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DIFFERENTIABILITY OF NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION WITH RESPECT TO THE INITIAL DATA

Zakaria Boumezbeur and Hacène Boutabia*

Abstract. This paper deals with differentiability of solutions of neutral stochastic differential equations with respect to the initial data in the G-framework. Since the initial data belongs to the space $BC\left([-r,0];\mathbb{R}^n\right)$ of bounded continuous $\mathbb{R}^n$-valued functions defined on $[-r,0]\,(r>0)$, the derivative belongs to the Banach space $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$ of linear bounded operators from $BC\left([-r,0];\mathbb{R}^n\right)$ to $\mathbb{R}^n$. We give the neutral stochastic differential equation of the derivative. In addition, we exhibit two examples confirming the accuracy of the obtained results.

Motivation

In the theory of differential equations, the problem of differentiability with respect to initial data of system of the form

$$\begin{cases} dX\left(t\right)=f\left(t,X\left(t\right)\right)dt \\ \quad X\left(t_0\right)=x \end{cases},$$

is investigated by many authors such as Slavík [16]. It is known that the derivative satisfies the so called variational equation, which is used in determining the value of the derivative. Hartung in [7] obtained the differentiability of neutral differential equations (NDEs) with respect to initial data and parameters. The differentiability with respect to initial data is needed in order to guarantee the existence of a C^1 smooth solution semiflow, see [8]. Furthermore, problems of differentiability of stochastic differential equations (SDEs) are also studied, see [2, 10].

To motivate our work, we consider a neutral differential equation which describes the dynamique of heat conduction material with memory where the

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*Corresponding author

memory argument arises in the derivative of state variable as well as independent variable. When the noise is big we can not obtain a complete information about the parameters e.g. the diffusion coefficient σ, but only we know that it belongs to an interval $[\sigma_{\min}, \sigma_{\max}]$, under these circumstances, if we want to measure the sensitivity of the solution of the stochastic system with respect to the initial data, this will lead us to investigate the problem of differentiability of solution with respect to its initial data, and a typical formulation of such problems can be done by using neutral stochastic differential equations driven by the G-Brownian motion.

1. Introduction

Motivated by the uncertainty problems, risk measures and the super-hedging in finance, Peng in [12] introduced a sublinear expectation, called G-expectation generated by a nonlinear heat equation with a given generator function G, G-Brownian motion $(B_t)_{t\geq 0}$ and the corresponding stochastic calculus. Subsequently, SDEs driven by G-Brownian motion (G-SDEs) was introduced by [13]. Since then, there is an increasing interest in G-SDEs, see [17] and references therein.

Among the obtained results on the G-SDEs, we mention the differentiability of the solutions with respect to initial data, which was studied first by Lin [11] in 2013, and recently generalized to d-dimensional case by Bougherra et al. [3]. The differentiability might be helpful to obtain maximum principle for stochastic optimal control systems.

In this context, we consider the problem of differentiability of solutions of neutral stochastic differential equations driven by d-dimensional G-Brownian motion (G-NSDEs) with respect to the initial data, namely

$$\begin{cases} d\left[X(t) - Q(t, X_t)\right] = \sum_{l=0}^{d} A_l(t, X_t) dB_t^l + \sum_{i,j=1}^{d} A_{i,j}(t, X_t) d\left\langle B^i, B^j \right\rangle_t \\ X_0 = x := (x(\theta))_{-r\leq\theta\leq 0} \in BC\left([-r, 0]; \mathbb{R}^n\right) \end{cases}$$

where $B_t^0 = t$, $X_t = (X(t+\theta))_{-r\leq\theta\leq 0} \in BC\left([-r, 0]; \mathbb{R}^n\right)$ and the functions

$$A_l, A_{i,j}, Q : [0, T] \times BC\left([-r, 0]; \mathbb{R}^n\right) \times \Omega \to \mathbb{R}^n,$$

fulfill some of the regularity conditions which we will state later. Unlike the aforementioned studies, the initial data here is not a point of the Euclidean space $\mathbb{R}^n$, but a function of the Banach space $BC\left([-r, 0]; \mathbb{R}^n\right)$ which makes this research different and more general. Let $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$ be the space of linear bounded operators from $BC\left([-r, 0]; \mathbb{R}^n\right)$ to $\mathbb{R}^n$. The novelty in this paper is as follows: Firstly, we define the G-NSDEs where the solutions belong to $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$. Secondly, we prove under suitable hypotheses that the solution $X\left(t, x\right)$ is differentiable with respect to $x \in BC\left([-r, 0]; \mathbb{R}^n\right)$. In this paper we mean by differentiability, Fréchet-differentiability.

Through this paper, we have established a link between our result and the findings obtained by Bougherra et al. Therefore, the current research is an extension of previous studies to the class of G-NSDEs.

The present paper organized as follows. In Section 2, we recall briefly some preliminary results in the G-framework. In Section 3, we formulate the problem and the hypotheses. Also, we give the G-NSDE of the derivative $Y(t,x)$. Section 4 is dedicated to the main results and the proofs. We prove that the solution $X(t,x)$ is differentiable with respect to x. Moreover, the G-NSDE of the derivative is $Y(t,x)$. In the last section, we present two examples to illustrate the theoretically generated results.

2. Preliminaries

This section is devoted to introduce some notations, basic definitions and lemmas, concerned the theory of G-expectation and the related stochastic calculus. The reader interested in a more details, is referred to [9, 13, 14, 15].

Let Ω be a nonempty set, and $\mathcal{H}$ be a linear space of real valued functions defined on Ω, such that $\mathcal{H}$ satisfies $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$. $\mathcal{H}$ is the space of random variables.

Definition 2.1. *A functional $\mathbb{E} : \mathcal{H} \longrightarrow \mathbb{R}$ is called sublinear expectation, if for all X, $Y \in \mathcal{H}$, $c \in \mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:*
(i) *(Monotonicity):*

$$\text{if } X \geq Y \text{ then } \mathbb{E}[X] \geq \mathbb{E}[Y],$$

(ii) *(Constant preserving):*

$$\mathbb{E}[c] = c,$$

(iii) *(Sub-additivity):*

$$\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y],$$

(iv) *(Positive homogeneity):*

$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X].$$

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called sublinear expectation space. We assume that: if $X_1, X_2, ..., X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, ..., X_n) \in \mathcal{H}$, for each $\varphi \in C_{l,lip}(\mathbb{R}^n)$, where $C_{l,lip}(\mathbb{R}^n)$ is the space defined by:

$$\begin{aligned} C_{l,lip}(\mathbb{R}^n) &= \{\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \exists C > 0,\, m \in \mathbb{N}^* : \\ &\quad |\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)\,|x-y|\}. \end{aligned}$$

Definition 2.2. *Let X and Y be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega, \mathcal{H}, \mathbb{E})$ and $\left(\widehat{\Omega}, \widehat{\mathcal{H}}, \widehat{\mathbb{E}}\right)$. They are called identically distributed, represented as $X \overset{d}{=} Y$, if*

$$\mathbb{E}[\varphi(X)] = \widehat{\mathbb{E}}[\varphi(Y)], \quad \forall \varphi \in C_{l,lip}(\mathbb{R}^n).$$

Definition 2.3. *A random vector $Y = (Y_1, ..., Y_n) \in \mathcal{H}^n$ is said to be independent of $X \in \mathcal{H}^m$ under $\mathbb{E}$ if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^{n+m})$,*

$$\mathbb{E}\left[\varphi(X, Y)\right] = \mathbb{E}\left[\mathbb{E}\left(\varphi(x, Y)\right)_{x=X}\right].$$

$\tilde{X}$ is a copy of X, if $\tilde{X} \overset{d}{=} X$ and $\tilde{X}$ is independent from X.

Let X be a d-dimensional vector of random variables and let the mapping $G = G_X : \mathbb{S}_d \to \mathbb{R}$ defined by:

$$G(A) = \frac{1}{2}\mathbb{E}\left[(AX, X)\right],$$

where $\mathbb{S}_d$ is the set of $d \times d$ symmetric matrices and $(.,.)$ is the inner product on $\mathbb{R}^d$. It is easy to show that the function G is a sublinear and monotonic function. In addition, from [9] there exists a closed, bounded and convex subset Γ of $\mathbb{S}_d$ such that, for each $A \in \mathbb{S}_d$,

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr\left[\gamma\gamma^T A\right],$$

where γ^T is the transpose of the matrix γ. Let

$$\Delta = \left\{AA^T : A \in \Gamma\right\}.$$

Definition 2.4. *The random vector X is $N(0, \Delta)$-distributed, if for each $\varphi \in C_{l,lip}(\mathbb{R}^d)$, the function $u(t, x) = \mathbb{E}\left(\varphi\left(x + \sqrt{t}X\right)\right)$ is the unique viscosity solution of the following parabolic equation, called the G-heat equation*

$$\begin{cases} \frac{\partial u}{\partial t} = G(D^2 u) \\ u(0, x) = \varphi(x) \end{cases}, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

where $D^2 u = (\partial^2_{x_i x_j} u)_{i,j}$ is the Hessian matrix of u.

Definition 2.5. *A d-dimensional process $(B_t)_{t \geqslant 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-Brownian motion if the following properties are satisfied:*
(i): $B_0 = 0$,
(ii): For each $t, s \geqslant 0$, the increment $B_{t+s} - B_t$ is $N(0, s\Delta)$-distributed and independent from $(B_{t_1}, B_{t_2}, ..., B_{t_n})$, for each $n \in \mathbb{N}$ and $t_1, t_2, ..., t_n \in [0, t]$.

We denote by $\Omega = C_0^d(\mathbb{R}^+)$ the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with following distance:

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\left[\left(\max_{t \in [0,i]}\left|\omega_t^1 - \omega_t^2\right|\right) \wedge 1\right].$$

For each $t \in [0, \infty[$, we set $\Omega_t := \{\omega_{.\wedge t} : \omega \in \Omega\}$. Consider the canonical process defined by $B_t(\omega) = \omega_t$, for $\omega \in \Omega$ and $t \in [0, \infty[$. Now let $T \in [0, \infty[$, be a time horizon and let the sets:

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, ..., B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, ..., t_n \in [0, \infty[, \\ \varphi \in \mathbb{C}_{l.L_{ip}}(\mathbb{R}^d)^n\},$$

and

$$L_{ip}(\Omega) := \bigcup_{m=1}^{\infty} L_{ip}(\Omega_m).$$

It is clear that for all $t \leqslant T$, $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$. Peng constructed a sublinear expectation $\mathbb{E}$ defined on $(\Omega, L_{ip}(\Omega))$ under which the canonical process $(B_t)_{t\geqslant 0}$ is a G-Brownian motion (for more details see [12]). In what follows we consider this G-Brownian motion.

For $p \geq 1$, we consider the Banach spaces $L_G^p(\Omega)$ (resp. $L_G^p(\Omega_T)$) of random variables X defined on $L_{ip}(\Omega)$ (resp. $L_{ip}(\Omega_T)$) such that

$$\|X\|_p := \left[\mathbb{E}\left(|X|^p\right)\right]^{1/p} < \infty.$$

We consider the following type of simple processes for given partition $\pi_T = \{t_0, ..., t_N\}$ of $[0,T]$

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, ..., N-1$. The collection of these processes is denoted by $M_G^{p,0}(0,T)$. The completion of $M_G^{p,0}(0,T)$ under the following norm:

$$\|\eta\|_{p,T} = \left[\frac{1}{T}\int_0^T \mathbb{E}\left(|\eta_t|^p\right) dt\right]^{\frac{1}{p}},$$

is denoted by $M_G^p(0,T)$. Note that

$$M_G^q(0,T) \subset M_G^p(0,T), \text{ for } 1 \leq p \leq q.$$

It was proved in [14] that the process $B_t^{\boldsymbol{a}} := (a, B_t)$ is a real G-Brownian motion for each $a \in \mathbb{R}^d$. In particular all the components $B_t^{\boldsymbol{i}}$ of B_t are real G-Brownian motion.

Definition 2.6. *For each process $\eta \in M_G^{2,0}(0,T)$ of the form*

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t),$$

we define,

$$I(\eta) = \int_0^T \eta(s) dB_s^i := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}}^i - B_{t_k}^i).$$

The mapping $\eta \longrightarrow I(\eta)$ can be extended continuously to $M_G^2(0,T)$.

Definition 2.7. *The process defined by*

$$\left\langle B^i \right\rangle_t := \left(B_t^i\right)^2 - 2\int_0^t B_s^i dB_s^i,$$

called the quadratic variation process of B^i.

Definition 2.8. *The mutual variation process of* B^i *and* B^j *is defined by*

$$\langle B^i, B^j \rangle = \frac{1}{4} \left(\langle B^i + B^j \rangle - \langle B^i - B^j \rangle \right).$$

In particular $\langle B^i, B^i \rangle = \langle B^i \rangle$.

There exists a weakly compact collection $\mathcal{P}$ of probability measures P, defined on $(\Omega, \mathfrak{B}(\Omega))$ and the capacity $\widehat{C}(.)$ with respect to $\mathcal{P}$ is defined by:

$$\widehat{C}(A) = \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathfrak{B}(\Omega),$$

where $\mathfrak{B}(\Omega)$ is the Borel σ-algebra of Ω, see [4].

Definition 2.9. *A set* A *is called a polar if* $\widehat{C}(A) = 0$. *A property holds quasi-surely (q.s.), if it holds outside a polar.*

Some lemmas which required in the following.

Lemma 2.10 (Grönwall's inequality)**.** *Let* Φ *be a real and continuous function on* $[a, b]$ *and let* λ, $\mu \geq 0$, *such that*

$$\Phi(t) \leq \lambda + \mu \int_a^t \Phi(s) ds,$$

for each $a \leq t \leq b$. *Then*

$$\Phi(t) \leq \lambda e^{\mu(t-a)}.$$

The G-Itô formula [15], we use the Einstein notation.

Lemma 2.11. *Let* $X_t = (X_t^\gamma)_{\eta=1}^d$ *be a* d*-dimensional* G*-Itô process defined by*

$$X_t^\gamma = X_0^\gamma + \int_0^t \alpha_s^\gamma ds + \int_0^t \beta_s^{\gamma j} dB_s^j + \int_0^t \eta_s^{\gamma ij} d \langle B^i, B^j \rangle_s,$$

where $\eta^{\gamma ij}, \alpha^\gamma \in M_G^1(0, T)$ *and* $\beta^{\gamma j} \in M_G^2(0, T)$ *are bounded processes for* $\gamma \in \overline{1, d}$, $i, j \in \overline{1, d}$. *Let the real function* $\psi \in C^2(\mathbb{R}^d)$ *be bounded with bounded derivatives such that* $\left\{ \partial^2_{x^\nu x^\gamma} \psi \right\}_{\nu, \gamma = 1}^d$ *are uniformly Lipschitz. Then*

$$\begin{aligned} \psi(X_t) - \psi(X_0) &= \int_0^t \partial_{x^\gamma} \psi(X_s) \beta_s^{\gamma j} dB_s^j + \int_0^t \partial_{x^\gamma} \psi(X_s) \alpha_s^\gamma ds \\ &\quad + \int_0^t \left[\partial_{x^\gamma} \psi(X_s) \eta_s^{\gamma ij} + \frac{1}{2} \partial^2_{x^\nu x^\gamma} \psi(X_s) \beta_s^{\gamma j} \beta_s^{\nu j} \right] d \langle B^i, B^j \rangle_s, \end{aligned}$$

this equality holds in the sense of $L_G^2(\Omega_t)$.

Remark 2.12. *The G-Itô formula can be written in differential form*

$$d\psi(X_t) = \partial_{x^\gamma}\psi(X_t)\,dX_t^\gamma + \frac{1}{2}\partial^2_{x^\nu x^\gamma}\psi(X_t)\,dX_t^\gamma dX_t^\nu.$$

Thus, if (X_t), (Y_t) are two 1-dimensional G-Itô processes and $\psi(x,y) = xy$, we get the G-integration by part formula:

$$d(X_tY_t) = X_t dY_t + dX_t Y_t + dX_t dY_t.$$

The Burkholder-Davis-Gundy inequalities (G-BDG inequalities).

Lemma 2.13. *Let $p \geq 2$, $\varphi \in M_G^p([0,T])$. Then for all $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u \varphi_v dB_v^i\right|^p\right) \leq K_1 (t-s)^{\frac{p}{2}-1}\int_s^t \mathbb{E}(|\varphi_v|^p)\,dv,$$

where K_1 is a positive constant independent of φ.

Lemma 2.14. *Let $p \geq 1$, $\varphi \in M_G^p([0,T])$. Then for all $0 \leq s \leq t \leq T$, we have*

$$\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u \varphi_v d\left\langle B^i, B^j\right\rangle_v\right|^p\right) \leq K_2 (t-s)^{p-1}\int_s^t \mathbb{E}(|\varphi_v|^p)\,dv,$$

where K_2 is a positive constant independent of φ.

3. Problem statement and the G-NSDE of the derivative

Let $BC([-r,0];\mathbb{R}^n)$ be the family of bounded continuous $\mathbb{R}^n$-valued functions x defined on $[-r,0]$ equipped with the norm:

$$\|x\| = \sup_{-r\leq\theta\leq 0}|x(\theta)|,$$

where $r > 0$ is a fixed positive time. Consider the following n-dimensional G-NSDE:

$$(1)\quad \begin{cases} d\left[X(t) - Q(t,X_t)\right] = \sum_{l=0}^{d} A_l(t,X_t)dB_t^l + \sum_{i,j=1}^{d} A_{i,j}(t,X_t)d\left\langle B^i,B^j\right\rangle_t, \\ X_0 = x := (x(\theta))_{-r\leq\theta\leq 0} \in BC([-r,0];\mathbb{R}^n) \end{cases}$$

All the functions $A_l(.,x)$, $A_{i,j}(.,x) \in M_G^2([-r,T];\mathbb{R}^n)$, for $l \in \overline{0,d}$, $i,j \in \overline{1,d}$ and $x \in BC([-r,0];\mathbb{R}^n)$. The equation (1) can be written in integral form:

$$(2)\qquad X(t) = Q(t,X_t) + x(0) - Q(0,x) + \sum_{l=0}^{d}\int_0^t A_l(s,X_s)dB_s^l + \sum_{i,j=1}^{d}\int_0^t A_{i,j}(s,X_s)d\left\langle B^i,B^j\right\rangle_s$$

The objective is to prove that, if $X(t,x)$ denotes the value of the solution at time t through the initial data x, then the function $x \longmapsto X(t,x)$ is $q.s.$ Fréchet-differentiable.

We equip the space $\mathcal{L}_{BC}(\mathbb{R}^n)$ by the norm:

$$|L|_r := \sup_{\|h\|\leq 1} |Lh|.$$

Consider the space $BC([-r,0];\mathcal{L}_{BC}(\mathbb{R}^n))$ of $\mathcal{L}_{BC}(\mathbb{R}^n)$-valued bounded functions on $[-r,0]$ equipped with the norm:

$$\|T\|_r := \sup_{-r\leq\theta\leq 0} |T(\theta)|_r.$$

We denote by $A'(t,x)$ the Fréchet-derivative of $A(t,x)$ with respect to x. Assume that all the coefficients A_l, $A_{i,j}$, Q are Fréchet-differentiable with respect to the second variable, and for all $x, y \in BC([-r,0];\mathbb{R}^n)$, $l \in \overline{0,d}$ and $i,j \in \overline{1,d}$ we have:

(H1) :

$$|Q'(t,x)|_r \leq \kappa \text{ and } |H'(t,x)|_r \leq C_1 \quad q.s.,$$

uniformly with respect to t, where $0 < \kappa < \frac{1}{2}$, C_1 is a positive constant and $H = A_l,\ A_{i,j}$.

(H2) :

$$\sup_{0\leq t\leq T} |I(t,0)| < \infty \quad q.s.,$$

where $I = A_l, A_{i,j}, Q$.

(H3) :

$$|J'(t,x) - J'(t,y)| \leq C_2 \|x-y\| \left(1 + \|x\|^{n_0} + \|y\|^{n_0}\right) \quad q.s.,$$

uniformly with respect to t, where C_2 is a positive constant, $n_0 \geq 1$ and $J = A_l, A_{i,j}, Q$.

Remark 3.1. *Condition **(H1)** implies that*

$$|Q(t,x) - Q(t,y)| \leq \kappa \|x-y\|$$

and

$$|H(t,x) - H(t,y)| \leq C_1 \|x-y\|,$$

uniformly with respect to t, for $H = A_l$, $A_{i,j}$ for $l \in \overline{0,d}$ and $i,j \in \overline{1,d}$.

Remark 3.2. *Conditions **(H1)** and **(H2)** imply that*

$$|I(t,x)|^2 \leq C_3 \left(1 + \|x\|^2\right),$$

uniformly with respect to t, where C_3 is a positive constant, for $I = A_l, A_{i,j}, Q$, $l \in \overline{0,d}$ and $i,j \in \overline{1,d}$.

It follows from remarks 3.1 and 3.2 that the equation (1) admits a unique solution, see [1, 6].

3.1. The G-NSDE of the derivative

Since the initial data belongs to the space $BC\left([-r,0]\,;\mathbb{R}^n\right)$, then the derivative belongs to the Banach space $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$. We mean by stochastic operators the following:

Definition 3.3. *Let the $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$-valued process U, V and W such that*

$$(U_t h)_{0\le t\le T}\,,\ (V_t h)_{0\le t\le T} \in M_G^1([0,T]\,;\mathbb{R}^n)$$

and

$$(W_t h)_{0\le t\le T} \in M_G^2([0,T]\,;\mathbb{R}^n),$$

for each $h \in BC\left([-r,0]\,;\mathbb{R}^n\right)$. We define the following stochastic operators by

$$\left(\int_0^t U_s d\left\langle B^i, B^j\right\rangle_s\right)(h) := \int_0^t U_s\left(h\right) d\left\langle B^i, B^j\right\rangle_s,$$

$$\left(\int_0^t V_s ds\right)(h) := \int_0^t V_s\left(h\right) ds,$$

and

$$\left(\int_0^t W_s dB_s^i\right)(h) := \int_0^t W_s\left(h\right) dB_s^i.$$

Denote by ϕ, the linear operator of $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$ defined by $\phi\left(h\right) = h\left(0\right)$. The goal is to prove that the solution of (2) is differentiable and the derivative is given by the following G-NSDE:

$$\left\{\begin{aligned} d\left[Y\left(t\right) - Q'(t,X_t)\left(Y_t\right)\right] &= \sum_{l=0}^{d} A_l'(t,X_t)\left(Y_t\right) dB_t^l \\ &+ \sum_{i,j=1}^{d} A_{i,j}'(t,X_t)\left(Y_t\right) d\left\langle B^i, B^j\right\rangle_t\,, \\ Y_0 &= Id \end{aligned}\right. \tag{3}$$

where the solution $Y(t)$ belongs to $\mathcal{L}_{BC}\left(\mathbb{R}^n\right)$ and $Y_t = \left(Y\left(t+\theta\right)\right)_{-r\le\theta\le 0}$ such that $Y_0 = Id$ is the identity operator on $BC\left([-r,0]\,;\mathbb{R}^n\right)$.

4. Main results and proofs

In this section, we present the main results and we give the proofs. We start by existence-uniqueness theorem.

Theorem 4.1. *Under the hypotheses (**H1**) and (**H2**), the derivative* (3) *admits a unique solution.*

Proof. Consider the following G-NSDE:

$$
(4)\quad \begin{cases} d\left[Z(t)-N(t,Z_t)\right]=\sum\limits_{l=0}^{d}D_l(t,Z_t)dB_t^l+\sum\limits_{i,j=1}^{d}D_{i,j}(t,Z_t)d\left\langle B^i,B^j\right\rangle_t, \\ Z_0=h\in BC\left([-r,0];\mathbb{R}^n\right), \end{cases}
$$

where the functions $N, D_l, D_{i,j}:[0,T]\times BC\left([-r,0];\mathbb{R}^n\right)\times\Omega\to\mathbb{R}^n$ are defined by

$$
\begin{aligned}
N(t,z) &= Q'(t,X_t)(z), \\
D_l(t,z) &= A_l'(t,X_t)(z),\ l\in\overline{0,d}
\end{aligned}
$$

and

$$
D_{i,j}(t,z)=A_{i,j}'(t,X_t)(z),\ i,j\in\overline{1,d}.
$$

Observe that the process $Y(t)$ is solution of the G-NSDE (3) if and only if the process $Y(t)(h)$ is solution of the G-NSDE (4). On the other hand, it is easy to check that the functions N, D_l and $D_{i,j}$ satisfy the conditions (**H1**) and (**H2**). Thus, G-NSDE (4) admits a unique solution. The proof is complete. □

The following lemma extends the G-BDG inequalities to $\mathcal{L}_{BC}(\mathbb{R}^n)$-valued process.

Lemma 4.2. *For each $p\geq 1$ and $s<t$, we have*
(i):

$$
\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u U_s d\left\langle B^i,B^j\right\rangle_s\right|_r^p\right)\leq 2^pK_2(t-s)^{p-1}\int_s^t\mathbb{E}\left(|U_s|_r^p\right)ds.
$$

For each $p\geq 2$ and $s<t$, we have
(ii):

$$
\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u W_s dB_s^i\right|_r^p\right)\leq 2^pK_1(t-s)^{\frac{p}{2}-1}\int_s^t\mathbb{E}\left(|W_s|_r^p\right)ds,
$$

(iii):

$$
\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_0^t V_s ds\right|_r^p\right)\leq 2^pT^{p-1}\int_0^T\mathbb{E}\left(|V_s|_r^p\right)ds.
$$

Proof. **(i):** Let $\varepsilon>0$. There exists $h_\varepsilon\in BC\left([-r,0];\mathbb{R}^n\right)$ such that $\|h_\varepsilon\|\leq 1$ and

$$
\left|\left(\int_0^t U_s d\left\langle B^i,B^j\right\rangle_s\right)\right|_r\leq\varepsilon+\left|\int_0^t U_s(h_\varepsilon)d\left\langle B^i,B^j\right\rangle_s\right|,
$$

by using the inequality $(|a|+|b|)^p \leq 2^p\left(|a|^p+|b|^p\right)$ and lemma 2.14, for each $p \geq 1$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u U_s d\left\langle B^i,B^j\right\rangle_s\right|_r^p\right)\\
&\leq 2^p\left(\varepsilon^p+\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u U_s\left(h_\varepsilon\right)d\left\langle B^i,B^j\right\rangle_s\right|^p\right)\right)\\
&\leq 2^p\left(\varepsilon^p+K_2\left(t-s\right)^{p-1}\int_s^t \mathbb{E}\left(\left|U_s\right|_r^p\left\|h_\varepsilon\right\|^p\right)ds\right).
\end{aligned}$$

It follows that

$$\mathbb{E}\left(\sup_{s\leq u\leq t}\left|\int_s^u U_s d\left\langle B^i,B^j\right\rangle_s\right|_r^p\right)\leq 2^pK_2\left(t-s\right)^{p-1}\int_s^t\mathbb{E}\left(\left|U_s\right|_r^p\right)ds.$$

This proves **(i)**. Similarly, by using lemma 2.13 (resp. Hölder's inequality), we obtain **(ii)** (resp. **(iii)**). The proof is complete. □

Some auxiliary propositions and lemmas that needed for the proof of the main theorem.

Proposition 4.3. *For all $x,y\in BC\left([-r,0];\mathbb{R}^n\right)$ and for all $p\geq 2$, there exists a positive constant L_1 such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left\|X_t(x)-X_t(y)\right\|^p\right)\leq L_1\left\|x-y\right\|^p.$$

Proof. Note that

$$\begin{aligned}
\left\|X_t(x)-X_t(y)\right\| &\leq \sup_{-r\leq s\leq T}\left|X(s,x)-X(s,y)\right|\\
&\leq \sup_{-r\leq s\leq 0}\left|X(s,x)-X(s,y)\right|+\sup_{0\leq s\leq T}\left|X(s,x)-X(s,y)\right|\\
&\leq \left\|x-y\right\|+\sup_{0\leq s\leq T}\left|X(s,x)-X(s,y)\right| \qquad (5)
\end{aligned}$$

Then from equation (2), remark 3.1 and formula (5) we obtain

$$\begin{aligned}|X(t,x)-X(t,y)| &\leq \kappa \|X_t(x)-X_t(y)\| + |x(0)-y(0)| + \kappa\|x-y\| \\ &+\sum_{l=0}^{d}\left|\int_0^t [A_l(s,X_s(x))-A_l(s,X_s(y))]\, dB_s^l\right| \\ &+\sum_{i,j=1}^{d}\left|\int_0^t [A_{i,j}(s,X_s(x))-A_{i,j}(s,X_s(y))]\, d\left\langle B^i,B^j\right\rangle_s\right| \\ &\leq \kappa\left(\|x-y\| + \sup_{0\leq u\leq T}|X(u,x)-X(u,y)|\right) + (\kappa+1)\|x-y\| \\ &+\sum_{l=0}^{d}\left|\int_0^t [A_l(s,X_s(x))-A_l(s,X_s(y))]\, dB_s^l\right| \\ &+\sum_{i,j=1}^{d}\left|\int_0^t [A_{i,j}(s,X_s(x))-A_{i,j}(s,X_s(y))]\, d\left\langle B^i,B^j\right\rangle_s\right|,\end{aligned}$$

thus

$$\begin{aligned}(1-\kappa)\sup_{0\leq t\leq T}|X(t,x)-X(t,y)| &\leq (2\kappa+1)\|x-y\| \\ &+\sum_{l=0}^{d}\sup_{0\leq t\leq T}\left|\int_0^t [A_l(s,X_s(x))-A_l(s,X_s(y))]\, dB_s^l\right| \\ &+\sum_{i,j=1}^{d}\sup_{0\leq t\leq T}\left|\int_0^t [A_{i,j}(s,X_s(x))-A_{i,j}(s,X_s(y))]\, d\left\langle B^i,B^j\right\rangle_s\right|.\end{aligned}$$

We use the inequality

$$\left(\sum_{i=1}^{n}|a_i|\right)^p \leq n^p\sum_{i=1}^{n}|a_i|^p,$$

and we take the G-expectation of both sides, we can write

$$\begin{aligned}\frac{(1-\kappa)^p}{C_4}\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t,x)-X(t,y)|^p\right) &\leq (2\kappa+1)^p\|x-y\|^p \\ &+\sum_{l=0}^{d}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t (A_l(s,X_s(x))-A_l(s,X_s(y)))\, dB_s^l\right|^p\right) \\ &+\sum_{i,j=1}^{d}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t (A_{i,j}(s,X_s(x))-A_{i,j}(s,X_s(y)))\, d\left\langle B^i,B^j\right\rangle_s\right|^p\right),\end{aligned}$$

where $C_4 = \left(d^2 + d + 2\right)^p$. We apply the Hölder's inequality, G-BDG inequalities then hypothesis (**H1**), we get

$$\frac{(1-\kappa)^p}{C_4}\mathbb{E}\left(\sup_{0\le t\le T}|X(t,x)-X(t,y)|^p\right)$$
$$\le (2\kappa+1)^p\|x-y\|^p + C_5\int_0^T \mathbb{E}\left(\sup_{0\le t\le T}|X(s,x)-X(s,y)|^p\right)ds,$$

where $C_5 = \left(T^{p-1} + dK_1T^{\frac{p}{2}-1} + d^2K_2T^{p-1}\right)C_1^p$. Hence, by setting

$$u(t) := \mathbb{E}\left(\sup_{0\le s\le t}|X(s,x)-X(s,y)|^p\right),$$

yields

$$u(T) \le \frac{C_4}{(1-\kappa)^p}\left((2\kappa+1)^p\|x-y\|^p + C_5\int_0^T u(s)\,ds\right).$$

By applying Grönwall's inequality we infer that

$$u(T) \le C_6\|x-y\|^p,$$

where

$$C_6 = \frac{C_4(2\kappa+1)^p}{(1-\kappa)^p}e^{\frac{C_4C_5}{(1-\kappa)^p}T}.$$

On the other hand, by the inequality (5), we have

$$\|X_t(x)-X_t(y)\|^p \le 2^p\left(\|x-y\|^p + \sup_{0\le u\le T}|X(u,x)-X(u,y)|^p\right),$$

consequently

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}\|X_t(x)-X_t(y)\|^p\right) &\le 2^p\left(\|x-y\|^p + u(T)\right)\\ &\le L_1\|x-y\|^p,\end{aligned}$$

where $L_1 = 2^p(1+C_6)$. The proof is complete. □

Proposition 4.4. *For all $p\ge 2$, we have*

$$\mathbb{E}\left(\sup_{0\le t\le T}\|Y_t\|_r^p\right) \le L_2,$$

where L_2 is a positive constant.

Proof. First, observe that

$$\begin{aligned}\|Y_t\|_r &\le \sup_{-r\le u\le T}|Y(u)|_r\\ &\le \sup_{-r\le u\le 0}|Y(u)|_r + \sup_{0\le u\le T}|Y(u)|_r\\ &\le 1 + \sup_{0\le u\le T}|Y(u)|_r \qquad (6)\end{aligned}$$

From the equation (3), we have

$$\begin{aligned}
|Y(t)|_r &\le |Q'(X_t)|_r \, \|Y_t\|_r + 1 + |Q'(x)|_r + \sum_{l=0}^{d} \left| \int_0^t A'_l(s, X_s)(Y_s)\, dB_s^l \right|_r \\
&\quad + \sum_{i,j=1}^{d} \left| \int_0^t A'_{i,j}(s, X_s)(Y_s)\, d\left\langle B^i, B^j \right\rangle_s \right|_r , \\
&\le \kappa \left(1 + \sup_{0\le u\le T} |Y(u)|_r \right) + 1 + \kappa + \sum_{l=0}^{d} \left| \int_0^t A'_l(s, X_s)(Y_s)\, dB_s^l \right|_r \\
&\quad + \sum_{i,j=1}^{d} \left| \int_0^t A'_{i,j}(s, X_s)(Y_s)\, d\left\langle B^i, B^j \right\rangle_s \right|_r ,
\end{aligned}$$

thus

$$\begin{aligned}
(1-\kappa) \sup_{0\le u\le T} |Y(u)|_r &\le 1 + 2\kappa + \sum_{l=0}^{d} \sup_{0\le t\le T} \left| \int_0^t A'_l(s, X_s)(Y_s)\, dB_s^l \right|_r \\
&\quad + \sum_{i,j=1}^{d} \sup_{0\le t\le T} \left| \int_0^t A'_{i,j}(s, X_s)(Y_s)\, d\left\langle B^i, B^j \right\rangle_s \right|_r .
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{(1-\kappa)^p}{C_4} \sup_{0\le u\le T} |Y(u)|_r^p &\le (1+2\kappa)^p + \sum_{l=0}^{d} \sup_{0\le t\le T} \left| \int_0^t A'_l(s, X_s)(Y_s)\, dB_s^l \right|_r^p \\
&\quad + \sum_{i,j=1}^{d} \sup_{0\le t\le T} \left| \int_s^t A'_{i,j}(s, X_s)(Y_s)\, d\left\langle B^i, B^j \right\rangle_s \right|_r^p .
\end{aligned}$$

Taking the G-expectation of the inequality above and using lemma 4.2, we can write

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le u\le T} |Y(u)|_r^p \right) &\le \frac{C_4}{(1-\kappa)^p} \left((1+2\kappa)^p + C_7 \int_0^T \mathbb{E}\left(\|Y_s\|_r^p \right) ds \right) \\
&\le \frac{C_4}{(1-\kappa)^p} \left((1+2\kappa)^p + C_7 T + C_7 \int_0^T \mathbb{E}\left(\sup_{0\le u\le s} |Y(u)|_r^p \right) ds \right),
\end{aligned}$$

where $C_7 = \left(T^{p-1} + dK_1 T^{\frac{p}{2}-1} + d^2 K_2 T^{p-1}\right)(2C_1)^p$. Consequently, by Grönwall's inequality

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y(t)|_r^p\right) \le C_8 \tag{7}$$

where

$$C_8 = \frac{C_4\left((1+2\kappa)^p + C_7 T\right)}{(1-\kappa)^p} e^{\frac{C_4 C_7}{(1-\kappa)^p} T}.$$

We deduce from the inequalities (6) and (7), that

$$\sup_{0\le u\le T} \|Y_t\|_r^p \le 2^p \left(1 + \sup_{0\le u\le T} |Y(u)|_r^p\right),$$

we take the G-expectation and set $L_2 = 2^p (C_8 + 1)$, the result follows. □

Lemma 4.5. *We assume that hypotheses (**H1**) and (**H3**) hold. Let $x, h \in BC([-r,0];\mathbb{R})$ and let the processes*

$$Z(t) := \frac{X(t, x+h) - X(t,x) - Y(t)(h)}{\|h\|},$$

$$\xi_t := \frac{1}{\|h\|}\left(Q(t, X_t(x+h)) - Q(t, X_t(x)) - Q'(t, X_t(x))(Y_t)(h)\right)$$

and

$$\xi_t^R := \frac{1}{\|h\|}\left(R(t, X_t(x+h)) - R(t, X_t(x)) - R'(t, X_t(x))(Y_t)(h)\right)$$

for $R = A_l, A_{i,j}$, $l \in \overline{0,d}$ and $i,j \in \overline{1,d}$. Then we have for all $\varepsilon > 0$,
***(i)**:*

$$\mathbb{E}\left(\sup_{0\le u\le t} |\xi_u|^2\right) \le 2\kappa^2 \mathbb{E}\left(\sup_{0\le u\le t} |Z(u)|^2\right) + \varepsilon L_2 + \frac{\Phi(\|h\|)}{\varepsilon},$$

***(ii)**:*

$$\mathbb{E}\left(\sup_{0\le u\le t} \left|\xi_u^R\right|^2\right) \le 2C_1^2 \mathbb{E}\left(\sup_{0\le u\le t} |Z(u)|^2\right) + \varepsilon L_2 + \frac{\Phi(\|h\|)}{\varepsilon},$$

where Φ is a positive function independent of ε such that $\lim_{t\downarrow 0} \Phi(t) = 0$.

Proof. For clarity we put the following:

$$\widetilde{X}(t) = X(t, x+h) \text{ and } Z(t) = \frac{\widetilde{X}(t) - X(t) - Y(t)(h)}{\|h\|}$$

(i): Since $Z(u)=0$ for each $u\in[-r,0]$, then

$$\begin{aligned}\left\|\widetilde{X}_t-X_t-Y_t(h)\right\| &= \|h\|\sup_{-r\leq\theta\leq 0}|Z(t+\theta)| \\ &\leq \|h\|\left(\sup_{-r\leq u\leq t}|Z(u)|\right) \\ &\leq \|h\|\left(\sup_{-r\leq u\leq 0}|Z(u)|+\sup_{0\leq u\leq t}|Z(u)|\right) \\ &\leq \|h\|\sup_{0\leq u\leq t}|Z(u)|.\end{aligned}$$

By the finite-increments theorem, we have for some $\tau\in(0,1)$,

$$Q(t,X_t)-Q\left(t,\widetilde{X}_t\right)=Q'\left(t,X_t+\tau\left(\widetilde{X}_t-X_t\right)\right)\left(\widetilde{X}_t-X_t\right),$$

it follows from hypotheses (**H1**) and (**H3**), that

$$\begin{gathered}\|h\||\xi_t|=\left|Q'\left(t,X_t+\tau\left(\widetilde{X}_t-X_t\right)\right)\left(\widetilde{X}_t-X_t\right)-Q'(t,X_t)(Y_t(h))\right| \\ \leq\left|Q'\left(t,X_t+\tau\left(\widetilde{X}_t-X_t\right)\right)\left(\widetilde{X}_t-X_t-Y_t(h)\right)\right| \\ +\left|\left(\left(Q'\left(t,X_t+\tau\left(\widetilde{X}_t-X_t\right)\right)-Q'(t,X_t)\right)(Y_t)\right)(h)\right| \\ \leq\kappa\left\|\widetilde{X}_t-X_t-Y_t(h)\right\| \\ +\|h\|\left|Q'\left(t,X_t+\tau\left(\widetilde{X}_t-X_t\right)\right)-Q'(t,X_t)\right|_r\|Y_t\|_r \\ \leq\|h\|\left(\kappa\sup_{0\leq u\leq t}|Z(u)|\right. \\ \left.+C_2\left\|\widetilde{X}_t-X_t\right\|\left(1+\|X_t\|^{n_0}+\left\|X_t+\tau\left(\widetilde{X}_t-X_t\right)\right\|^{n_0}\right)\|Y_t\|_r\right),\end{gathered}$$

by using the inequality $(a+b)^{n_0}\leq 2^{n_0}\left(a^{n_0}+b^{n_0}\right)$, we have

$$\begin{gathered}|\xi_t|\leq\kappa\sup_{0\leq u\leq t}|Z(u)|+ \\ C_2\left\|\widetilde{X}_t-X_t\right\|\left(1+(2^{n_0}+1)\|X_t\|^{n_0}+2^{n_0}\left\|\widetilde{X}_t-X_t\right\|^{n_0}\right)\|Y_t\|_r,\end{gathered}$$

then, we can write

$$\begin{gathered}|\xi_t|^2\leq 2\kappa^2\sup_{0\leq u\leq t}|Z(u)|^2 \\ +2\left(C_2\left\|\widetilde{X}_t-X_t\right\|\left(1+(2^{n_0}+1)\|X_t\|^{n_0}+2^{n_0}\left\|\widetilde{X}_t-X_t\right\|^{n_0}\right)\right)^2\|Y_t\|_r^2 \qquad (8)\end{gathered}$$

We use the inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, we obtain

$$2\left(C_2\left\|\widetilde{X}_t - X_t\right\|\left(1+(2^{n_0}+1)\|X_t\|^{n_0}+2^{n_0}\left\|\widetilde{X}_t - X_t\right\|^{n_0}\right)\right)^2\|Y_t\|_r^2$$

$$\leq \frac{1}{\varepsilon}\left(C_2\left\|\widetilde{X}_t - X_t\right\|\left(1+(2^{n_0}+1)\|X_t\|^{n_0}+2^{n_0}\left\|\widetilde{X}_t - X_t\right\|^{n_0}\right)\right)^4 +\varepsilon\|Y_t\|_r^4$$

$$\leq \frac{J(t,h)}{\varepsilon} + \varepsilon\|Y_t\|_r^4 \tag{9}$$

where

$$J(t,h) := (3C_2)^4\left\|\widetilde{X}_t - X_t\right\|^4 \times\left(1+(2^{n_0}+1)^4\|X_t\|^{4n_0}+2^{4n_0}\left\|\widetilde{X}_t - X_t\right\|^{4n_0}\right) \tag{10}$$

by combination of the inequalities (8), (9) and (10), we get

$$\sup_{0\leq u\leq t}|\xi_u|^2 \leq 2\kappa^2\sup_{0\leq u\leq t}|Z(u)|^2 + \varepsilon\sup_{0\leq u\leq t}\|Y_u\|_r^4 + \sup_{0\leq u\leq t}\frac{J(u,h)}{\varepsilon} \tag{11}$$

On the other hand, by G-Hölder's inequality, we can write

$$\mathbb{E}\left[J(u,h)\right] \leq (3C_2)^4\sqrt{\mathbb{E}\left(\left\|\widetilde{X}_u - X_u\right\|^8\right)} \times\sqrt{\mathbb{E}\left(\left(1+(2^{n_0}+1)^4\|X_u\|^{4n_0}+2^{4n_0}\left\|\widetilde{X}_u - X_u\right\|^{4n_0}\right)^2\right)}$$

$$\leq 3\,(3C_2)^4\sqrt{\mathbb{E}\left(\left\|\widetilde{X}_u - X_u\right\|^8\right)} \times\sqrt{\mathbb{E}\left(1+(2^{n_0}+1)^8\|X_u\|^{8n_0}+2^{8n_0}\left\|\widetilde{X}_u - X_u\right\|^{8n_0}\right)}$$

It was proved in [5] that there exists a positive constant C_9 such that

$$\mathbb{E}\left(\sup_{-r\leq t\leq T}|X(t)|^p\right) \leq C_9,$$

for each $p \geq 2$. It follows that by proposition 4.3,

$$\mathbb{E}\left(\sup_{0\leq u\leq t}J(u,h)\right) \leq \Phi(\|h\|) \tag{12}$$

where

$$\Phi(\|h\|) := 3\,(3C_2)^4\sqrt{L_1}\,\|h\|^4\sqrt{1+(2^{n_0}+1)^8C_9+(2\,\|h\|)^{8n_0}L_1}$$

by applying the G-expectation of both sides of the inequality (11), using the inequality (12) and proposition 4.4, we obtain

$$\mathbb{E}\left(\sup_{0\leq u\leq t}|\xi_u|^2\right)\leq 2\kappa^2\mathbb{E}\left(\sup_{0\leq u\leq t}|Z(u)|^2\right)+\varepsilon L_2+\frac{\Phi(\|h\|)}{\varepsilon}$$

Similarly, by the same arguments we prove **(ii)**. The proof is complete. □

Now, we are able to state the main theorem.

Theorem 4.6. *Under the hypotheses (**H1**)-(**H3**), the solution $X(t)$ of the G-NSDE (2) is differentiable with respect to x and the derivative is $Y(t)$, q.s.*

Proof. We adopt the same notations used in the previous proof. From the equations (2) and (3), we can write

$$|Z(t)|\leq \xi_t+\frac{|Q(t,x+h)-Q(t,x)-Q'(t,x)(h)|}{\|h\|}+\sum_{l=0}^{d}\left|\int_0^t \xi_s^{A_l}dB_s^l\right|$$
$$+\sum_{i,j=1}^{d}\left|\int_0^t \xi_s^{A_{i,j}}d\left\langle B^i,B^j\right\rangle_s\right|.$$

It follows that

$$|Z(t)|^2\leq 2|\xi_t|^2+\frac{2}{\|h\|^2}\Bigg(|Q(t,x+h)-Q(t,x)-Q'(t,x)(h)|$$
$$+\sum_{l=0}^{d}\left|\int_0^t \xi_s^{A_l}dB_s^l\right|+\sum_{i,j=1}^{d}\left|\int_0^t \xi_s^{A_{i,j}}d\left\langle B^i,B^j\right\rangle_s\right|\Bigg)^2.$$

Taking the G-expectation of the above inequality then applying Hölder's inequality, G-BDG inequalities, yields

$$\mathbb{E}\left(\sup_{0\leq u\leq T}|Z(u)|^2\right)\leq 2\mathbb{E}\left(\sup_{0\leq u\leq T}|\xi_u|^2\right)$$
$$+C_{10}\Bigg(\frac{|Q(t,x+h)-Q(t,x)-Q'(t,x)(h)|^2}{\|h\|^2}+T\int_0^T\mathbb{E}\left(\sup_{0\leq u\leq s}\left|\xi_u^{A_0}\right|^2\right)ds$$
$$+K_1\sum_{l=0}^{d}\int_0^t\mathbb{E}\left(\sup_{0\leq u\leq s}\left|\xi_u^{A_l}\right|^2\right)ds+TK_2\sum_{i,j=1}^{d}\int_0^t\mathbb{E}\left(\sup_{0\leq u\leq s}\left|\xi_u^{A_{i,j}}\right|^2\right)ds\Bigg),$$

where $C_{10} = 2\left(d^2+d+2\right)^2$. Therefore by lemma 4.5, we obtain

$$\mathbb{E}\left(\sup_{0\le u\le T}|Z(u)|^2\right) \le 4\kappa^2\mathbb{E}\left(\sup_{0\le u\le T}|Z\left(u\right)|^2\right) + 2\left(\varepsilon L_2 + \frac{\Phi(\|h\|)}{\varepsilon}\right)$$
$$+C_{10}\left(\frac{|Q(t,x+h)-Q(t,x)-Q'\left(t,x\right)\left(h\right)|^2}{\|h\|^2}\right.$$
$$\left.+\left(T+dK_1+d^2K_2T\right)\int_0^T\left(2\kappa^2\mathbb{E}\left(\sup_{0\le u\le s}|Z(u)|^2\right)+\varepsilon L_2+\frac{\Phi(\|h\|)}{\varepsilon}\right)ds\right),$$

thus

$$\left(1-4\kappa^2\right)\mathbb{E}\left(\sup_{0\le u\le T}|Z(u)|^2\right) \le C_{10}\frac{|Q(t,x+h)-Q(t,x)-Q'\left(t,x\right)\left(h\right)|^2}{\|h\|^2}$$
$$+\left(2+C_{10}T\left(T+dK_1+d^2K_2T\right)\right)\left(\varepsilon L_2+\frac{\Phi(\|h\|)}{\varepsilon}\right)$$
$$+C_{11}\int_0^t\mathbb{E}\left(\sup_{0\le u\le s}|Z(u)|^2\right)ds,$$

where $C_{11} = 2\left(T+dK_1+d^2K_2T\right)\kappa^2C_{10}$, which implies that for each $\varepsilon>0$,

$$\mathbb{E}\left(\sup_{0\le u\le T}|Z(u)|^2\right) \le \frac{1}{1-4\kappa^2}$$
$$\times\left(C_{10}\frac{|Q(t,x+h)-Q(t,x)-Q'\left(t,x\right)h|^2}{\|h\|^2}+A_\varepsilon\left(h\right)\right.$$
$$\left.+C_{11}\int_0^T\mathbb{E}\left(\sup_{0\le u\le s}|Z(u)|^2\right)ds\right),$$

where $A_\varepsilon\left(h\right) = \left(2+C_{10}T\left(T+dK_1+d^2TK_2\right)\right)\left(\varepsilon L_2+\frac{\Phi(\|h\|)}{\varepsilon}\right)$. It follows by Grönwall's lemma

$$\mathbb{E}\left(\sup_{0\le u\le T}|Z(u)|^2\right) \le f\left(\varepsilon,h\right), \tag{13}$$

where

$$f\left(\varepsilon,h\right) := \frac{1}{1-4\kappa^2}$$
$$\times\left(C_{10}\frac{\|Q(t,x+h)-Q(t,x)-Q'\left(t,x\right)\left(h\right)\|^2}{\|h\|^2}+A_\varepsilon\left(h\right)\right)e^{\gamma T},$$
$$\gamma = \frac{C_{11}}{1-4\kappa^2}.$$

Since

$$\lim_{\|h\|\to 0} f(\varepsilon, h) = \frac{C_{11}e^{\gamma T}}{1-4\kappa^2}\left(2 + C_{10}T\left(T + dK_1 + d^2TK_2\right)\right)\varepsilon L_2,$$

and

$$\lim_{\varepsilon\to 0}\left(\lim_{\|h\|\to 0} f(\varepsilon, h)\right) = 0,$$

then, by the inequality (13)

$$\underset{\|h\|\to 0}{lim}\ \mathbb{E}\left(\sup_{0\le u\le T}|Z(u)|^2\right) = \lim_{\|h\|\to 0}\left(\lim_{\varepsilon\to 0} f(\varepsilon, h)\right) = 0,$$

which implies that

$$\underset{\|h\|\to 0}{lim}\frac{X(t,x) - X(t,x+h)}{\|h\|} = Y(t,x) \quad q.s. \text{ for each } t\in[0,T],$$

this completes the proof. □

5. Application

In this section, we give two examples, the first is an application of theorem 4.6. The second shows the link between our result and that of Bougherra et al. [3].

Example 5.1. *In this example we consider only one-dimensional case. Since the explicit solutions of neutral equations are generally cannot be obtained, we first choose the term neutral Q and a process $X(t)$ which is supposed to be the solution of the G-NSDE, then we give the G-SDE satisfied by $X(t) - Q(t, X_t)$, which allows us to find the coefficients of dt, dB_t and $d\langle B\rangle_t$. For example, we set*

$$X(t) = x(0) + \sin B_t \ \text{ and } \ Q(t,x) = \frac{x(0)(1+\sin B_t)}{8},$$

*where $x \in BC([-r,0];\mathbb{R})$ and (B_t) is a G-Brownian motion. Note that Q satisfies the hypotheses **(H1)**-**(H3)** and $Q'(t,x) = \frac{\phi(1+\sin B_t)}{8}$. Then we have*

$$\begin{aligned} X(t) - Q(t,X_t) &= X(t) - \frac{X(t)(1+\sin B_t)}{8} \\ &= \frac{X(t)(7-\sin B_t)}{8}. \end{aligned}$$

By the G-Itô formula,

$$dX(t) = \cos B_t dB_t - \frac{1}{2}\sin B_t d\langle B\rangle_t$$

and

$$d\left(\frac{(7-\sin B_t)}{8}\right) = -\frac{1}{8}\left(\cos B_t dB_t - \frac{1}{2}\sin B_t d\langle B\rangle_t\right).$$

Thanks to G-integration by part formula, we have

$$d\left(X\left(t\right)-Q\left(t,X_t\right)\right)=X\left(t\right)d\left(\frac{\left(7-\sin B_t\right)}{8}\right)+\left(\frac{\left(7-\sin B_t\right)}{8}\right)dX\left(t\right)$$
$$+dX\left(t\right)d\left(\frac{\left(7-\sin B_t\right)}{8}\right)$$
$$=\frac{1}{8}\left(-X\left(t\right)+\frac{\left(7-\sin B_t\right)}{8}\right)\left(\cos B_t dB_t-\frac{1}{2}\sin B_t d\left\langle B\right\rangle_t\right)$$
$$-\frac{1}{8}\cos^2 B_t d\left\langle B\right\rangle_t$$
$$=\frac{1}{8}\left[\left(-X\left(t\right)+\frac{\left(7-\sin B_t\right)}{8}\right)\cos B_t\right]dB_t$$
$$-\frac{1}{8}\left[\frac{1}{2}\sin B_t\left(-X\left(t\right)+\frac{\left(7-\sin B_t\right)}{8}\right)+\cos^2 B_t\right]d\left\langle B\right\rangle_t$$

This means that $X(t)$ is the solution of the following G-NSDE

$$(14)\qquad \begin{cases} d\left(X\left(t\right)-Q\left(t,X_t\right)\right)=A_1\left(t,X_t\right)dB_t+A_2\left(t,X_t\right)d\left\langle B\right\rangle_t, \\ X_0=x \end{cases}$$

where

$$A_1\left(t,x\right)=\frac{1}{8}\left[\cos B_t\left(-x\left(0\right)+\frac{\left(7-\sin B_t\right)}{8}\right)\right]$$

and

$$A_2\left(t,x\right)=-\frac{1}{8}\left[\frac{1}{2}\sin B_t\left(-x\left(0\right)+\frac{\left(7-\sin B_t\right)}{8}\right)+\cos^2 B_t\right].$$

Since

$$A_1'\left(t,x\right)=-\frac{\phi}{8}\cos B_t \text{ and } A_2'\left(t,x\right)=\frac{\phi}{16}\sin B_t,$$

*then A_1 and A_2 satisfy conditions (**H1**)-(**H3**). It follows that $X(t)$ is the unique solution of the G-NSDE* (14). *Then by applying theorem 4.6, the derivative $Y(t)$ of $X(t)$ is the unique solution of the following G-NSDE*

$$(15)\qquad \begin{cases} d\left(Y\left(t\right)-Q'\left(t,X_t\right)\left(Y_t\right)\right)=A_1'\left(t,X_t\right)\left(Y_t\right)dB_t \\ +A_2\left(t,X_t\right)\left(Y_t\right)d\left\langle B\right\rangle_t, \\ Y_0=Id \end{cases}$$

Since

$$Q'\left(t,X_t\right)\left(Y_t\right)=\frac{Y\left(t\right)\left(1+\sin B_t\right)}{8},\ A_1'\left(t,X_t\right)\left(Y_t\right)=-\frac{Y\left(t\right)}{8}\cos B_t,$$

and

$$A_2'\left(t,X_t\right)\left(Y_t\right)=\frac{Y\left(t\right)}{16}\sin B_t,$$

then the explicit expression of equation (15) *is,*

$$\begin{cases} d\left(\frac{Y(t)(1+\sin B_t)}{8}\right)=-\frac{Y(t)}{8}\cos B_t dB_t+\frac{Y(t)}{16}\sin B_t d\left\langle B\right\rangle_t \\ Y_0=Id \end{cases}$$

It is easy to check that the derivative $Y(t) = \phi$ of $X(t)$ is the trivial unique solution of (15).

Example 5.2. *Consider the following G-SDE:*

$$\text{(16)} \qquad \begin{cases} dX(t) = \sum_{l=0}^{d} f_l(X(t))dB_t^l + \sum_{i,j=1}^{d} \sigma_{i,j}(X(t))d\left\langle B^i, B^j\right\rangle_t, \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

where $B_t^0 = t$ and the functions f_l, $\sigma_{i,j} : \mathbb{R}^n \longmapsto \mathbb{R}^n$, $l \in \overline{0,d}$ and $i,j \in \overline{1,d}$ are differentiable with respect to $x \in \mathbb{R}^n$, satisfy the following hypotheses:

For $H = f_l, \sigma_{i,j}$ and for all $x, y \in \mathbb{R}^n$:

($\widetilde{H}1$):

$$|H'(x)| \leq C_1,$$

($\widetilde{H}2$):

$$|H(0)| < \infty,$$

($\widetilde{H}3$):

$$H' \in C_{l,lip}(\mathbb{R}^n),$$

where $H'(x)$ denote the derivatives with respect to x (Jacobian matrices). Now, consider the functions A_l, $A_{i,j} : BC([-r,0];\mathbb{R}^n) \to \mathbb{R}^n$ defined as follows:

$$A_l(x) := f_l(x(0)) = f_l(\phi(x)) \text{ and } A_{i,j}(x) := \sigma_{i,j}(x(0)) = \sigma_{i,j}(\phi(x)).$$

*It easy to check that the functions A_l, $A_{i,j}$ are differentiable and satisfy hypotheses (**H1**)-(**H3**). Then the following G-NSDE, with neutral term $Q := 0$,*

$$\text{(17)} \qquad \begin{cases} dZ(t) = \sum_{l=0}^{d} A_l(Z_t)dB_t^l + \sum_{i,j=1}^{d} A_{i,j}(Z_t)d\left\langle B^i, B^j\right\rangle_t, \\ Z_0 = z \in BC([-r,0];\mathbb{R}^n) \end{cases}$$

admits a unique solution. Therefore, the equation (16) *admits a unique solution*

$$X(t,x) := Z(t, \rho(x)),$$

where $\rho(x) \in BC([-r,0];\mathbb{R}^n)$ defined by $\rho(x)(\theta) = x$ for each $\theta \in [-r,0]$. The derivatives are given by

$$A_l'(z) = f_l'(\phi(z)) \circ \phi \text{ and } A_{i,j}'(z) = \sigma_{i,j}'(\phi(z)) \circ \phi.$$

Hence, by theorem 4.6, the solution of equation (17) *is differentiable with respect to z and the derivative is the following G-SDE*

$$\text{(18)} \qquad \begin{cases} dV(t) = \sum_{l=0}^{d} A_l'(Z_t)(V_t)dB_t^l + \sum_{i,j=1}^{d} A_{i,j}'(Z_t)(V_t)d\left\langle B^i, B^j\right\rangle_t, \\ V(0) = \phi \end{cases}$$

It follows that $X(t,x)$ is differentiable with respect to $x \in \mathbb{R}^n$ and the derivative is

$$Y(t,x) = V(t, \rho(x)) \circ \rho$$

On the other hand, we have

$$Y\left(0,x\right)=\phi\circ\rho\left(x\right)=I \text{ the identity on } \mathbb{R}^{n},$$

which shows that the solution of equation (16) *is differentiable and the derivative is the unique solution of the following equation*

$$\text{(19)}\qquad \begin{cases} dY(t,x)=\sum\limits_{l=0}^{d}f_{l}'\left(X\left(t,x\right)\right)\left(Y\left(t,x\right)\right)dB_{t}^{l} \\ +\sum\limits_{i,j=1}^{d}\sigma_{i,j}'\left(X\left(t,x\right)\right)\left(Y\left(t,x\right)\right)d\left\langle B^{i},B^{j}\right\rangle_{t}, \\ Y\left(0,x\right)=I \end{cases}$$

Finally, owing to the fact that $f_{l}'\left(X\left(t,x\right)\right)$ *and* $\sigma_{i,j}'\left(X\left(t,x\right)\right)$ *are matrices processes, then* $Y(t,x)$ *must be a matrix-valued stochastic processes. Thus the G-SDE* (19) *is a matrix-valued stochastic differential equation. Indeed, this is one of the main results which obtained by Bougherra et al., under different hypotheses.*

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Zakaria Boumezbeur
Department of Mathematics, Faculty of Science,
LaPS Laboratory, Badji Mokhtar University,
Annaba, Algeria.
E-mail: boumezbeurza@gmail.com

Hacène Boutabia
Department of Mathematics, Faculty of Science,
LaPS Laboratory, Badji Mokhtar University,
Annaba, Algeria.
E-mail: boutabiah@yahoo.fr