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THE GRAM AND HANKEL MATRICES VIA SPECIAL NUMBER SEQUENCES

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Abstract. In this study, we consider the Hankel and Gram matrices which are defined by the elements of special number sequences. Firstly, the eigenvalues, determinant, and norms of the Hankel matrix defined by special number sequences are obtained. Afterwards, using the relationship between the Gram and Hankel matrices, the eigenvalues, determinants, and norms of the Gram matrices defined by number sequences are given.

1. Introduction

Many researchers have studied special matrices and some number sequences until today. Let's give some studies associated with the circulant and Hankel matrices. Solak has defined the circulant matrices with Fibonacci and Lucas numbers and investigated the norms of these matrices in [27]. Alptekin has considered the circulant and semicirculant matrices with Horadam's numbers in [2]. The other articles related to circulant matrices which are defined by various number sequences can be found in [3,15,25,29–31]. It is possible to see some of the studies related to $k-$circulant ($r-$circulant) matrices in [4,17–24,33]. Also, Kızılates and Tuglu have defined the geometric circulant matrices in [12,14].

In [11], the Hankel matrix $H = (h_{ij})$ has been defined as

$$h_{ij} = s_{i+j-1}$$

where the sequence $\{s_i\}$ and $i, j = 1, 2, \ldots$. The Hankel matrices, which are defined by special number sequences, and their properties have been studied in [1, 28, 34, 35].

Sequences of integer have an important place in literature. The most famous of these sequences are the Fibonacci and Lucas sequences. In [9,10], the authors have given some generalized number sequences and obtained some properties of these sequences. Now, let's give the definitions and properties of these generalized number sequences.

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The sequence $\{u_n\}$ is defined by the following recurrence relation for $n \geq 2$,

$$u_n = pu_{n-1} + u_{n-2} \tag{1}$$

with the initial conditions $u_0 = 0$, $u_1 = 1$.

Similarly, the sequence $\{v_n\}$ is defined by the following recurrence relation for $n \geq 2$,

$$v_n = pv_{n-1} + v_{n-2} \tag{2}$$

with the initial conditions $v_0 = 2$, $v_1 = p$. Note that, taking $p = 1$ in (1) and (2), the Fibonacci and Lucas numbers are obtained. Also, if we take $p = 2$ in (1) and (2), we have Pell and Pell-Lucas numbers. For more information, please refer to [9, 10, 13] and closely related references therein.

The characteristic equation of recurrences (1) and (2) is given as

$$\lambda^2 - p\lambda - 1 = 0. \tag{3}$$

The Binet's formulas of the $\{u_n\}$ and $\{v_n\}$ sequences are obtained as

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n$$

where α and β are roots of the characteristic equation (3).

Also, some identities between $\{u_n\}$ and $\{v_n\}$ sequences are given as follows:

$$u_{2n} = u_n v_n \tag{4}$$

$$v_{n+m} + (-1)^m v_{n-m} = v_m v_n \tag{5}$$

$$u_{n+m} - (-1)^m u_{n-m} = u_m v_n \tag{6}$$

$$v_{n+m} - (-1)^m v_{n-m} = \left(p^2 + 4\right) u_m u_n \tag{7}$$

$$\left(p^2 + 4\right) u_n^2 - v_n^2 = 4(-1)^{n+1} \tag{8}$$

$$v_n - pu_n = 2u_{n-1} \tag{9}$$

$$u_{n-1} + u_{n+1} = v_n \tag{10}$$

$$v_{2n} + (-1)^n 2 = v_n^2 \tag{11}$$

In [8], the Gram matrix $B = (b_{ik})$ has been defined by $b_{ik} = \langle x_i, x_k \rangle$ such that $x_1, x_2, \ldots, x_m$ are vectors in some Hilbert space and $i, k = 1, 2, \ldots, m$. B has been denoted $G(x_1, x_2, \ldots, x_m)$. Also, Halperin has introduced the Gram matrix and given the connection between the Gram and hermitian definite matrices in [8].

For an $m \times m$ matrix A with columns $x_1, x_2, \ldots, x_m$, the matrix product $G = AA^*$ is called the Gram matrix associated with the set $m-$ vectors $x_1, x_2, \ldots, x_m$.

In [16], the explicit expressions for the inverse of the Gram matrix of the Bernstein basis and its principal submatrices are given. Sreeram and Agathoklis have obtained new properties of the Gram matrix. Also, they have given new techniques for the computation of the Gram matrix and characteristic equation of the systems by using these properties in [32]. In [6], the author has stated that Gram matrices provide a natural link between positive semidefinite matrices and systems of vectors in an Euclidean space. Also, some examples of the interplay between matrix theory, graph theory, and $n-$ dimensional Euclidean geometry have been presented. For more information, please refer to [5, 7, 26].

Motivated by some of the above recent papers, we consider the Hankel matrices which are defined by the elements of the sequences $\{u_n\}$ and $\{v_n\}$ in the second section. Also, we give the eigenvalues and norms of the Gram matrix, which is defined by the elements of the sequences $\{u_n\}$ and $\{v_n\}$ in the third section.

2. Hankel Matrices with $\{u_n\}$ and $\{v_n\}$

In this section, we consider the Hankel matrices, which are defined by elements of the sequences $\{u_n\}$ and $\{v_n\}$. Firstly, we give the Hankel matrix H_u with elements of the sequence $\{u_n\}$ as follows:

$$H_u = (h_{i,j}) \text{ and } h_{i,j} = u_{i+j} \tag{12}$$

for $i, j = 0, 1, \ldots, n-1$.

Theorem 2.1. *The characteristic polynomial of the Hankel matrix H_u is given as follows:*

$$|\lambda I - H_u| = \begin{cases} \lambda^{n-2}\left(\lambda^2 - \frac{u_{2n-1}-1}{p}\lambda + \frac{2-v_{2n}}{p^2(p^2+4)}\right), & n \text{ is even} \\ \lambda^{n-2}\left(\lambda^2 - \frac{u_{2n-1}-1}{p}\lambda + \frac{p^2+2-v_{2n}}{p^2(p^2+4)}\right), & n \text{ is odd} \end{cases}. \tag{13}$$

Proof. If we apply the row and column operations to the Hankel matrix H_u, we obtain that the rank of H_u is 2. Therefore, the characteristic equation of the matrix H_u is

$$\lambda^n - A\lambda^{n-1} + B\lambda^{n-2} = 0.$$

In this equation, $A = trH_u$ and B equals the sum of second-order principal minor. Firstly, let's prove the following identity:

$$A = trH_u = \sum_{k=o}^{n-1} u_{2k}.$$

Using Binet's formula for the sequence $\{u_n\}$ and (1), we have

$$\sum_{k=0}^{n-1} u_{2k} = \frac{-1}{p^2}\left(u_{2n-2} - u_{2n} + u_2\right) = \frac{u_{2n-1}-1}{p}.$$

The sum of the second-order principal minors is

$$B=\sum_{k=1}^{n-1}(k-n)\,u_k^2.$$

Using Binet's formula and the recurrence relation of the sequence $\{u_n\}$, we have

$$\sum_{k=1}^{n-1}ku_k^2=\begin{cases}\dfrac{1}{p^2+4}\left(\dfrac{npv_{2n-1}-v_{2n}+2}{p^2}+n\right), & n \text{ is even}\\ \dfrac{1}{p^2+4}\left(\dfrac{npv_{2n-1}-v_{2n}+2}{p^2}+1-n\right), & n \text{ is odd}\end{cases}$$

and

$$\sum_{k=1}^{n-1}-nu_k^2=\begin{cases}\frac{-nv_{2n-1}-np}{(p^2+4)p}, & n \text{ is even}\\ \frac{-nv_{2n-1}+np}{(p^2+4)p}, & n \text{ is odd}\end{cases}.$$

Hereby, the characteristic polynomial of the matrix H_u is given as

$$|\lambda I-H_u|=\begin{cases}\lambda^{n-2}\left(\lambda^2-\dfrac{u_{2n-1}-1}{p}\lambda+\dfrac{2-v_{2n}}{p^2(p^2+4)}\right), & n \text{ is even}\\ \lambda^{n-2}\left(\lambda^2-\dfrac{u_{2n-1}-1}{p}\lambda+\dfrac{p^2+2-v_{2n}}{p^2(p^2+4)}\right), & n \text{ is odd}\end{cases}.$$

□

Using (13), (10), and (8), the eigenvalues of the Hankel matrix H_u are obtained as

$$\lambda_{1,2}=\begin{cases}\frac{1}{2p}\left(u_{2n-1}-1\mp(v_{2n-1}+p)\sqrt{\frac{1}{p^2+4}}\right), & n \text{ is even}\\ \frac{1}{2p}\left(u_{2n-1}-1\mp\sqrt{\dfrac{(v_{2n-1}+p)^2-4p^2}{p^2+4}}\right), & n \text{ is odd}\end{cases} \tag{14}$$

and $\lambda_m=0$ for $m=3,4,\ldots,n$.

Taking $p=1$ in (14), the eigenvalues of the Hankel matrix, which is defined by Fibonacci numbers are obtained as follows:

$$\lambda_{1,2}=\begin{cases}\frac{1}{2}\left(F_{2n-1}-1\mp\frac{1}{\sqrt{5}}\left(L_{2n-1}+1\right)\right), & n \text{ is even}\\ \frac{1}{2}\left(F_{2n-1}-1\mp\frac{1}{\sqrt{5}}\sqrt{L_{2n-1}^2+2L_{2n-1}-3}\right), & n \text{ is odd}\end{cases}$$

and $\lambda_m=0$ for $m=3,4,\ldots,n$ (see [28]). Similarly, taking $p=2$ in (14), the eigenvalues of the Hankel matrix H_P, which is defined by the Pell numbers are given as

$$\lambda_{1,2}=\begin{cases}\frac{1}{4}\left(P_{2n-1}-1\mp\frac{1}{2\sqrt{2}}\left(Q_{2n-1}+2\right)\right), & n \text{ is even}\\ \frac{1}{4}\left(P_{2n-1}-1\mp\frac{1}{2\sqrt{2}}\sqrt{Q_{2n-1}^2+4Q_{2n-1}-12}\right), & n \text{ is odd}\end{cases}$$

and $\lambda_m=0$ for $m=3,4,\ldots,n$.

Naturally, the determinant of the Hankel matrix H_u is $\det H_u = -1$ for $n = 2$. For other n, $\det H_u = 0$.

Since the Hankel matrix H_u is symmetric matrix, the spectral norm of H_u is the maximum eigenvalue of H_u. So, the spectral norm of the Hankel matrix H_u is obtained as follows:

$$(15)\qquad \|H_u\|_2 = \begin{cases} \dfrac{1}{2p}\left(u_{2n-1} - 1 + (v_{2n-1} + p)\sqrt{\dfrac{1}{p^2+4}}\right), & n \text{ is even} \\ \dfrac{1}{2p}\left(u_{2n-1} - 1 + \sqrt{\dfrac{(v_{2n-1} + p)^2 - 4p^2}{p^2+4}}\right), & n \text{ is odd} \end{cases}.$$

Therefore, the spectral norm of the Hankel matrix, which is defined by the Fibonacci numbers is stated as

$$\|H_F\|_2 = \begin{cases} \frac{1}{2}\left(F_{2n-1} - 1 + \frac{1}{\sqrt{5}}\left(L_{2n-1} + 1\right)\right), & n \text{ is even} \\ \frac{1}{2}\left(F_{2n-1} - 1 + \frac{1}{\sqrt{5}}\sqrt{L_{2n-1}^2 + 2L_{2n-1} - 3}\right), & n \text{ is odd} \end{cases}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers (see [28]). The spectral norm of the Hankel matrix, which is defined by the Pell numbers is given as

$$\|H_P\|_2 = \begin{cases} \frac{1}{4}\left(P_{2n-1} - 1 + \frac{1}{2\sqrt{2}}\left(Q_{2n-1} + 2\right)\right), & n \text{ is even} \\ \frac{1}{4}\left(P_{2n-1} - 1 + \frac{1}{2\sqrt{2}}\sqrt{Q_{2n-1}^2 + 4Q_{2n-1} - 12}\right), & n \text{ is odd} \end{cases}$$

where P_n and Q_n are the nth Pell and Pell-Lucas numbers (see [1]).

Theorem 2.2. *The Euclidean norm of the Hankel matrix H_u is given as follows:*

$$(16)\qquad \|H_u\| = \begin{cases} \sqrt{\dfrac{v_{4n-2} - 2v_{2n-2} + p^2 + 2}{p^2\left(p^2+4\right)}}, & n \text{ is even} \\ \sqrt{\dfrac{v_{4n-2} - 2v_{2n-2} - p^2 + 2}{p^2\left(p^2+4\right)}}, & n \text{ is odd} \end{cases}.$$

Proof. Considering (12) and the Binet's formula of the sequence $\{u_n\}$, we have

$$\begin{aligned} \sum_{i,j=0}^{n-1} u_{i+j}^2 &= \frac{1}{(\alpha-\beta)^2}\sum_{i,j=0}^{n-1}\left(\alpha^{i+j} - \beta^{i+j}\right)^2 \\ &= \frac{1}{p^2+4}\sum_{i,j=0}^{n-1}\left(\alpha^{2i+2j} + \beta^{2i+2j} - 2(-1)^{i+j}\right). \end{aligned}$$

Using the recurrence relation of the sequence $\{v_n\}$, the result is clear. □

Considering different values of n and p in (15) and (16), we obtain the following table:

n	p	$\|H_u\|_2$	$\|H_u\|$
2	1	1.618	1.732
	2	2.414	2.449
3	1	4.646	4.690
	2	14.416	14.422
4	1	12.708	12.728
	2	84.426	84.427
5	1	33.712	33.719
	2	492.426	492.427
10	1	4180.724	4180.724
	2	3312554.427	3312554.427

Table 1. The spectral and Euclidean norms of H_u

Now, we consider $H_v = (h_{i,j})$ Hankel matrix, which is defined by the elements of the sequence $\{v_n\}$ as follows:

$$h_{i,j} = v_{i+j} \tag{17}$$

for $i, j = 0, 1, \ldots, n-1$. Similar to the proof of (13), we have the characteristic polynomial of the Hankel matrix H_v as follows:

$$|\lambda I - H_v| = \begin{cases} \lambda^{n-2}\left(\lambda^2 - \dfrac{v_{2n-1}+p}{p}\lambda + \dfrac{v_{2n}-2}{p^2}\right), & n \text{ is even} \\ \lambda^{n-2}\left(\lambda^2 - \dfrac{v_{2n-1}+p}{p}\lambda + \dfrac{v_{2n}-p^2-2}{p^2}\right), & n \text{ is odd} \end{cases}.$$

Therefore, the eigenvalues of the Hankel matrix H_v are given as

$$\lambda_{1,2} = \begin{cases} \frac{1}{2p}\left(v_{2n-1}+p \mp (u_{2n-1}-1)\sqrt{p^2+4}\right), & n \text{ is even} \\ \frac{1}{2p}\left(v_{2n-1}+p \mp \sqrt{(u_{2n-1}-1)^2(p^2+4)+4p^2}\right), & n \text{ is odd} \end{cases} \tag{18}$$

and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

If we take $p = 1$ and $p = 2$ in (18), we obtain the eigenvalues of the Hankel matrix with Lucas and Pell-Lucas numbers, respectively.

From (18), the determinant of the Hankel matrix H_v is $\det H_v = p^2 + 4$ for $n = 2$.

Using the eigenvalues of the Hankel matrix H_v, the spectral norm of H_v is obtained as follows:

$$\|H_v\|_2 = \begin{cases} \frac{1}{2p}\left(v_{2n-1}+p+(u_{2n-1}-1)\sqrt{p^2+4}\right), & n \text{ is even} \\ \frac{1}{2p}\left(v_{2n-1}+p+\sqrt{(u_{2n-1}-1)^2(p^2+4)+4p^2}\right), & n \text{ is odd} \end{cases}. \tag{19}$$

Taking $p = 1$ and $p = 2$ in (19), the special cases of the spectral norm of the Hankel matrix H_v are given as follows:

$$\|H_L\|_2 = \begin{cases} \frac{1}{2}\left(L_{2n-1} + 1 + \sqrt{5}\,(F_{2n-1} - 1)\right), & n \text{ is even} \\ \frac{1}{2}\left(L_{2n-1} + 1 + \sqrt{5\,(F_{2n-1} - 1)^2 + 4}\right), & n \text{ is odd} \end{cases}$$

where F_n and L_n are the nth Fibonacci and Lucas numbers and

$$\|H_Q\|_2 = \begin{cases} \frac{1}{4}\left(Q_{2n-1} + 2 + 2\sqrt{2}\,(P_{2n-1} - 1)\right), & n \text{ is even} \\ \frac{1}{4}\left(Q_{2n-1} + 2 + \sqrt{8\,(P_{2n-1} - 1)^2 + 16}\right), & n \text{ is odd} \end{cases}$$

where P_n and Q_n are the nth Pell and Pell-Lucas numbers.

Theorem 2.3. *The Euclidean norm of the Hankel matrix H_v is given as follows:*

$$\|H_v\| = \begin{cases} \sqrt{\dfrac{v_{4n-2} - 2v_{2n-2} + p^2 + 2}{p^2}}, & n \text{ is even} \\ \sqrt{\dfrac{v_{4n-2} - 2v_{2n-2} + 3p^2 + 2}{p^2}}, & n \text{ is odd} \end{cases}. \tag{20}$$

Proof. Considering the definition of the Hankel matrix H_v and the Binet's formula of the sequence $\{v_n\}$, we have

$$\begin{aligned} \sum_{i,j=0}^{n-1} v_{i+j}^2 &= \sum_{i,j=0}^{n-1} \left(\alpha^{i+j} + \beta^{i+j}\right)^2 \\ &= \sum_{i,j=0}^{n-1} \left(\alpha^{2i+2j} + \beta^{2i+2j} + 2(-1)^{i+j}\right). \end{aligned}$$

Using the recurrence relation of the sequence $\{v_n\}$, the result is clear. □

For the different values n and p, the Euclidean and spectral norms of the Hankel matrix H_v are given in the following table.

n	p	$\|H_v\|_2$	$\|H_v\|$
2	1	3.618	3.873
	2	6.828	6.928
3	1	10.583	10.677
	2	40.824	40.841
4	1	28.416	28.460
	2	238.794	238.797
5	1	75.409	75.425
	2	1392.794	1392.794
10	1	9348.382	9348.382
	2	9369318.793	9369318.793

Table 2. The spectral and Euclidean norms of H_v

3. Gram Matrices with $\{u_n\}$ and $\{v_n\}$

Definition 3.1. *The Gram matrix $G_u = (g_{i,j})$ is defined by the elements of the sequence $\{u_n\}$, as follows:*

$$g_{i,j} = \langle x_i, x_j \rangle$$

for $x_i = (u_i, u_{i+1}, \ldots, u_{n+i-1})$ where u_i is the ith elements of the sequence $\{u_n\}$.

Also, the elements of the Gram matrix G_u can be given as follows:

$$g_{i,j} = \sum_{k=0}^{n-1} u_{k+i} u_{k+j} \tag{21}$$

for $i, j = 0, 1, \ldots, n-1$. From (21), (5), (7) and the Binet's formula of $\{u_n\}$, $g_{i,j}$ can be considered as follows:

$$g_{i,j} = \begin{cases} \frac{u_n u_{n+i+j-1}}{p}, & n \text{ is even} \\ \frac{v_n v_{n+i+j-1} + p(-1)^{j+1} v_{i-j}}{p(p^2+4)}, & n \text{ is odd} \end{cases} \tag{22}$$

for $i, j = 0, 1, \ldots, n-1$. If we take $p = 1$ and $p = 2$ in (22), we obtain the Gram matrices which are defined by the Fibonacci and Pell numbers, respectively.

Using (12) and (21), we have $G_u = H_u H_u^T$, where the Hankel matrix H_u is defined with the sequence $\{u_n\}$. Since the matrix H_u is a symmetric matrix, we have

$$G_u = H_u^2. \tag{23}$$

Theorem 3.2. *The eigenvalues of the Gram matrix G_u are obtained as*

$$\lambda_{1,2} = \frac{1}{2p^2(p^2+4)} \left(v_{2n-1}^2 - 2v_{2n-2} + p^2 + 4 \mp (u_{4n-2} - 2u_{2n-2} - p) \sqrt{p^2+4} \right)$$

for even n and

$$\lambda_{1,2} = \frac{1}{2p^2(p^2+4)} \left(v_{2n-1}^2 - 2v_{2n-2} - p^2 + 4 \mp (u_{2n-1} - 1) \sqrt{\left((v_{2n-1} + p)^2 - 4p^2\right)(p^2+4)} \right)$$

for odd n . Also, $\lambda_m = 0$ for $m = 3, 4, \ldots, n$,

Proof. Using (14) and (23), the result is clear. □

Taking $p = 1$ in the above theorem, the eigenvalues of the Gram matrix G_F with Fibonacci numbers are given as

$$\lambda_{1,2} = \begin{cases} \frac{1}{10} \left(L_{2n-1}^2 - 2L_{2n-2} + 5 \mp \sqrt{5} (F_{4n-2} - 2F_{2n-2} - 1) \right), & n \text{ is even} \\ \frac{1}{10} \left(L_{2n-1}^2 - 2L_{2n-2} + 3 \mp (F_{2n-1} - 1) \sqrt{5\left((L_{2n-1} + 1)^2 - 4\right)} \right), & n \text{ is odd} \end{cases}$$

and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

For $p = 2$, the eigenvalues of the Gram matrix G_P with Pell numbers are obtained as

$$\lambda_{1,2} = \begin{cases} \frac{1}{64} \left(Q_{2n-1}^2 - 2Q_{2n-2} + 8 \mp 2\sqrt{2} (P_{4n-2} - 2P_{2n-2} - 2) \right), & n \text{ is even} \\ \frac{1}{64} \left(Q_{2n-1}^2 - 2Q_{2n-2} \mp (P_{2n-1} - 1) \sqrt{8\left((Q_{2n-1} + 2)^2 - 16\right)} \right), & n \text{ is odd} \end{cases}$$

and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

Clearly, the determinant of the Gram matrix G_u is $\det G_u = 1$ for $n = 2$. Since the Gram matrix G_u is a symmetric matrix, the spectral norm of the matrix G_u is the maximum eigenvalue of the matrix G_u. So, we can give the following theorem.

Theorem 3.3. *The spectral norm of the Gram matrix G_u is given as*

$$\|G_u\|_2 = \frac{1}{2p^2\left(p^2+4\right)}\left(v_{2n-1}^2 - 2v_{2n-2} + p^2 + 4 + \left(u_{4n-2} - 2u_{2n-2} - p\right)\sqrt{p^2+4}\right)$$

for n even and

$$\|G_u\|_2 = \frac{1}{2p^2\left(p^2+4\right)}\left(v_{2n-1}^2 - 2v_{2n-2} - p^2 + 4 + \left(u_{2n-1} - 1\right)\sqrt{\left(\left(v_{2n-1}+p\right)^2 - 4p^2\right)\left(p^2+4\right)}\right)$$

for n odd.

Let's give the special cases of the above theorem for $p = 1$ and $p = 2$. The spectral norms of the Gram matrices G_F and G_P which are defined by the Fibonacci and Pell numbers are given as follows:

$$\|G_F\|_2 = \begin{cases} \frac{1}{10}\left(L_{2n-1}^2 - 2L_{2n-2} + 5 + \sqrt{5}\left(F_{4n-2} - 2F_{2n-2} - 1\right)\right), & n \text{ is even} \\ \frac{1}{10}\left(L_{2n-1}^2 - 2L_{2n-2} + 3 + \left(F_{2n-1} - 1\right)\sqrt{5\left(\left(L_{2n-1}+1\right)^2 - 4\right)}\right), & n \text{ is odd} \end{cases}$$

and

$$\|G_P\|_2 = \begin{cases} \frac{1}{64}\left(Q_{2n-1}^2 - 2Q_{2n-2} + 8 + 2\sqrt{2}\left(P_{4n-2} - 2P_{2n-2} - 2\right)\right), & n \text{ is even} \\ \frac{1}{64}\left(Q_{2n-1}^2 - 2Q_{2n-2} + \left(P_{2n-1} - 1\right)\sqrt{8\left(\left(Q_{2n-1}+2\right)^2 - 16\right)}\right), & n \text{ is odd} \end{cases}.$$

Theorem 3.4. *The Euclidean norm of the Gram matrix G_u is given as follows:*

$$\|G_u\| = \begin{cases} \dfrac{u_n^2}{p^2}\sqrt{v_{4n-4}}, & n \text{ is even} \\ \dfrac{\sqrt{v_n^2\left(v_n^2 v_{4n-4} + 4p^2\left(1 - v_{2n-2}\right)\right) + 2p^4}}{p^2\left(p^2+4\right)}, & n \text{ is odd} \end{cases} \tag{24}$$

Proof. For even n, using Binet's formula of the sequence $\{u_n\}$ and (22), we have,

$$\begin{aligned} \|G_u\|^2 &= \sum_{i,j=0}^{n-1} \frac{1}{p^2} u_n^2 u_{n+i+j-1}^2 \\ &= \frac{u_n^2}{p^2\left(p^2+4\right)} \sum_{i,j=0}^{n-1}\left(\alpha^{2n+2i+2j-2} + \beta^{2n+2i+2j-2} - 2\left(-1\right)^{n+i+j-1}\right). \end{aligned}$$

From (1),(6) and (7), we obtain

$$\|G_u\| = \frac{u_n^2}{p^2}\sqrt{v_{4n-4}}.$$

Let's take as an n odd. Similarly, using Binet's formula of the sequence $\{v_n\}$ and (22), we have

$$\begin{aligned}\|G_u\|^2 &= \sum_{i,j=0}^{n-1}\left(\frac{v_n v_{n+i+j-1}+p\,(-1)^{j+1}\,v_{i-j}}{p\,(p^2+4)}\right)^2\\ &= \frac{1}{p^2\,(p^2+4)^2}\sum_{i,j=0}^{n-1}\left(v_n^2 v_{n+i+j-1}^2+p^2 v_{i-j}^2+2v_n v_{n+i+j-1}p\,(-1)^{j+1}\,v_{i-j}\right)\end{aligned}$$

for odd n. Now, if we consider the sums separately and use (5), (7), and (11), we obtain

$$\begin{aligned}&\textstyle\sum_{i,j=0}^{n-1} v_n^2 v_{n+i+j-1}^2 = \frac{1}{p^2} v_n^4 v_{4n-4} + 2v_n^2,\\ &\textstyle\sum_{i,j=0}^{n-1} 2v_n v_{n+i+j-1} p\,(-1)^{j+1}\,v_{i-j} = -4v_n^2 v_{2n-2},\\ &\textstyle\sum_{i,j=0}^{n-1} p^2 v_{i-j}^2 = 2v_{2n} + 2p^2 - 4.\end{aligned}$$

Hence, the Euclidean norm of the matrix G_u is obtained as follows:

$$\|G_u\| = \frac{\sqrt{v_n^2\,(v_n^2 v_{4n-4} + 4p^2\,(1 - v_{2n-2})) + 2p^4}}{p^2\,(p^2+4)}.$$

□

Taking $p = 1$ and $p = 2$ in (24), the Euclidean norms are obtained as

$$\|G_F\| = \begin{cases} F_n^2\sqrt{L_{4n-4}}, & n \text{ is even}\\ \frac{1}{5}\sqrt{L_n^2\,(L_n^2 L_{4n-4} + 4 - 4L_{2n-2}) + 2}, & n \text{ is odd}\end{cases}$$

and

$$\|G_P\| = \begin{cases} \frac{1}{4}P_n^2\sqrt{Q_{4n-4}}, & n \text{ is even}\\ \frac{1}{32}\sqrt{Q_n^2\,(Q_n^2 Q_{4n-4} + 16 - 16Q_{2n-2}) + 32}, & n \text{ is odd}\end{cases}$$

where G_F and G_P are the Gram matrices, which are defined by the Fibonacci and Pell numbers, respectively.

We give the numerical examples for the Euclidean and spectral norms of the Gram matrix G_u in the following table:

n	p	$\|G_u\|_2$	$\|G_u\|$
2	1	2.618	2.646
	2	5.828	5.831
3	1	21.583	21.587
	2	207.827	207.827
4	1	161.498	161.499
	2	7127.818	7127.818
5	1	1136.493	1136.493
	2	242483.818	242483.818
10	1	17478449.476	17478449.476
	2	10973016830357.818	10973016830357.818

Table 3. The spectral and Euclidean norms of G_u

Until now, we have investigated the Gram matrix with elements of the sequence $\{u_n\}$. Now, we give a definition of the Gram matrix, which is defined by elements of the sequence $\{v_n\}$. We consider

$$g_{ij} = \langle x_i, x_j \rangle$$

where $x_i = (v_i, v_{i+1}, \ldots, v_{n+i-1})$ where v_i is the *ith* element of the sequence $\{v_n\}$. Also, the elements of the Gram matrix are given as follows:

$$g_{i,j} = \sum_{k=0}^{n-1} v_{k+i} v_{k+j}. \tag{25}$$

From (25), (5), (7) and the Binet's formula of $\{v_n\}$, we have

$$g_{i,j} = \begin{cases} \dfrac{\left(p^2+4\right) u_n u_{n+i+j-1}}{p}, & n \text{ is even} \\ \dfrac{v_n v_{n+i+j-1} + p\left(-1\right)^j v_{i-j}}{p}, & n \text{ is odd} \end{cases} \tag{26}$$

and for $i, j = 0, 1, \ldots, n-1$.

Using the relationship $G_v = H_v^2$ and (18), the eigenvalues of the matrix G_v are given in the following theorem.

Theorem 3.5. *Let the matrix G_v be a Gram matrix with elements of the sequence $\{v_n\}$. The eigenvalues of matrix G_v are given as*

$$\lambda_{1,2} = \frac{1}{2p^2}\left(v_{2n-1}^2 - 2v_{2n-2} + p^2 + 4 \mp \left(u_{4n-2} - 2u_{2n-2} - p\right)\sqrt{p^2+4}\right) \tag{27}$$

for even n and

$$\lambda_{1,2} = \frac{1}{2p^2}\left(v_{2n-1}^2 - 2v_{2n-2} + 3p^2 + 4 \mp \left(v_{2n-1} + p\right)\sqrt{\left(p^2+4\right)\left(u_{2n-1}-1\right)^2 + 4p^2}\right) \tag{28}$$

for odd n and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

For $p = 1$ in (27) and (28), the eigenvalues of the Gram matrix G_L with Lucas numbers are obtained as

$$\lambda_{1,2} = \begin{cases} \frac{1}{2}\left(L_{2n-1}^2 - 2L_{2n-2} + 5 \mp \sqrt{5}\left(F_{4n-2} - 2F_{2n-2} - 1\right)\right), & n \text{ is even} \\ \frac{1}{2}\left(L_{2n-1}^2 - 2L_{2n-2} + 7 \mp \left(L_{2n-1}+1\right)\sqrt{5\left(F_{2n-1}-1\right)^2 + 4}\right), & n \text{ is odd} \end{cases}$$

and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

Taking $p = 2$ in (27) and (28), the eigenvalues of the Gram matrix G_Q with Pell-Lucas numbers are given as

$$\lambda_{1,2} = \begin{cases} \frac{1}{8}\left(Q_{2n-1}^2 - 2Q_{2n-2} + 8 \mp 2\sqrt{2}\left(P_{4n-2} - 2P_{2n-2} - 2\right)\right), & n \text{ is even} \\ \frac{1}{8}\left(Q_{2n-1}^2 - 2Q_{2n-2} + 16 \mp \left(Q_{2n-1}+2\right)\sqrt{8\left(P_{2n-1}-1\right)^2 + 16}\right), & n \text{ is odd} \end{cases}$$

and $\lambda_m = 0$ for $m = 3, 4, \ldots, n$.

Clearly, the determinant of the Gram matrix G_v is $\det G_v = p^4 + 8p^2 + 16$ for $n = 2$.

Since the Gram matrix G_v is a symmetric matrix, the spectral norm of G_v is the maximum eigenvalue of the matrix G_v. So, we can give the following theorem:

Theorem 3.6. *The spectral norm of the Gram matrix G_v is given as*

$$\|G_v\|_2 = \frac{1}{2p^2}\left(v_{2n-1}^2 - 2v_{2n-2} + p^2 + 4 + \left(u_{4n-2} - 2u_{2n-2} - p\right)\sqrt{p^2+4}\right) \tag{29}$$

for even n

$$\|G_v\|_2 = \frac{1}{2p^2}\left(v_{2n-1}^2 - 2v_{2n-2} + 3p^2 + 4 + \left(v_{2n-1} + p\right)\sqrt{\left(p^2+4\right)\left(u_{2n-1}-1\right)^2 + 4p^2}\right) \tag{30}$$

for odd n.

Taking $p = 1$ and $p = 2$ in (29) and (30), the spectral norms of G_L and G_Q matrices are obtained as follows:

$$\|G_L\|_2 = \begin{cases} \frac{1}{2}\left(L_{2n-1}^2 - 2L_{2n-2} + 5 + \sqrt{5}\left(F_{4n-2} - 2F_{2n-2} - 1\right)\right), & n \text{ is even} \\ \frac{1}{2}\left(L_{2n-1}^2 - 2L_{2n-2} + 7 + \left(L_{2n-1}+1\right)\sqrt{5\left(F_{2n-1}-1\right)^2 + 4}\right), & n \text{ is odd} \end{cases}$$

and

$$\|G_Q\|_2 = \begin{cases} \frac{1}{8}\left(Q_{2n-1}^2 - 2Q_{2n-2} + 8 + 2\sqrt{2}\left(P_{4n-2} - 2P_{2n-2} - 2\right)\right), & n \text{ is even} \\ \frac{1}{8}\left(Q_{2n-1}^2 - 2Q_{2n-2} + 16 + \left(Q_{2n-1}+2\right)\sqrt{8\left(P_{2n-1}-1\right)^2 + 16}\right), & n \text{ is odd} \end{cases}.$$

Theorem 3.7. *The Euclidean norm of the Gram matrix G_v is given as follows:*

$$\|G_v\| = \begin{cases} \frac{\left(p^2+4\right)}{p^2}u_n^2\sqrt{v_{4n-4}}, & n \text{ is even} \\ \frac{\sqrt{v_n^2\left(v_n^2 v_{4n-4} + 4p^2\left(1+v_{2n-2}\right)\right) + 2p^4}}{p^2}, & n \text{ is odd} \end{cases}.$$

Proof. For even n, using Binet's formula of the sequence $\{v_n\}$ and (26), we have

$$\begin{aligned} \|G_v\|^2 &= \sum_{i,j=0}^{n-1} \frac{\left(p^2+4\right)^2}{p^2} u_n^2 u_{n+i+j-1}^2 \\ &= \frac{\left(p^2+4\right)^2 u_n^2}{p^2} \sum_{i,j=0}^{n-1}\left(\alpha^{2n+2i+2j-2} + \beta^{2n+2i+2j-2} - 2\left(-1\right)^{n+i+j-1}\right). \end{aligned}$$

Using (6) and (7), we obtain

$$\|G_u\| = \frac{\left(p^2+4\right)}{p^2}u_n^2\sqrt{v_{4n-4}}.$$

For odd n, using Binet's formula of the sequence $\{v_n\}$ and (26), we have

$$\begin{aligned} \|G_v\|^2 &= \sum_{i,j=0}^{n-1}\left(\frac{v_n v_{n+i+j-1} + p\left(-1\right)^j v_{i-j}}{p}\right)^2 \\ &= \frac{1}{p^2}\sum_{i,j=0}^{n-1} v_n^2 v_{n+i+j-1}^2 + p^2 v_{i-j}^2 + 2v_n v_{n+i+j-1} p\left(-1\right)^j v_{i-j}. \end{aligned}$$

Now, if we consider the sums separately and use (5), (7), and (11), we obtain

$$\sum_{i,j=0}^{n-1} v_n^2 v_{n+i+j-1}^2 = \frac{1}{p^2} v_n^4 v_{4n-4} + 2v_n^2,$$
$$\sum_{i,j=0}^{n-1} 2v_n v_{n+i+j-1} p\left(-1\right)^j v_{i-j} = 4v_n^2 v_{2n-2},$$
$$\sum_{i,j=0}^{n-1} p^2 v_{i-j}^2 = 2v_{2n} + 2p^2 - 4.$$

Therefore, the Euclidean norm of the matrix G_v is obtained as follows:

$$\|G_v\| = \frac{\sqrt{v_n^2\left(v_n^2 v_{4n-4} + 4p^2\left(1 + v_{2n-2}\right)\right) + 2p^4}}{p^2}.$$

□

For $p = 1$ and $p = 2$, the Euclidean norms of the Gram matrix G_L and G_Q are given as

$$\|G_L\| = \begin{cases} 5F_n^2\sqrt{L_{4n-4}}, & n \text{ is even} \\ \sqrt{L_n^2\left(L_n^2 L_{4n-4} + 4 + 4L_{2n-2}\right) + 2}, & n \text{ is odd} \end{cases}$$

and

$$\|G_Q\| = \begin{cases} 2P_n^2\sqrt{Q_{4n-4}}, & n \text{ is even} \\ \frac{1}{4}\sqrt{Q_n^2\left(Q_n^2 Q_{4n-4} + 16 + 16Q_{2n-2}\right) + 32}, & n \text{ is odd} \end{cases}$$

where F_n, L_n, P_n and Q_n are the nth Fibonacci, Lucas, Pell and Pell-Lucas numbers, respectively.

Considering different values of n and p, we have the following table:

n	p	$\|G_v\|_2$	$\|G_v\|$
2	1	13.090	13.229
	2	46.627	46.648
3	1	111.991	112.009
	2	1666.618	1666.618
4	1	807.492	807.496
	2	57022.545	57022.545
5	1	5686.468	5686.468
	2	1939874.545	1939874.545
10	1	87392247.382	87392247.382
	2	87784134642862.543	87784134642862.543

Table 4. The spectral and Euclidean norms of G_v

4. Conclusion

In this study, we considered the sequences $\{u_n\}$ and $\{v_n\}$ which are defined by the recurrence relations (1) and (2). These sequences are a general form of the Fibonacci, Pell, Lucas, and Pell-Lucas sequences. Afterwards, we studied the Gram and Hankel matrices, which are defined by the elements of the sequences $\{u_n\}$ and $\{v_n\}$. The eigenvalues, determinants, and various norms of these matrices are obtained. When we give special cases of all the results obtained, we can obtain the eigenvalues, determinants, and norms of the Gram and Hankel matrices, which are defined by the Fibonacci, Pell, Lucas, Pell-Lucas numbers.

Our suggestion for researchers interested in this subject is to define the Gram and Hankel matrices with Horadam numbers, and investigate the properties of these matrices.

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