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TATE-SHAFAREVICH GROUPS OVER THE COMMUTATIVE DIAGRAM OF 8 ABELIAN VARIETIES

Hoseog Yu

Abstract. Suppose that there are 8 abelian varieties defined over a number field K which satisfy a commutative diagram. We show that if we know that three out of four short exact sequences satisfy the rate formula of Tate-Shafarevich groups, then the unknown short exact sequence satisfies the rate formula of Tate-Shafarevich groups, too.

1. Introduction

Let K be a number field. Write $\overline{K}$, G_K, M_K, K_v for the algebraic closure of K, $\mathrm{Gal}(\overline{K}/K)$, a complete set of places on K, the completion of K at the place $v \in M_K$, respectively.

Let A be an abelian variety defined over K and let $Ш(A/K)$ denote the Tate-Shafarevich group of A over K. We assume throughout that the Tate-Shafarevich groups are finite for any abelian varieties. We write $[X]$ for the order of a finite abelian group X. For a morphism $f\colon A \to B$ defined over K, write f_K for the restriction morphism $f_K\colon A(K) \to B(K)$.

Given a short exact sequence of abelian varieties A, B and C defined over K

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0, \tag{1}$$

we define that (1) satisfies the rate formula of Tate-Shafarevich groups if it holds that

$$\frac{[Ш(A/K)][Ш(C/K)]}{[Ш(B/K)]} = \frac{[\mathrm{Coker}(g_K)][\mathrm{Coker}(f_K^\vee)]}{\prod_{v\in M_K}[\mathrm{Coker}(g_{K_v})]},$$

where $f^\vee\colon B^\vee \to A^\vee$ is the dual morphism of $f\colon A \to B$. Denote the quotient $\dfrac{[\mathrm{Coker}(g_K))][\mathrm{Coker}(f_K^\vee)]}{\prod_{v\in M_K}[\mathrm{Coker}(g_{K_v})]}$ by $TS(f,g)$.

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Suppose that A_{ij}'s are abelian varieties defined over K which satisfy the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A_{12} & \xrightarrow{f} & A_{13} & & \\
 & & & & a\downarrow & & m\downarrow & & \\
0 & \longrightarrow & A_{21} & \xrightarrow{p} & A_{22} & \xrightarrow{q} & A_{23} & \longrightarrow & 0 \\
 & & g\downarrow & & b\downarrow & & n\downarrow & & \\
0 & \longrightarrow & A_{31} & \xrightarrow{s} & A_{32} & \xrightarrow{t} & A_{33} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2}
$$

where f and g are isogenies. Note that in the above commutative diagram we have four short exact sequences:

$$0 \longrightarrow A_{21} \xrightarrow{p} A_{22} \xrightarrow{q} A_{23} \longrightarrow 0,$$
$$0 \longrightarrow A_{31} \xrightarrow{s} A_{32} \xrightarrow{t} A_{33} \longrightarrow 0,$$
$$0 \longrightarrow A_{12} \xrightarrow{a} A_{22} \xrightarrow{b} A_{32} \longrightarrow 0,$$

and

$$0 \longrightarrow A_{13} \xrightarrow{m} A_{23} \xrightarrow{n} A_{33} \longrightarrow 0.$$

Main Theorem. *Suppose that three short sequences in the above four short exact sequences satisfy the rate formula of Tate-Shafarevich groups. Then the remaining short exact sequence satisfies the rate formula of Tate-Shafarevich groups, too.*

Proof. It is clear from Theorem 5. □

2. Proof

From the commutative diagram (2) we have the following commutative diagram

$$
\begin{array}{ccccccc}
 & & A_{12}(K) & \xrightarrow{f_K} & A_{13}(K) & \twoheadrightarrow & \operatorname{Coker}(f_K) \\
 & & \downarrow{\scriptstyle a_K} & & \downarrow{\scriptstyle m_K} & & \downarrow{\scriptstyle \widehat{m}} \\
A_{21}(K) & \overset{p_K}{\hookrightarrow} & A_{22}(K) & \xrightarrow{q_K} & A_{23}(K) & \twoheadrightarrow & \operatorname{Coker}(q_K) \\
\downarrow{\scriptstyle g_K} & & \downarrow{\scriptstyle b_K} & & \downarrow{\scriptstyle n_K} & & \downarrow{\scriptstyle \widehat{n}} \\
A_{31}(K) & \overset{s_K}{\hookrightarrow} & A_{32}(K) & \xrightarrow{t_K} & A_{33}(K) & \twoheadrightarrow & \operatorname{Coker}(t_K) \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Coker}(g_K) & \overset{\widehat{s}}{\dashrightarrow} & \operatorname{Coker}(b_K) & \overset{\widehat{t}}{\dashrightarrow} & \operatorname{Coker}(n_K), & &
\end{array}
$$

where $\widehat{m}$, $\widehat{n}$, $\widehat{s}$ and $\widehat{t}$ are induced morphisms from $m_K,\ n_K,\ s_K,\ t_K$.

Lemma 1. *We get*

$$\frac{[\operatorname{Coker}(f_K)][\operatorname{Coker}(t_K)]}{[\operatorname{Coker}(q_K)]} = \frac{[\operatorname{Coker}(g_K)][\operatorname{Coker}(n_K)]}{[\operatorname{Coker}(b_K)]}.$$

Proof. From diagram chasing we can show the following three isomorphisms $\operatorname{Ker}(\widehat{m}) \cong \operatorname{Ker}(\widehat{s})$, $\operatorname{Coker}(\widehat{n}) \cong \operatorname{Coker}(\widehat{t})$ and

$$\operatorname{Ker}(\widehat{n})/\widehat{m}(\operatorname{Coker}(f_K)) \cong \operatorname{Ker}(\widehat{t})/\widehat{s}(\operatorname{Coker}(g_K)).$$

Because

$$\frac{[\operatorname{Coker}(f_K)][\operatorname{Coker}(t_K)]}{[\operatorname{Coker}(q_K)]} = \frac{[\operatorname{Ker}(\widehat{m})][\operatorname{Coker}(\widehat{n})]}{[\operatorname{Ker}(\widehat{n})/\widehat{m}(\operatorname{Coker}(f_K))]},$$

the lemma follows. □

From the dual commutative diagram of (2), we have the dual commutative diagram

(3)
$$
\begin{array}{ccccccc}
A_{33}^\vee(K) & \overset{t_K^\vee}{\hookrightarrow} & A_{32}^\vee(K) & \xrightarrow{s_K^\vee} & A_{31}^\vee(K) & \twoheadrightarrow & \operatorname{Coker}(s_K^\vee) \\
\downarrow{\scriptstyle n_K^\vee} & & \downarrow{\scriptstyle b_K^\vee} & & \downarrow{\scriptstyle g_K^\vee} & & \downarrow{\scriptstyle \widehat{g^\vee}} \\
A_{23}^\vee(K) & \overset{q_K^\vee}{\hookrightarrow} & A_{22}^\vee(K) & \xrightarrow{p_K^\vee} & A_{21}^\vee(K) & \twoheadrightarrow & \operatorname{Coker}(p_K^\vee) \\
\downarrow{\scriptstyle m_K^\vee} & & \downarrow{\scriptstyle a_K^\vee} & & & & \\
A_{13}^\vee(K) & \xrightarrow{f_K^\vee} & A_{12}^\vee(K) & & & & \\
\downarrow & & \downarrow & & & & \\
\operatorname{Coker}(m_K^\vee) & \overset{\widehat{f^\vee}}{\dashrightarrow} & \operatorname{Coker}(a_K^\vee). & & & &
\end{array}
$$

where $\widehat{f^\vee}$ and $\widehat{g^\vee}$ are induced morphism from $f_K^\vee,\ g_K^\vee$.

From (3) we have following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & s_K^\vee(A_{32}^\vee(K)) & \longrightarrow & A_{31}^\vee(K) & \longrightarrow & \mathrm{Coker}(s_K^\vee) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle g_K^\vee} & & \downarrow{\scriptstyle \widehat{g^\vee}} & & \\ 0 & \longrightarrow & p_K^\vee(A_{22}^\vee(K)) & \longrightarrow & A_{21}^\vee(K) & \longrightarrow & \mathrm{Coker}(p_K^\vee) & \longrightarrow & 0, \end{array}$$

where T is the restriction morphism of $g_K^\vee$ on $s_K^\vee(A_{32}^\vee(K))$.

Then the snake lemma becomes

$$(4)\quad 0 \to \mathrm{Ker}(T) \to \mathrm{Ker}(g_K^\vee) \to \mathrm{Ker}(\widehat{g^\vee}) \to \mathrm{Coker}(T) \to \mathrm{Coker}(g_K^\vee) \to \mathrm{Coker}(\widehat{g^\vee}) \to 0.$$

Similarly we have

$$(5)\quad 0 \to \mathrm{Ker}(S) \to \mathrm{Ker}(f_K^\vee) \to \mathrm{Ker}(\widehat{f^\vee}) \to \mathrm{Coker}(S) \to \mathrm{Coker}(f_K^\vee) \to \mathrm{Coker}(\widehat{f^\vee}) \to 0$$

from

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ m_K^\vee(A_{23}^\vee(K)) & \xrightarrow{S} & a_K^\vee(A_{22}^\vee(K)) \\ \downarrow & & \downarrow \\ A_{13}^\vee(K) & \xrightarrow{f_K^\vee} & A_{12}^\vee(K) \\ \downarrow & & \downarrow \\ \mathrm{Coker}(m_K^\vee) & \xrightarrow{\widehat{f^\vee}} & \mathrm{Coker}(a_K^\vee) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where S is the restriction morphisam of $f_K^\vee$ on $m_K^\vee(A_{23}^\vee(K))$.

Through diagram chasing in (3) we can show that $\mathrm{Ker}(T) \cong \mathrm{Ker}(S)$ and $\mathrm{Coker}(T) \cong \mathrm{Coker}(S)$. From the exact sequences (4) and (5)

$$\frac{[\mathrm{Ker}(T)]}{[\mathrm{Coker}(T)]} = \frac{[\mathrm{Ker}(g_K^\vee)]}{[\mathrm{Coker}(g_K^\vee)]} \cdot \frac{[\mathrm{Coker}(\widehat{g^\vee})]}{[\mathrm{Ker}(\widehat{g^\vee})]} = \frac{[\mathrm{Ker}(g_k^\vee)]}{[\mathrm{Coker}(g_K^\vee)]} \cdot \frac{[\mathrm{Coker}(p_K^\vee)]}{[\mathrm{Coker}(s_K^\vee)]}$$

and

$$\frac{[\mathrm{Ker}(S)]}{[\mathrm{Coker}(S)]} = \frac{[\mathrm{Ker}(f_K^\vee)]}{[\mathrm{Coker}(f_K^\vee)]} \cdot \frac{[\mathrm{Coker}(a_K^\vee)]}{[\mathrm{Coker}(m_K^\vee)]}.$$

Lemma 2. *We get the equality*

$$\frac{[\mathrm{Ker}(f_K^\vee)]}{[\mathrm{Coker}(f_K^\vee)]}\cdot\frac{[\mathrm{Coker}(s_K^\vee)]}{[\mathrm{Coker}(p_K^\vee)]}=\frac{[\mathrm{Ker}(g_K^\vee)]}{[\mathrm{Coker}(g_K^\vee)]}\cdot\frac{[\mathrm{Coker}(m_K^\vee)]}{[\mathrm{Coker}(a_K^\vee)]}.$$

Lemma 3. *For an isogeny $f\colon A\to B$ defined over K, we have*

$$\frac{[Ш(B/K)]}{[Ш(A/K)]}=\frac{[\mathrm{Ker}(f_K^\vee)]}{[\mathrm{Coker}(f_K^\vee)]}\cdot\frac{[\mathrm{Coker}(f_K)]}{[\mathrm{Ker}(f_K)]}\cdot\prod_{v\in M_K}\frac{[\mathrm{Ker}(f_{K_v})]}{[\mathrm{Coker}(f_{K_v})]}.$$

Proof. See [2, p.98]. □

Lemma 4. *We get*

$$\frac{[Ш(A_{13}/K)]}{[Ш(A_{12}/K)]}\cdot\frac{TS(s,t)}{TS(p,q)}=\frac{[Ш(A_{31}/K)]}{[Ш(A_{21}/K)]}\cdot\frac{TS(m,n)}{TS(a,b)}.$$

Proof. We have the following equality

$$\begin{aligned}
&\frac{[Ш(A_{13}/K)]}{[Ш(A_{12}/K)]}\cdot\frac{TS(s,t)}{TS(p,q)}\\
&=\frac{[\mathrm{Ker}(f_K^\vee)]}{[\mathrm{Coker}(f_K^\vee)]}\cdot\frac{[\mathrm{Coker}(f_K)]}{[\mathrm{Ker}(f_K)]}\cdot\prod_{v\in M_K}\frac{[\mathrm{Ker}(f_{K_v})]}{[\mathrm{Coker}(f_{K_v})]}\\
&\quad\times\frac{[\mathrm{Coker}(t_K)][\mathrm{Coker}(s_K^\vee)]}{\prod_{v\in M_K}[\mathrm{Coker}(t_{K_v})]}\cdot\frac{\prod_{v\in M_K}[\mathrm{Coker}(q_{K_v})]}{[\mathrm{Coker}(q_K)][\mathrm{Coker}(p_K^\vee)]}\\
&=\frac{[\mathrm{Coker}(f_K)][\mathrm{Coker}(t_K)]}{[\mathrm{Coker}(q_K)]}\cdot\frac{[\mathrm{Ker}(f_K^\vee)]}{[\mathrm{Coker}(f_K^\vee)]}\cdot\frac{[\mathrm{Coker}(s_K^\vee)]}{[\mathrm{Coker}(p_K^\vee)]}\\
&\quad\times\prod_{v\in M_K}\frac{[\mathrm{Coker}(q_{K_v})]}{[\mathrm{Coker}(f_{K_v})][\mathrm{Coker}(t_{K_v})]}\cdot\frac{\prod_{v\in M_K}[\mathrm{Ker}(f_{K_v})]}{[\mathrm{Ker}(f_K)]}\\
&=\frac{[\mathrm{Coker}(g_K)][\mathrm{Coker}(n_K)]}{[\mathrm{Coker}(b_K)]}\cdot\frac{[\mathrm{Ker}(g_K^\vee)]}{[\mathrm{Coker}(g_K^\vee)]}\cdot\frac{[\mathrm{Coker}(m_K^\vee)]}{[\mathrm{Coker}(a_K^\vee)]}\\
&\quad\times\prod_{v\in M_K}\frac{[\mathrm{Coker}(b_{K_v})]}{[\mathrm{Coker}(g_{K_v})][\mathrm{Coker}(n_{K_v})]}\cdot\frac{\prod_{v\in M_K}[\mathrm{Ker}(g_{K_v})]}{[\mathrm{Ker}(g_K)]}\\
&=\frac{[Ш(A_{31}/K)]}{[Ш(A_{21}/K)]}\cdot\frac{TS(m,n)}{TS(a,b)}.
\end{aligned}$$

from lemmas. Note $\mathrm{Ker}(f_K)\cong\mathrm{Ker}(g_K)$ and $\mathrm{Ker}(f_{K_v})\cong\mathrm{Ker}(g_{K_v})$). □

Theorem 5. *If three equalities out of following four equalities hold, then the remaining one holds, too.*

$$TS(m,n)=\frac{[Ш(A_{13}/K)][Ш(A_{33}/K)]}{[Ш(A_{23}/K)]}$$

$$TS(a,b)=\frac{[Ш(A_{12}/K)][Ш(A_{32}/K)]}{[Ш(A_{22}/K)]}$$

$$TS(p,q)=\frac{[Ш(A_{21}/K)][Ш(A_{23}/K)]}{[Ш(A_{22}/K)]}$$

$$TS(s,t) = \frac{[Ш(A_{31}/K)][Ш(A_{33}/K)]}{[Ш(A_{32}/K)]}$$

Proof. From the previous lemma we get

$$\begin{aligned}\frac{TS(m,n)}{TS(a,b)} \cdot \frac{TS(p,q)}{TS(s,t)} &= \frac{[Ш(A_{13}/K)][Ш(A_{21}/K)]}{[Ш(A_{12}/K)][Ш(A_{31}/K)]} \\ &= \frac{[Ш(A_{13}/K)][Ш(A_{33}/K)]}{[Ш(A_{23}/K)]} \frac{[Ш(A_{22}/K)]}{[Ш(A_{12}/K)][Ш(A_{32}/K)]} \\ &\quad \times \frac{[Ш(A_{21}/K)][Ш(A_{23}/K)]}{[Ш(A_{22}/K)]} \frac{[Ш(A_{32}/K)]}{[Ш(A_{31}/K)][Ш(A_{33}/K)]}.\end{aligned}$$

Thus the theorem holds. □

3. Application

Let m and n be positive integers. Let L/K be a cyclic extension of number fields with Galois group G of order mn. Let H be the subgroup of G of order m and let $F = L^H$. Fix a generator $\sigma \in G$.

Let A be an abelian variety defined over K. For positive integers k, denote $x^k - 1$ by φ_k. For a monic polynomial $g(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, denote by $M(g)$ the companion matrix of $g(x)$

$$M(g) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{k-2} & -a_{k-1} \end{pmatrix} \in \mathrm{End}_K(A^k),$$

where $\mathrm{End}_K(A^k)$ is the endomorphism ring of A^k defined over K and 1 is the identity automorphism of A. Let $A(g)$ be an abelian variety defined over K satisfying that there is an isomorphism $\widehat{g}\colon A^k \to A(g)$ defined over L such that $\sigma(\widehat{g})^{-1} \circ \widehat{g} = M(g)$. For the existence and the uniqueness up to K-isomorphism of such a variety $A(g)$, see [1, §2]. Let $Res_{L/K}(A)$ be the restriction of scalars of A from L to K. Note that $A(\varphi_n) = Res_{F/K}(A)$ and $A(\varphi_{mn}) = Res_{L/K}(A)$.

We know that two short exact sequences

$$0 \longrightarrow A \longrightarrow Res_{L/K}(A) \longrightarrow A(\varphi_{mn}/\varphi_1) \longrightarrow 0$$

and

$$0 \longrightarrow A \longrightarrow Res_{F/K}(A) \longrightarrow A(\varphi_n/\varphi_1) \longrightarrow 0,$$

satisfy the rate formula of Tate-Shafarevich groups from [3, Main Theorem].

In the commutative diagram

$$\begin{array}{ccccc}
A(\varphi_{mn}/\varphi_n) & \hookrightarrow & Res_{L/K}(A) & \twoheadrightarrow & Res_{F/K}(A) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
Res_{F/K}(A(\varphi_m/\varphi_1)) & \hookrightarrow & Res_{F/K}(Res_{L/F}(A)) & \twoheadrightarrow & Res_{F/K}(A)
\end{array}$$

note that the upper short exact sequence satisfies the rate formula of Tate-Shafarevich groups because the lower one does.

Denote the k-dimensional column vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ by $\mathbf{1}_k$ and $\mathbf{0}_k$, respectively.

Denote by E_k the $k \times k$ identity matrix. For a matrix M_1 and M_2, denote by M_1^T the transpose of the matrix M_1 and denote by $M_1 \otimes M_2$ the Kronecker product of M_1 and M_2.

For a $k_1 \times k_2$ matrix M, define $(k_1 - 1) \times k_2$ matrix $\underline{M}$ by removing the bottom row from M and define $(k_1 - 1) \times (k_2 - 1)$ matrix $\underline{M}|$ by removing the right end column from $\underline{M}$.

With matrices $F_{12} = E_{(m-1)n} - M(\varphi_{mn}/\varphi_n)$, $F_{22} = \begin{pmatrix} E_{mn-1} & \mathbf{0}_{mn-1} \end{pmatrix} - \begin{pmatrix} \mathbf{0}_{mn-1} & E_{mn-1} \end{pmatrix}$ and $F_{32} = \begin{pmatrix} E_{n-1} & \mathbf{0}_{n-1} \end{pmatrix} - \begin{pmatrix} \mathbf{0}_{n-1} & E_{n-1} \end{pmatrix}$: Make decent the long formula

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A^{(m-1)n} & \xrightarrow{F_{12}} & A^{(m-1)n} & & \\
 & & & & \begin{pmatrix} E_{(m-1)n} \\ -\mathbf{1}_{m-1}^T \otimes E_n \end{pmatrix} \downarrow & & \downarrow \underline{\begin{pmatrix} E_{(m-1)n} \\ -\mathbf{1}_{m-1}^T \otimes E_n \end{pmatrix}} & & \\
0 & \longrightarrow & A & \xrightarrow{\mathbf{1}_{mn}} & A^{mn} & \xrightarrow{F_{22}} & A^{mn-1} & \longrightarrow & 0 \\
 & & \times m \downarrow & & \mathbf{1}_m^T \otimes E_n \downarrow & & \downarrow \underline{\mathbf{1}_m^T \otimes E_n}| & & \\
0 & \longrightarrow & A & \xrightarrow{\mathbf{1}_n} & A^n & \xrightarrow{F_{32}} & A^{n-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{6}$$

From the commutative diagram (6), it is a commutative diagram of abelian varieties defined over K that follows:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A(\varphi_{mn}/\varphi_n) & \longrightarrow & A(\varphi_{mn}/\varphi_n) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & Res_{L/K}(A) & \longrightarrow & A(\varphi_{mn}/\varphi_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & Res_{F/K}(A) & \longrightarrow & A(\varphi_n/\varphi_1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Thus from the main theorem the short exact sequence

$$0 \longrightarrow A(\varphi_{mn}/\varphi_n) \longrightarrow A(\varphi_{mn}/\varphi_1) \longrightarrow A(\varphi_n/\varphi_1) \longrightarrow 0$$

satisfies the rate formula of Tate-Shafarevich groups, which is the generalization of the main theorem in [4].

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Hoseog Yu
Department of Mathematics and Statistics, Sejong University,
Seoul, 05006, Korea.
E-mail: hsyu@sejong.ac.kr