j, m \le i, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}\right| = 0$$

for each $x \in \mathcal{X}$ and the notation $A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}$ is used for this case. The class of all asymptotical statistical equivalence double set sequences in the Wijsman sense is denoted by $\{W_2^\eta S\}$.

The deferred Cesàro mean $(D_{\varphi,\,\phi})$ in the Wijsman sense of a double set sequence $\mathcal{A} = \{A_{nm}\}$ is defined by

$$\begin{aligned}(D_{\varphi,\,\phi}\,\mathcal{A})_{vu} &= \frac{1}{\varphi(v)\phi(u)} \sum_{n=p(v)+1}^{r(v)} \sum_{m=q(u)+1}^{s(u)} \rho_x(A_{nm}) \\ &:= \frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1\\ m=q(u)+1}}^{r(v),s(u)} \rho_x(A_{nm})\end{aligned}$$

for each $x \in \mathcal{X}$, where $(p(v))$, $(r(v))$, $(q(u))$ and $(s(u))$ are sequences of non-negative integers satisfying following conditions:

$$p(v) < r(v),\ \lim_{v\to\infty} r(v) = \infty;\quad q(u) < s(u),\ \lim_{u\to\infty} s(u) = \infty \quad (1)$$

and

$$r(v) - p(v) = \varphi(v);\quad s(u) - q(u) = \phi(u). \quad (2)$$

Throughout the paper, unless otherwise specified, $(p(v))$, $(r(v))$, $(q(u))$ and $(s(u))$ are considered as sequences of non-negative integers satisfying (1) and (2).

A double set sequence $\{A_{nm}\}$ is said to be deferred Cesàro summable to a set A in the Wijsman sense if

$$\lim_{v,u\to\infty} \frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1\\ m=q(u)+1}}^{r(v),s(u)} \rho_x(A_{nm}) = \rho_x(A)$$

for each $x \in \mathcal{X}$.

A double set sequence $\{A_{nm}\}$ is said to be deferred statistically convergent to a set A in the Wijsman sense if for every $\delta > 0$,

$$\lim_{v,u\to\infty} \frac{1}{\varphi(v)\phi(u)} \Big|\big\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, |\rho_x(A_{nm}) - \rho_x(A)| \ge \delta\big\}\Big| = 0$$

for each $x \in \mathcal{X}$.

2. Main Results

In this section, we first define the concept of asymptotical deferred statistical equivalence in the Wijsman sense for double set sequences. Then, we give some properties of this concept and prove some theorems associated with this concept. Also, we examine the connection between the concepts of asymptotical deferred statistical and Cesàro equivalence in the Wijsman sense for double set sequences.

Throughout the section, we regard that $\rho_x(A_{nm}) > 0$ and $\rho_x(B_{nm}) > 0$ for each $x \in \mathcal{X}$ and each $n, m \in \mathbb{N}$.

Definition 2.1. The double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are said to be asymptotically deferred statistical equivalent to multiple η in the Wijsman sense if for every $\delta > 0$

$$\lim_{v,u\to\infty} \frac{1}{\varphi(v)\phi(u)} \left| \left\{ (n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left| \rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta \right| \geq \delta \right\} \right| = 0$$

for each $x \in \mathcal{X}$. The notation $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$ is used for this case and simply call these sequences asymptotically deferred statistical equivalent in the Wijsman sense if $\eta = 1$.

The class of all double set sequences that asymptotically deferred statistical equivalent (to multiple η) in the Wijsman sense is denoted by $\{W_2^\eta DS\}$.

Remark 2.2.

(i) For $p(v) = 0$, $r(v) = v$ and $q(u) = 0$, $s(u) = u$, the concept of asymptotically deferred statistical equivalence (to multiple η) in the Wijsman sense coincides with the concept of Wijsman asymptotically statistical equivalence (to multiple η) for double set sequences in [14].

(ii) For $p(v) = k_{v-1}$, $r(v) = k_v$ and $q(u) = j_{u-1}$, $s(u) = j_u$ where $\{(k_v, j_u)\}$ is a double lacunary sequence, the concept of asymptotically deferred statistical equivalence (to multiple η) in the Wijsman sense coincides with the concept of Wijsman asymptotically lacunary statistical equivalence (to multiple η) for double set sequences in [14].

Example 2.3. Let take $\mathcal{X} = \mathbb{R}^2$ and define the double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ as follows:

$$A_{nm} := \begin{cases} \left\{(x_1, y_1) \in \mathbb{R}^2 : x_1^2 + (y_1 + \frac{3}{2})^2 = \frac{1}{nm}\right\} & ; \text{ if } p(v) < n \leq r(v), \\ & \quad q(u) < m \leq s(u) \text{ and} \\ & \quad n, m \text{ are square integers} \\ \{(0,0)\} & ; \text{ in other cases} \end{cases}$$

and

$$B_{nm} := \begin{cases} \left\{(x_1,y_1)\in\mathbb{R}^2 : x_1^2+(y_1-\frac{3}{2})^2=\frac{1}{nm}\right\} & ; \text{ if } p(v)<n\le r(v),\\ & \quad q(u)<m\le s(u) \text{ and}\\ & \quad n,m \text{ are square integers}\\ \{(0,0)\} & ; \text{ in other cases}. \end{cases}$$

Since

$$\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-1\right|\ge\delta\right\}\right|$$
$$\le \frac{\sqrt{\varphi(v)\phi(u)}}{\varphi(v)\phi(u)} \to 0 \quad (v,u\to\infty)\,,$$

we get $A_{nm} \overset{W_2^1 DS}{\sim} B_{nm}$.

Corollary 2.4. If $A_{nm} \overset{W_2^\eta}{\sim} B_{nm}$, then $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$.

Theorem 2.5. *Let $A_{nm}\subseteq B_{nm}$ for all $n,m\in\mathbb{N}$. If $A_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$, then $B_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$.*

Proof. Since $A_{nm}\subseteq B_{nm}$ for all $n,m\in\mathbb{N}$, we have

$$A_{nm}\subseteq B_{nm} \Rightarrow \rho_x(B_{nm})\le\rho_x(A_{nm}) \quad (\text{for each } x\in\mathcal{X})$$
$$\Rightarrow \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big)-\eta\right| \le \left|\rho_x\Big(\frac{A_{nm}}{C_{nm}}\Big)-\eta\right|.$$

Suppose that $A_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$. For every $\delta>0$, the inclusion

$$\left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big)-\eta\right|\ge\delta\right\}$$
$$\subseteq \left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{C_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

is hold. Hence, the following inequality is obtained:

$$\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big)-\eta\right|\ge\delta\right\}\right|$$
$$\le \frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{C_{nm}}\Big)-\eta\right|\ge\delta\right\}\right|.$$

Since by our assumption, right side of the above inequality convergent to 0 for $v,u\to\infty$. Thus, we get $B_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$. □

Theorem 2.6. *Let $A_{nm}\subseteq C_{nm}$ for all $n,m\in\mathbb{N}$. If $B_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$, then $B_{nm} \overset{W_2^\eta DS}{\sim} A_{nm}$.*

Proof. Since $A_{nm} \subseteq C_{nm}$ for all $n, m \in \mathbb{N}$, we have

$$A_{nm} \subseteq C_{nm} \Rightarrow \rho_x(C_{nm}) \leq \rho_x(A_{nm}) \quad (\text{for each } x \in \mathcal{X})$$
$$\Rightarrow \left|\rho_x\Big(\frac{B_{nm}}{A_{nm}}\Big) - \eta\right| \leq \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big) - \eta\right|.$$

Suppose that $B_{nm} \overset{W_2^\eta DS}{\sim} C_{nm}$. For every $\delta > 0$, the inclusion

$$\left\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{A_{nm}}\Big) - \eta\right| \geq \delta\right\}$$
$$\subseteq \left\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big) - \eta\right| \geq \delta\right\}$$

is hold. Hence, the following inequality is obtained:

$$\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{A_{nm}}\Big) - \eta\right| \geq \delta\right\}\right|$$
$$\leq \frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{C_{nm}}\Big) - \eta\right| \geq \delta\right\}\right|.$$

Since by our assumption, right side of the above inequality convergent to 0 for $v, u \to \infty$. Thus, we get $B_{nm} \overset{W_2^\eta DS}{\sim} A_{nm}$. □

Theorem 2.7. *Let $\{A_{nm}\}$, $\{B_{nm}\}$ and $\{C_{nm}\}$ be double set sequences such that $A_{nm} \subset B_{nm} \subset C_{nm}$ for all $n, m \in \mathbb{N}$. If*

$$A_{nm} \overset{W_2^\eta DS}{\sim} F_{nm} \text{ and } C_{nm} \overset{W_2^\eta DS}{\sim} F_{nm}\,,$$

then, $B_{nm} \overset{W_2^\eta DS}{\sim} F_{nm}$.

Proof. Since $A_{nm} \subset B_{nm} \subset C_{nm}$ for all $n, m \in \mathbb{N}$, we have

$$A_{nm} \subset B_{nm} \subset C_{nm}$$
$$\Rightarrow \rho_x(C_{nm}) \leq \rho_x(B_{nm}) \leq \rho_x(A_{nm}) \quad (\text{for each } x \in \mathcal{X})$$
$$\Rightarrow \left|\rho_x\Big(\frac{C_{nm}}{F_{nm}}\Big) - \eta\right| \leq \left|\rho_x\Big(\frac{B_{nm}}{F_{nm}}\Big) - \eta\right| \leq \left|\rho_x\Big(\frac{A_{nm}}{F_{nm}}\Big) - \eta\right|.$$

Suppose that $A_{nm} \overset{W_2^\eta DS}{\sim} F_{nm}$ and $C_{nm} \overset{W_2^\eta DS}{\sim} F_{nm}$. Considering the above inequality, for every $\delta > 0$, we can write

$$\left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{F_{nm}}\Big) - \eta\right| \ge \delta\right\}$$

$$= \left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \rho_x\Big(\frac{B_{nm}}{F_{nm}}\Big) \ge \eta + \delta\right\}$$

$$\cup \left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \rho_x\Big(\frac{B_{nm}}{F_{nm}}\Big) \le \eta - \delta\right\}$$

$$\subseteq \left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \rho_x\Big(\frac{A_{nm}}{F_{nm}}\Big) \ge \eta + \delta\right\}$$

$$\cup \left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \rho_x\Big(\frac{C_{nm}}{F_{nm}}\Big) \le \eta - \delta\right\}.$$

So, the following inequality is hold:

$$\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \left|\rho_x\Big(\frac{B_{nm}}{F_{nm}}\Big) - \eta\right| \ge \delta\right\}\right|$$

$$\le \frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{F_{nm}}\Big) - \eta\right| \ge \delta\right\}\right|$$

$$+ \frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m) : p(v) < n \le r(v),\, q(u) < m \le s(u),\, \left|\rho_x\Big(\frac{C_{nm}}{F_{nm}}\Big) - \eta\right| \ge \delta\right\}\right|.$$

Since by our assumption, both terms on right side of the above inequality convergent to 0 for $v, u \to \infty$. Thus, we get $B_{nm} \overset{W_2^\eta DS}{\sim} F_{nm}$. □

In the next two theorems, we compare the sets $\{W_2^\eta DS\}$ and $\{W_2^\eta S\}$.

Theorem 2.8. *Let* $\left(\frac{p(v)}{\varphi(v)}\right)$ *and* $\left(\frac{q(u)}{\phi(u)}\right)$ *be bounded. If* $A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}$, *then* $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$.

Proof. Assume that $A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}$. Then, for every $\delta > 0$, as per our assumption

$$\lim_{j,i\to\infty} \frac{1}{j\,i}\left|\left\{(n,m) : n \le j,\, m \le i,\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right| \ge \delta\right\}\right| = 0$$

for each $x \in \mathcal{X}$. Also, we can easily write the following limit

$$\lim_{v,u\to\infty} \frac{1}{r(v)s(u)}\left|\left\{(n,m) : n \le r(v),\, m \le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right| \ge \delta\right\}\right| = 0. \quad (3)$$

Since

$$\Big\{(n,m): p(v) < n \le r(v),\, q(u) < m \le s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \ge \delta\Big\}$$
$$\subseteq \Big\{(n,m): n \le r(v),\, m \le s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \ge \delta\Big\},$$

we have following result considering the limit (3)

$$\frac{1}{\varphi(v)\phi(u)}\Big|\Big\{(n,m): p(v) < n \le r(v),\, q(u) < m \le s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \ge \delta\Big\}\Big|$$
$$\le \frac{r(v)s(u)}{\varphi(v)\phi(u)}\,\frac{1}{r(v)s(u)}\Big|\Big\{(n,m): n \le r(v),\, m \le s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \ge \delta\Big\}\Big|$$
$$= \Big(1 + \frac{p(v)}{\varphi(v)}\Big)\Big(1 + \frac{q(u)}{\phi(u)}\Big)\cdot$$
$$\frac{1}{r(v)s(u)}\Big|\Big\{(n,m): n \le r(v),\, m \le s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \ge \delta\Big\}\Big|.$$

Since by our assumptions, right side of the above inequality convergent to 0 for $v, u \to \infty$. Thus, we get $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$. □

Theorem 2.9. *Let $r(v) = v$ and $s(u) = u$ for all $v, u \in \mathbb{N}$. If $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$, then $A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}$.*

Proof. Assume that $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$. Applying the technique given in [1], the following sequences can be defined

$$p(v) = v_{(1)} > p(v_{(1)}) = v_{(2)} > p(v_{(2)}) = v_{(3)} > \dots$$
$$q(u) = u_{(1)} > q(u_{(1)}) = u_{(2)} > q(u_{(2)}) = u_{(3)} > \dots$$

for all $v, u \in \mathbb{N}$.

For every $\delta > 0$, we have

$$\begin{aligned}
&\left\{(n,m) : n \le v,\, m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&= \left\{(n,m) : n \le v_{(1)},\, m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : v_{(1)} < n \le v,\, m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : n \le v_{(1)},\, u_{(1)} < m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : v_{(1)} < n \le v,\, u_{(1)} < m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}
\end{aligned}$$

for each $x \in \mathcal{X}$. With similar logic, some of the above sets can be written as:

$$\begin{aligned}
&\left\{(n,m) : n \le v_{(1)},\, m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&= \left\{(n,m) : n \le v_{(2)},\, m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : v_{(2)} < n \le v_{(1)},\, m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : n \le v_{(2)},\, u_{(2)} < m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : v_{(2)} < n \le v_{(1)},\, u_{(2)} < m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\},
\end{aligned}$$

$$\begin{aligned}
&\left\{(n,m) : v_{(1)} < n \le v,\, m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&= \left\{(n,m) : v_{(1)} < n \le v,\, m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\} \\
&\cup \left\{(n,m) : v_{(1)} < n \le v,\, u_{(2)} < m \le u_{(1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\},
\end{aligned}$$

and

$$\left\{(n,m): n \le v_{(1)},\, u_{(1)} < m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$= \left\{(n,m): n \le v_{(2)},\, u_{(1)} < m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): v_{(2)} < n \le v_{(1)},\, u_{(1)} < m \le u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}.$$

Similarly,

$$\left\{(n,m): n \le v_{(2)},\, m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$= \left\{(n,m): n \le v_{(3)},\, m \le u_{(3)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): v_{(3)} < n \le v_{(2)},\, m \le u_{(3)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): n \le v_{(3)},\, u_{(3)} < m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): v_{(3)} < n \le v_{(2)},\, u_{(3)} < m \le u_{(2)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}.$$

If this process is continued in this way, then

$$\left\{(n,m): n \le v_{(\alpha-1)},\, m \le u_{(\beta-1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$= \left\{(n,m): n \le v_{(\alpha)},\, m \le u_{(\beta)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): v_{(\alpha)} < n \le v_{(\alpha-1)},\, m \le u_{(\beta)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): n \le v_{(\alpha)},\, u_{(\beta)} < m \le u_{(\beta-1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

$$\cup \left\{(n,m): v_{(\alpha)} < n \le v_{(\alpha-1)},\, u_{(\beta)} < m \le u_{(\beta-1)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \ge \delta\right\}$$

is obtained for fixed positive integers α, β such that $v_{(\alpha)} \ge 1,\ v_{(\alpha+1)} = 0$ and $u_{(\beta)} \ge 1,\ u_{(\beta+1)} = 0$.

Thus, for all $v, u \in \mathbb{N}$, we have

$$\frac{1}{vu}\left|\left\{(n,m) : n \leq v,\, m \leq u,\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \geq \delta\right\}\right| = \sum_{(t,\,s)=(0,0)}^{(\alpha+1,\ \beta+1)} \frac{(v_{(t)} - v_{(t+1)})(u_{(s)} - u_{(s+1)})}{vu} M_{vu}$$

where

$$M_{vu} := \frac{\left|\left\{(n,m) : v_{(t+1)} < n \leq v_{(t)},\, u_{(s+1)} < m \leq u_{(s)},\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \geq \delta\right\}\right|}{(v_{(t)} - v_{(t+1)})(u_{(s)} - u_{(s+1)})}$$

for each $x \in \mathcal{X}$. Let set a following matrix $\mathcal{H}_{vu}^{ts}$,

$$\mathcal{H}_{vu}^{ts} := \begin{cases} \dfrac{(v_{(t)} - v_{(t+1)})(u_{(s)} - u_{(s+1)})}{vu} & , \quad \begin{array}{l} t = 0,1,2,\dots,\alpha \\ s = 0,1,2,\dots,\beta \end{array}, \\ 0 & , \quad if\, not, \end{cases}$$

where $v_{(0)} = v$ and $u_{(0)} = u$. Then, asymptotically statistical equivalence (to multiple η) of the double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ is equivalent to equivalence of the transform under the regular matrix $\mathcal{H}_{vu}^{ts}$ of the sequence $\{M_{vu}\}$. Since by our assumption, we get $A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}$ for $v, u \to \infty$. □

Considering both Theorem 2.8 and Theorem 2.9, the following corollary can be easily obtained.

Corollary 2.10. *Let* $\left\{\frac{p(v)}{v-p(v)}\right\}$ *and* $\left\{\frac{q(u)}{u-q(u)}\right\}$ *be bounded. Then,*

$$A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm} \Leftrightarrow A_{nm} \overset{W_2^\eta S}{\sim} B_{nm}.$$

Now, we will give some new theorems under the following constraints

$$p(v) \leq p'(v) < r'(v) \leq r(v) \quad \text{and} \quad q(u) \leq q'(u) < s'(u) \leq s(u)$$

for all $v, u \in \mathbb{N}$ where all of these are sequences of non-negative integers.

Theorem 2.11. *Let* $\left(\frac{\varphi(v)\,\phi(u)}{\varphi'(v)\,\phi'(u)}\right)$ *be bounded. Then,* $A_{nm} \overset{W_2^\eta DS_{[\varphi,\phi]}}{\sim} B_{nm}$ *implies that* $A_{nm} \overset{W_2^\eta DS_{[\varphi',\phi']}}{\sim} B_{nm}$.

Proof. Assume that $A_{nm} \overset{W_2^\eta DS_{[\varphi,\phi]}}{\sim} B_{nm}$. Also, for every $\delta > 0$ and each $x \in \mathcal{X}$, since

$$\left\{(n,m) : p'(v) < n \leq r'(v),\, q'(u) < m \leq s'(u),\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \geq \delta\right\}$$

$$\subset \left\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right) - \eta\right| \geq \delta\right\},$$

we have

$$\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m): p'(v)<n\le r'(v),\, q'(u)<m\le s'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}\right|$$

$$\le \frac{\varphi(v)\,\phi(u)}{\varphi'(v)\,\phi'(u)}\left(\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m):\right.\right.\right.$$

$$\left.\left.\left. p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}\right|\right).$$

Since by our assumptions, right side of the above inequality convergent to 0 for $v,u\to\infty$. Thus, we get $A_{nm} \overset{W_2^\eta DS_{[\varphi',\phi']}}{\sim} B_{nm}$. □

Theorem 2.12. *Let the sets* $\{n : p(v) < n \le p'(v)\}$, $\{n : r'(v) < n \le r(v)\}$, $\{m : q(u) < m \le q'(u)\}$ *and* $\{m : s'(u) < m \le s(u)\}$ *be finite for all* $v,u\in\mathbb{N}$. *Then,* $A_{nm} \overset{W_2^\eta DS_{[\varphi',\phi']}}{\sim} B_{nm}$ *implies that* $A_{nm} \overset{W_2^\eta DS_{[\varphi,\phi]}}{\sim} B_{nm}$.

Proof. Assume that $A_{nm} \overset{W_2^\eta DS_{[\varphi',\phi']}}{\sim} B_{nm}$. Also, for every $\delta>0$ and each $x\in\mathcal{X}$, since

$$\left\{(n,m): p(v)<n\le r(v),\, q(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$=\left\{(n,m): p(v)<n\le p'(v),\, q(u)<m\le q'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): p(v)<n\le p'(v),\, q'(u)<m\le s'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): p(v)<n\le p'(v),\, s'(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): p'(v)<n\le r'(v),\, q(u)<m\le q'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): p'(v)<n\le r'(v),\, q'(u)<m\le s'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): p'(v)<n\le r'(v),\, s'(u)<m\le s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m): r'(v)<n\le r(v),\, q(u)<m\le q'(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m):r'(v)<n\le r(v),\,q'(u)<m\le s'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}$$

$$\cup\left\{(n,m):r'(v)<n\le r(v),\,s'(u)<m\le s(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\},$$

we have

$$\begin{aligned}
&\frac{1}{\varphi(v)\phi(u)}\left|\left\{(n,m):p(v)<n\le r(v),\,q(u)<m\le s(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&\le\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p(v)<n\le p'(v),\,q(u)<m\le q'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p(v)<n\le p'(v),\,q'(u)<m\le s'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p(v)<n\le p'(v),\,s'(u)<m\le s(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p'(v)<n\le r'(v),\,q(u)<m\le q'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p'(v)<n\le r'(v),\,q'(u)<m\le s'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):p'(v)<n\le r'(v),\,s'(u)<m\le s(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):r'(v)<n\le r(v),\,q(u)<m\le q'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):r'(v)<n\le r(v),\,q'(u)<m\le s'(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|\\
&+\frac{1}{\varphi'(v)\phi'(u)}\left|\left\{(n,m):r'(v)<n\le r(v),\,s'(u)<m\le s(u),\,\left|\rho_x\left(\frac{A_{nm}}{B_{nm}}\right)-\eta\right|\ge\delta\right\}\right|.
\end{aligned}$$

Since by our assumptions, all terms on right side of the above inequality convergent to 0 for $v,u\to\infty$. Thus, we get $A_{nm}\overset{W_2^\eta DS_{[\varphi,\phi]}}{\sim}B_{nm}$. □

In the last two theorems, we present the relation between the sets $\{W_2^\eta DS\}$ and $\{W_2^\eta D\}$. For this, firstly, we recall the concept of asymptotical deferred Cesàro equivalence to multiple η in the Wijsman sense for double set sequences in [22].

Definition 2.13. [22] The double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are said to be asymptotically deferred Cesàro equivalent to multiple η in the Wijsman sense if

$$\lim_{v,u\to\infty} \frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1}}^{r(v),s(u)} \rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) = \eta$$

for each $x \in \mathcal{X}$. The notation $A_{nm} \overset{W_2^\eta D}{\sim} B_{nm}$ is used for this case and simply call these sequences asymptotically deferred Cesàro equivalent in the Wijsman sense if $\eta = 1$.

The class of all double set sequences that asymptotically deferred Cesàro equivalent (to multiple η) in the Wijsman sense is denoted by $\{W_2^\eta D\}$.

Theorem 2.14. *If sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are asymptotically deferred Cesàro equivalent to multiple η in the Wijsman sense, then these sequences are asymptotically deferred statistical equivalent to multiple η in the Wijsman sense.*

Proof. Suppose that $A_{nm} \overset{W_2^\eta D}{\sim} B_{nm}$. Then, for every $\delta > 0$ and each $x \in \mathcal{X}$ we have

$$\sum_{\substack{n=p(v)+1 \\ m=q(u)+1}}^{r(v),s(u)} \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big|$$

$$\geq \sum_{\substack{n=p(v)+1 \\ m=q(u)+1 \\ |\rho_x(\frac{A_{nm}}{B_{nm}})-\eta|\geq\delta}}^{r(v),s(u)} \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big|$$

$$\geq \delta \Big|\Big\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \geq \delta\Big\}\Big|$$

and so

$$\frac{1}{\delta}\frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1}}^{r(v),s(u)} \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big|$$

$$\geq \frac{1}{\varphi(v)\phi(u)}\Big|\Big\{(n,m) :$$

$$p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \geq \delta\Big\}\Big|.$$

Since by our assumption, left side of the above inequality convergent to 0 for $v, u \to \infty$. Thus, we get $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$. □

The converse of Theorem 2.14 is not true in general. We can consider the following example to explain this situation.

Example 2.15. Let take $\mathcal{X} = \mathbb{R}^2$ and define the double set sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ as follows:

$$A_{nm} := \begin{cases} \big\{(x_1, y_1) \in \mathbb{R}^2 : x_1^2 - nmx_1 + y_1^2 = 0\big\} & ; \text{ if } p(v) < n \le r(v), \\ & \quad q(u) < m \le s(u) \text{ and} \\ & \quad n, m \text{ are square integers} \\ \{(-\frac{1}{2}, 0)\} & ; \text{ in other cases} \end{cases}$$

and

$$B_{nm} := \begin{cases} \big\{(x_1, y_1) \in \mathbb{R}^2 : x_1^2 + nmx_1 + y_1^2 = 0\big\} & ; \text{ if } p(v) < n \le r(v), \\ & \quad q(u) < m \le s(u) \text{ and} \\ & \quad n, m \text{ are square integers} \\ \{(-\frac{1}{2}, 0)\} & ; \text{ in other cases}\,. \end{cases}$$

The sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are not bounded. Also, these are not asymptotically deferred Cesàro equivalent in the Wijsman sense since

$$\frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1}}^{r(v),s(u)} \rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - 1 \nrightarrow 0 \quad (v, u \to \infty).$$

But these sequences are asymptotically deferred statistical equivalent in the Wijsman sense, since

$$\frac{1}{\varphi(v)\phi(u)}\bigg|\bigg\{(n,m) : p(v) < n \le r(v), \\ q(u) < m \le s(u), \ \Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - 1\Big| \ge \delta\bigg\}\bigg| \to 0 \quad (v, u \to \infty).$$

Theorem 2.16. *Let $\{A_{nm}\}, \{B_{nm}\} \in L_\infty^2$. If sequences $\{A_{nm}\}$ and $\{B_{nm}\}$ are asymptotically deferred statistical equivalent to multiple η in the Wijsman sense, then these sequences are asymptotically deferred Cesàro equivalent to multiple η in the Wijsman sense.*

Proof. Since $\{A_{nm}\}, \{B_{nm}\} \in L_\infty^2$, there is a positive real number $\mathcal{T}$ such that

$$\Big|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\Big| \le \mathcal{T}$$

for each $x \in \mathcal{X}$ and each $n, m \in \mathbb{N}$.

Suppose that $A_{nm} \overset{W_2^\eta DS}{\sim} B_{nm}$. For every $\delta > 0$, we have

$$\frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1}}^{r(v),s(u)} \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right|$$

$$= \frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1 \\ |\rho_x(\frac{A_{nm}}{B_{nm}})-\eta| \geq \delta}}^{r(v),s(u)} \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right|$$

$$+ \frac{1}{\varphi(v)\phi(u)} \sum_{\substack{n=p(v)+1 \\ m=q(u)+1 \\ |\rho_x(\frac{A_{nm}}{B_{nm}})-\eta| < \delta}}^{r(v),s(u)} \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right|$$

$$\leq \frac{\mathcal{T}}{\varphi(v)\phi(u)} \Bigg|\Bigg\{(n,m) : p(v) < n \leq r(v),\, q(u) < m \leq s(u),\, \left|\rho_x\Big(\frac{A_{nm}}{B_{nm}}\Big) - \eta\right| \geq \delta\Bigg\}\Bigg| + \delta$$

for each $x \in \mathcal{X}$. Since by our assumption, right side of the above inequality convergent to 0 for $v, u \to \infty$. Thus, we get $A_{nm} \overset{W_2^\eta D}{\sim} B_{nm}$. □

Corollary 2.17. $\{W_2^\eta D\} \bigcap L_\infty^2 = \{W_2^\eta DS\} \bigcap L_\infty^2$.

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