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A RECENT GENERALIZATION OF COFINITELY INJECTIVE MODULES

Esra ÖZTÜRK SÖZEN

Abstract. Let R be an associative ring with identity and M be a left R-module. In this paper, we define modules that have the property (δ-CE) ((δ-CEE)), these are modules that have a δ-supplement (ample δ-supplements) in every cofinite extension which are generalized version of modules that have the properties (CE) and (CEE) introduced in [6] and so a generalization of Zöschinger's modules with the properties (E) and (EE) given in [23]. We investigate various properties of these modules along with examples. In particular we prove these: (1) a module M has the property (δ-CEE) if and only if every submodule of M has the property (δ-CE); (2) direct summands of a module that has the property (δ-CE) also have the property (δ-CE); (3) each factor module of a module that has the property (δ-CE) also has the property (δ-CE) under a special condition; (4) every module with composition series has the property (δ-CE); (5) over a δ-V-ring a module M has the property (δ-CE) if and only if M is cofinitely injective; (6) a ring R is δ-semiperfect if and only if every left R-module has the property (δ-CE).

1. Introduction

Throughout this study all rings are associative with identity and all modules are unitary left modules, unless otherwise specified. Let M be such a module and K be a *submodule* of M. In general it is denoted by $K \leq M$. Let $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ be a short exact sequence of modules. Then N is an *extension* of M by K. For a module M, N is called a *cofinite extension* of M whenever $M \leq N$ and $\frac{N}{M}$ is finitely generated *(see in* [6]*)*.

A submodule K of M is called *essential* in M if the intersection of K with every non-zero submodule of M is non-zero. A module M is called *singular* if $M \cong \frac{N}{L}$ for some module N and an essential submodule $L \trianglelefteq N$. Dually, a submodule K of M is called *small* in M if for any other submodule L of M provided $K + L = M$ implies that $L = M$. The usual notations prefered for essential and small submodules are $K \trianglelefteq M$ and $K \ll M$, respectively. By the *radical* of M, briefly $Rad(M)$, the intersection of all maximal submodules of

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M or, equivalently the sum of all small submodules of M is pointed. For a submodule K of M if there exists a submodule L of M such that $K+L=M$ and $K\cap L=\{0\}$ then each of K and L is a *direct summand* of M. By weakening this concept the *supplement submodule* is defined as that for a submodule K if there exists a minimal submodule L with respect to $M=K+L$, then L is a supplement of K in M. Equivalently this fact is given with $M=K+L$ and $K\cap L \ll L$. In particular, a submodule K of M has *ample supplements* in M if, L includes a supplement of K in M whereas $M=K+L$. We refer to [2] and [20] for detailed information of these concepts given here. It is a known fact that for a given module supplements need not exist. So to study on modules whose all submodules have a supplement give rise to the notion of the supplemented modules. A module M is called *(amply) supplemented* if every submodule of M has (ample supplements) a supplement in M.

A module epimorphism $f: P \longrightarrow M$, where P is projective, is called *projective cover* of M if $Ker(f) \ll P$. In one of his leading studies [4], Bass defined projective covers as above and following this he introduced *semiperfect rings* as those rings over which every cyclic module has a projective cover. With [9], Kasch and Mares introduced a pioneering work, where the excellent notion of supplemented modules is given. As a particular result it is presented that semiperfect rings are those rings whose every right ideal has a supplement.

Respectively in [10] and [21], the authors introduced the notions of δ-supplement submodules, δ-semiperfect rings and δ-supplemented modules as a generalization of small submodules, semiperfect rings and supplemented modules as following. A submodule N of M is called *δ-small* in M if $N+X\neq M$ for any proper submodule X of M with $\frac{M}{X}$ singular *see in* [7]. The sum of all δ-small submodules of M is denoted by $\delta(M)$. A projective module P is a *projective δ-cover* of M with an epimorphism $f: P\longrightarrow M$ such that $Ker(f) \ll_\delta P$; a ring R is defined to be a (right) *δ-semiperfect ring* if every simple (right) R-module has a projective δ-cover. Afterward, different generalizations of these modules are studied by [15].

Zöschinger generalized injective modules to modules with the property (E) *(see in* [23]*)*, that is to say, a module M has the property (E) $((EE))$ if M has a supplement (ample supplements) in its every extension. Thereafter the class of these modules was studied by many authors ([6], [11], [12], [13], [14], [17], [18]). In particular, as a fundamental reference of our study we like the mention the work by Çalışıcı and Türkmen in [6] where the important notion of modules with the properties (CE) and (CEE) is introduced and here it is proved that semiperfect rings are those rings whose all left modules have the property (CE) and also (CEE).

In this study we define modules that have a δ-supplement (ample δ-supplements) in every cofinite extension (briefly modules that have the property (δ-CE) ((δ-CEE)) as a generalization of modules that have the properties (CE) and (CEE). Thus we have the following implications on modules:

injective modules $\Longrightarrow$ modules with the property (E) $\Longrightarrow$ modules with the property (CE) $\Longrightarrow$ modules with the property $(\delta\text{-}CE)$

We investigate basic properties of them as following. The necessary and sufficient condition for a module M in order to have the property $(\delta\text{-}CEE)$ is that every submodule of M has the property $(\delta\text{-}CE)$. In general a module that has the property $(\delta\text{-}CE)$ need not has the property (CE). At the end we give an example which supports this reality. Every direct summand of a module that has the property $(\delta\text{-}CE)$ is also has the property $(\delta\text{-}CE)$. As a useful result we show that if a module has the property $(\delta\text{-}CE)$ then so is the every δ-cover of it. Moreover we prove that every module that has the property $(\delta\text{-}CE)$ is preserved under extensions. Following this we give an immediate consequence such that every module with a composition series has the property $(\delta\text{-}CE)$. As in stated in the abstract we wonder whether the factor module of a module that has the property $(\delta\text{-}CE)$ is also has the property $(\delta\text{-}CE)$. Following this we obtain a positive result under a special condition. Modules that have a δ-supplement in every cofinite extension are of course a new generalization of injective modules such as Zöschinger's modules with the property (E). In final we obtain the injectivity of a module and the sense of being with the property $(\delta\text{-}CE)$ are coincide over δ-V-rings. Even ring characterization of these modules is given as that δ-semiperfect rings are those rings whose all left modules have the property $(\delta\text{-}CE)$ and also $(\delta\text{-}CEE)$.

2. Preliminaries

In this section, we show basic properties of δ-small submodules and δ-supplemented modules in the following lemmas and theorems which are contained in [21] and [10] respectively.

2.1. δ-Small Submodules

Recall from [21] that a submodule $Z(M) = \{m \in M \mid Ann(m) \trianglelefteq {}_RR\}$ is called *singular* in M. In particular $Z_2(M)$ is defined by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$. If $M = Z_2(M)$,then M is called *Goldie torsion*.

Lemma 2.1. [21, Lemma 1.2] *Let N be a submodule of M. The following are equivalent:*

1. $N \ll_\delta M$.
2. *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*
3. *If $X + N = M$ with $\frac{M}{X}$ Goldie torsion, then $X = M$.*

Lemma 2.2. [21, Lemma 1.3] *Let M be a module.*

1. *For submodules N, K, L of M with $K \subseteq N$, we have*
   (a) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$.*
   (b) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*

2. *If* $K \ll_\delta M$ *and* $f : M \longrightarrow N$ *is a homomorphism, then* $f(K) \ll_\delta N$. *In particular, if* $K \ll_\delta M \subseteq N$, *then* $K \ll_\delta N$.
3. *Let* $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ *and* $M = M_1 \oplus M_2$. *Then* $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ *if and only if* $K_1 \ll_\delta M_1$ *and* $K_2 \ll_\delta M_2$.

2.2. δ-Supplemented Modules

In [21], a projective module P is called a *projective δ-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Ker(f) \ll_\delta M$, and a ring R is called *δ-perfect* (resp., δ-semiperfect) if every R-module (resp., every simple R-module) has a projective δ-cover. In addition, a module M is called δ-lifting if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ-small in B since B is a direct summand of M.

Theorem 2.3. [10, Theorem 3.3] *The following are equivalent for a ring* R :

1. R *is* δ*-semiperfect.*
2. *Every finitely generated module is* δ*-supplemented.*
3. *Every finitely generated projective module is* δ*-supplemented.*
4. *Every finitely generated projective module is* δ*-lifting.*
5. *Every left ideal of R has a* δ*-supplement in* $_RR$.

Theorem 2.4. [10, Theorem 3.4] *The following statements are equivalent for a ring* R :

1. R *is* δ*-perfect.*
2. *Every module is* δ*-supplemented.*
3. *Every projective module is* δ*-supplemented.*
4. *Every projective module is* δ*-lifting.*

3. Modules That Have the Properties (δ-CE) and (δ-CEE)

As we have pointed at the beginning of this study, we generalize the properties (δ-E) and (δ-EE) for a module M *(given in* [13]*)* as follows:

Definition 3.1. *A module* M *has the property* (δ-CE) *if it has a* δ*-supplement in every cofinite extension.*

Definition 3.2. *A module* M *has the property* (δ-CEE) *if it has ample* δ*-supplements in every cofinite extension.*

It is clear that every module that has the property (δ-E) has the property (δ-CE). Of course there exists the same relation between modules that have the properties (δ-EE) and (δ-CEE). At the end of this section, we shall give an example of a module which has the property (δ-CE) but not (δ-E).

On the other hand every module that has the property (CE) also has the property (δ-CE). In addition it is a known fact that every supplemented module is also δ-supplemented, but over singular modules these concepts coincide.

Now we give the following easy proposition to show the existence of analogous relation for our new modules.

Proposition 3.3. *Let M be a singular module that has the property (δ-CE). Then M also has the property (CE).*

Proof. It is obvious. □

Proposition 3.4. *Let R be a ring whose simple modules are singular. Then M has the property (δ-CE) if and only if M has the property (CE).*

Proof. It can be easily seen from [16]. □

Proposition 3.5. *Let R be a ring whose maximal ideals are essential in R.Then M has the property (δ-CE) if and only if M has the property (CE).*

Proof. It can be easily seen from [16]. □

If we consider R as a local ring or a commutative domain that is not a field, then the concepts of modules with the properties (δ-CE) and (CE) coincide for modules over these rings [16, Example 2.8].

Let consider the $\mathbb{Z}$-module $\mathbb{Z}$ and the submodule $2\mathbb{Z} \leq \mathbb{Z}$. Since $\mathbb{Z}$ is cofinitely injective (see in [6]), (i.e., $\mathbb{Z}$ is a direct summand of its every cofinite extension) $\mathbb{Z}$ has the property (δ-CE). However it is easy to see that $2\mathbb{Z}$ has no δ-supplement in $\mathbb{Z}$ as a cofinite extension of $2\mathbb{Z}$. So it does not be true in general a submodule of a module that has the property (δ-CE) also has the property (δ-CE).

Zöschinger proved in [23] that a module has the property (EE) if and only if every submodule has the property (E). We give an analogous characterization for our modules with the following proposition.

Proposition 3.6. *A module M has the property (δ-CEE) if and only if every submodule of M has the property (δ-CE).*

Proof. Let M be a module that has the property (δ-CEE) and N be any cofinite extension of M. Suppose that for a submodule $X \leq N$, $X + M = N$. By hypothesis, since $\frac{N}{M} = \frac{X+M}{M} \cong \frac{X}{X\cap M}$ is cofinite, the submodule $X \cap M$ of X has a δ-supplement V in X, that is, $(X \cap M) + V = X$ and $(X \cap M) \cap V \ll_\delta V$. Then, $N = M + X = M + [(X \cap M) + V] = M + V$ and $M \cap V = M \cap (V \cap X) = (X \cap M) \cap V \ll_\delta V$. Hence, V is a δ-supplement of M in N such that $V \leq X$.

Conversely, let M be a module that has the property (δ-CEE) and T be a submodule of M. For a cofinite extension N of T, let $F = \frac{M\oplus N}{H}$, where the submodule H is the set of a elements $(a, -a)$ of F with $a \in T$ and let $\alpha : M \longrightarrow F$ via $\alpha(m) = (m, 0) + H$, $\beta : N \longrightarrow F$ via $\beta(n) = (0, n) + H$ for all $m \in M$, $n \in N$. Then we can draw the following pushout:

$$\begin{array}{ccc} T & \xrightarrow{\mu_1} & N \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow{\alpha} & F \end{array}$$

μ_1 and μ_2 are inclusion homomorphisms in this diagram. Additionally $\alpha : M \longrightarrow F$ and $\beta : N \longrightarrow F$ are monomorphisms by the properties of push out. Let $\alpha(M) = M^{'} \subseteq F$ and $\beta(N) = N^{'} \subseteq F$. Then it can be easily shown that $F = M^{'} + N^{'}$. Now we define $\gamma : F \longrightarrow \frac{N}{T}$ by $\gamma((m,n) + H) = n + T$ for all $(m,n) + H \in F$. Then γ is an epimorphism. Note that $Ker(\gamma) = M^{'}$ and so $\frac{N}{T} \cong \frac{F}{M^{'}}$ is finitely generated. Since α is a monomorphism, by assumption, $M^{'} \cong M$ has the property δ-(CEE). So by using hypothesis, $M^{'}$has a δ-supplement V in F such that $V \leq N^{'}$, that is, $M^{'} + V = F$ and $M^{'} \cap V \ll_\delta V$. Hence,

$$(M^{'} \cap N^{'}) + V = (N^{'} \cap M^{'}) + V = N^{'} \cap (M^{'} + V) = N^{'} \cap F = N^{'}, \text{ and}$$

$$(M^{'} \cap N^{'}) \cap V = M^{'} \cap (N^{'} \cap V) = M^{'} \cap V \ll_\delta V.$$

So V is a δ-supplement of $M^{'} \cap N^{'}$ in $N^{'}$. Now we will show that $\beta^{-1}(V)$ is a δ-supplement of T in N. We have an isomorphism $\widetilde{\beta} : N \longrightarrow N^{'}$ defined as $\beta(x) = \widetilde{\beta}(x)$ for all $x \in N$, since β is a monomorphism. Using this, we obtain $\widetilde{\beta}^{-1}(V)$ is a δ-supplement of $\beta^{-1}(M^{'} \cap N^{'})$ in $\widetilde{\beta}^{-1}(N^{'})$ since V is a δ-supplement of $M^{'} \cap N^{'}$ in $N^{'}$. It can be seen that $\widetilde{\beta}^{-1}(V) = \beta^{-1}(V)$, $\beta^{-1}(N^{'}) = N$ and $\beta^{-1}(M^{'} \cap N^{'}) = T$. Thus $\beta^{-1}(V)$ is a δ-supplement of T in N. $\square$

Recall that a module M is *cofinitely (amply) δ-supplemented* if every cofinite submodule of M has a δ-supplement (ample δ-supplements) in M [1].

Corollary 3.7. *Every submodule of a module that has the property (δ-CEE) is cofinitely δ-supplemented. Moreover a module that has the property (δ-CEE) is amply cofinitely δ-supplemented.*

Recall that a module M is called a V-module if every simple R-module is M-injective. Dually a ring R is a left V-ring if ${}_RR$ is a V-module. As a generalization of this in [19] it is introduced that a ring R is a (right) δ-V ring if for any (right) R-module M, $\delta(M) = 0$.

Proposition 3.8. *Let R be δ-V-ring Then the following statements are equivalent for an R-module M:*

1. M has the property (δ-CE).
2. M is cofinitely injective.

Proof. (1) $\Longrightarrow$ (2) : Suppose that M has the property (δ-CE). Let N be any cofinite extension of M. So, there exists a δ-supplement V of M in N, that is, $M+V=N$ and $M\cap V \ll_\delta V$ and so $M\cap V \leq \delta(V)$. Since R is a δ-V ring, $\delta(V)=0$. So, $N=M\oplus V$. Therefore, M is injective.

(2) $\Longrightarrow$ (1) : Let M be a cofinitely injective module and N be any cofinite extension of M. Then there exists a submodule K of N such that $N=M\oplus K$. Hence K is a δ-supplement of M in N. □

It is shown in [23] that direct summands of modules that have the property (E) also have the property (E). Now we show that the property (δ-CE) is preserved by direct summands in the following proposition:

Proposition 3.9. *Let M be a module. If M has the property (δ-CE), then so does every direct summand.*

Proof. Let M be a module that has the property (δ-CE) and T be a direct summand of M. Then there exists a submodule D of M such that $M=T\oplus D$. Let N be any cofinite extension of T, N' be the external direct sum $N\oplus D$ and $\phi : M \longrightarrow N'$ be the canonical embedding. Then $M\cong \phi(M)$ has the property (δ-CE). Note that

$$\frac{N}{T}\cong\frac{N\oplus D}{\phi(M)}=\frac{N'}{\phi(M)}$$

is finitely generated. By hypothesis, $\phi(M)$ has the property (δ-CE), there exists a submodule supplement V of N' such that $\phi(M)+V=N'$ and $\phi(M)\cap V \ll_\delta V$. Consider the projection $g : N' \longrightarrow N$. It follows that $T+g(V)=N$ and since $Ker(g)\leq \phi(M)$

$$g(\phi(M)\cap V)=g(\phi(M))\cap g(V)=T\cap g(V)\ll_\delta g(V).$$

Hence, $g(V)$ is a δ-supplement of T in N. □

Proposition 3.10. *Let M be a module and K be a projective semisimple submodule of M. If $\frac{M}{K}$ has the property (δ-CE), then so does M.*

Proof. Let U be a cofinite extension of M. Without loss of generality we will assume that $K\leq M\leq U$. Note that

$$\frac{U}{M}\cong\frac{\frac{U}{K}}{\frac{M}{K}}$$

is finitely generated and so $\frac{U}{K}$ is a cofinite submodule of $\frac{M}{K}$. Since $\frac{M}{K}$ has the property (δ-CE) there is a submodule $\frac{V}{K}\leq\frac{U}{K}$, that is, $\frac{M}{K}+\frac{V}{K}=\frac{U}{K}$ and $\frac{M}{K}\cap\frac{V}{K}=\frac{M\cap V}{K}\ll_\delta\frac{V}{K}$. It is clear that $M+V=U$. Now we will show that $M\cap V\ll_\delta V$.*Let* $(M\cap V)+T=V$ for a submodule $T\leq V$ with $\frac{V}{T}$ singular. It follows that $\frac{M\cap V}{K}+\frac{T+K}{K}=\frac{V}{K}$ and since $\frac{V}{T+K}\cong\frac{\frac{V}{T}}{\frac{T+K}{T}}$ is singular and $\frac{M\cap V}{K}\ll_\delta\frac{V}{K}$, we have $\frac{T+K}{K}=\frac{V}{K}$ and so $T+K=V$ is obtained. Because K

is projective semisimple it can be easily seen $K \ll_\delta V$ and so we have $T = V$. Thus V is a δ-supplement of M in U. Therefore M has the property (δ-CE). □

Proposition 3.11. *Let B be a module and A be a submodule of B such that $\frac{B}{A}$ is Noetherian. If A and $\frac{B}{A}$ have the property (δ-CE), then so does B.*

Proof. Let N be any cofinite extension of B. By hypothesis, there exists a δ-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$. Then $\frac{N}{B} \cong \frac{\frac{V}{A}}{\frac{B \cap V}{A}}$ is finitely generated. Since $\frac{B}{A}$ is Noetherian, the submodule $\frac{B \cap V}{A}$ is finitely generated and so is $\frac{V}{A}$. Hence V is a cofinite extension of A. Since A has the property (δ-CE), then A has a δ-supplement in V, say T. However by the hypothesis, there is a δ-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$. We will show that T is a δ-supplement of B in N. We have δ-small epimorphisms $f : T \longrightarrow \frac{V}{A}$ and $g : \frac{V}{A} \longrightarrow \frac{N}{B}$ such that $Ker(f) = A \cap T \ll_\delta T$ and $Ker(g) = \frac{B}{A} \cap \frac{V}{A} \ll_\delta \frac{V}{A}$. This means that each of these epimorphisms are δ-small covers and so does $g \circ f$, i.e., $g \circ f : T \longrightarrow \frac{N}{B}$ is a δ-small epimorphism such that $T \cap B = Ker\ (g \circ f) \ll_\delta T$. Moreover, we have

$$B + T = (B + A) + T = B + (A + T) = B + V = N$$

since $\frac{V}{A}$ is a δ-supplement of $\frac{B}{A}$ in $\frac{N}{A}$. Finally T is a δ-supplement of B in its cofinite extension N. This completes the proof. □

Corollary 3.12. *If M_1 and M_2 have the property (δ-CE) with M_1 or M_2 Noetherian, so does $M_1 \oplus M_2$.*

Proof. Let $0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence. Result follows by Proposition 3.11. □

We give the following known fact for the completeness.

Lemma 3.13. *Every simple submodule S of a module M is either a direct summand of M or small (and so δ-small in M)(see in* [11]*).*

Proposition 3.14. *Every simple module has the property (δ-CE).*

Proof. Let S be a simple module and N be any cofinite extension of S. Then by Lemma 3.13, $S \ll N$ and so $S \ll_\delta N$ or $S \oplus S' = N$ for a submodule $S' \leq N$. If $S \ll_\delta N$, then N is a δ-supplement of S in N or if S is a direct summand of N then S' is a δ-supplement of S in N. So in each case S has a δ-supplement in N. This means that S has the property (δ-CE). □

Proposition 3.15. *Every projective semisimple module has the property (δ-CE).*

Proof. Let M be a projective semisimple module and N be any cofinite extension of M. $N \ll_\delta M$ since M is a projective semisimple module. So N is a δ-supplement of M in N. □

Locally projective modules first introduced by Zimmermann-Huisgen in [22] and we know from [8], that the module M is called locally projective if, for every submodule $N \leq M$, which is not small in M, there exists a projective direct summand $K \leq M$, $K \neq 0$ with $K \leq N$.

Corollary 3.16. *Let M be a locally projective module with $\delta(M) = M$. Since M is projective semisimple (see in* [3]*), then M has the property (δ-CE).*

Theorem 3.17. *Every module with composition series has the property (δ-CE).*

Proof. Let $0 = M_0 \leq M_1 \leq M_2 \leq ... \leq M_{n-1} \leq M_n = M$ be any composition series of a module M. We shall prove the theorem by induction on $n \epsilon \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple, and so M has the property (δ-CE) by Proposition 3.14. Assume that this is true for each $k \leq n-1$. Then M_{n-1} has the property (δ-CE). Since $\frac{M_n}{M_{n-1}}$ has the property (δ-CE) as a simple Noetherian module, $M_n = M$ has the property (δ-CE) by Proposition 3.11. □

Corollary 3.18. *Every finitely generated semisimple module has the property (δ-CE).*

Recall that over a left hereditary ring every factor module of an injective module is injective. In the following proposition we show that every factor module of a module that has the property (δ-CE) also has the property (δ-CE), under a special condition.

Proposition 3.19. *Let R be a left hereditary ring and M be an R-module. If M has the property (δ-CE), then every factor module of M has the property (δ-CE).*

Proof. For any submodule U of M, let N be any cofinite extension of $\frac{M}{U}$. Then N is cofinite. By $E(M)$, we denote the injective hull of M. Since R is left hereditary, $\frac{E(M)}{U}$ is injective, and so there exists a commutative diagram with exact rows in the following by [11, Lemma 2.16]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \overset{i_1}{\hookrightarrow} & M & \overset{\pi}{\longrightarrow} & M/U & \longrightarrow & 0\ . \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & U & \overset{f}{\longrightarrow} & K & \overset{\sigma}{\longrightarrow} & N & \longrightarrow & 0
\end{array}$$

i.e., $fI = \varphi i_1$ and $\sigma\varphi = i_2\pi$, where $\varphi : M \longrightarrow K$ is a monomorphism. We have that $N \cong \frac{K}{\varphi(M)} \cong \frac{K}{Ker(\sigma)}$ is cofinite. Since M has the property (δ-CE), $M \cong \varphi(M)$ has the property (δ-CE), i.e., $\varphi(M)$ has a δ-supplement V in K. In this case we have $\varphi(M) + V = K$ and $\varphi(M) \cap V \ll_\delta V$ and so $\frac{M}{U} + \sigma(V) = N$ and $\frac{M}{U} \cap \sigma(V) \ll_\delta \sigma(V)$ are obtained. Hence $\frac{M}{U}$ has the property (δ-CE). □

Proposition 3.20. *Let I be a two sided ideal in R and $\overline{R} = \frac{R}{I}$. If the ring $\overline{R}$ is left δ-perfect and $M \epsilon R$-Mod with $IM = 0$, then M has the property (δ-E) and so (δ-CE).*

Proof. Let N be an extension of R-module M. Since the factor module $\overline{R}$ is left δ-perfect, i.e. every left $\overline{R}$-module is δ-supplemented, then the submodule $\frac{M+IN}{IN}$ of $\frac{N}{IN}$ has a δ-supplement in $\frac{N}{IN}$, say $\frac{V}{IN}$. Then we have $\frac{M+IN}{IN} + \frac{V}{IN} = \frac{N}{IN}$ and $\frac{(M\cap V)+IN}{IN} \ll_\delta \frac{V}{IN}$. It is clear to see that $M + V = N$. Now we will claim that $M \cap V \ll_\delta V$. Assume that $(M \cap V) + T = V$ for a submodule T of V with $\frac{V}{T}$ singular. Following this we have $\frac{(M\cap V)+IN}{IN} + \frac{T+IN}{IN} = \frac{V}{IN}$.Then since $\frac{\frac{V}{T}}{\frac{T+IN}{T}} \cong \frac{V}{T+IN}$ is singular and $\frac{(M\cap V)+IN}{IN} \ll_\delta \frac{V}{IN}$ we obtain that $\frac{T+IN}{IN} = \frac{V}{IN}$ and so $T + IN = V$. It is obvious that $M + T = N$ since $M + (M \cap V) + T = M+V = N$. And from $IN = I(M+T) \leq IM+IT = IT \leq T$ and $T+IN = V$, we have $T = V$. □

Corollary 3.21. *It is sufficient for an R-module M which has the property (δ-E) that $\frac{R}{Ann_R(M)}$ is a δ-perfect ring.*

In commutative algebra, the Krull dimension of a commutative ring is the supremum of the lengths of all chains of prime ideals.

Corollary 3.22. *Let R be a Noetherian integral domain with Krull Dimension one and M be a bounded R-module. Then M has the property (δ-CE).*

Proof. By hypothesis, $Ann(M) \neq 0$ since M is bounded. Hence $\frac{R}{Ann(M)}$ is an Artinian ring and so it is δ-perfect. By Corollary 3.21, M has the property (δ-CE). □

Recall that a ring R is said to be almost perfect if all its proper homomorphic images are perfect [5].

Definition 3.23. *We define almost δ-perfect rings as rings whose all proper homomorphic images are δ-perfect, as a generalization of almost perfect rings.*

It is obvious that δ-perfect rings and almost perfect rings are almost δ-perfect. So we can add that every module over an almost δ-perfect ring has the property (δ-CE).

Recall from [6] that a ring R is semiperfect if and only if every left R-module has the property (CE). Now we give an analogous fact for δ-semiperfect rings.

Proposition 3.24. *A ring R is δ-semiperfect if and only if every left R-module has the property (δ-CE).*

Proof. Suppose that R is δ-semiperfect. Let M be an R-module and N be a cofinite extension of M. Then there exists a finitely generated submodule K of N such that $N = M + K$. Since R is δ-semiperfect, $\frac{N}{M}$ has a projective δ-cover. By [10], M has a δ-supplement in N. Hence M has the property (δ-CE). Conversely, suppose that every left R-module has the property (δ-CE). Then every left ideal of R has a δ-supplement in R as a left R-module. Hence the R-module ${}_RR$ is δ-supplemented and therefore R is δ-semiperfect by [10]. □

The next theorem give new characterizations of δ-semiperfect rings via their modules which have the property (δ-CE).

Theorem 3.25. *The following statements are equivalent for a ring R :*

1. R is δ-semiperfect.
2. ${}_RR$ has the property (δ-CEE).
3. ${}_RR$ has ample δ-supplements in every finitely generated extension.
4. Every left R-module has the property (δ-CE).
5. Every left R-module has the property (δ-CEE).

Proof. (1) $\Longleftrightarrow$ (4) is obvious by Proposition 3.24.
(5) $\Longrightarrow$ (3) $\Longrightarrow$ (2) are clear.
(2) $\Longrightarrow$ (1): If ${}_RR$ has the property (δ-CEE), then ${}_RR$ is δ-supplemented by Proposition 3.6. It follows from [10] that R is δ-semiperfect.
(4) $\Longrightarrow$ (5) follows from Proposition 3.6. □

Recall that the condition "ample" in the above theorem is necessary. For example, the left $\mathbb{Z}$-module $\mathbb{Z}$ has a δ-supplement in every finitely generated extension, since it is cofinitely injective. But it is not δ-semiperfect.

The following two examples show that in general a module that has the property (δ-CE) need not have the property (δ-E) and a module that has the property (δ-CE) need not have the property (CE), respectively.

Example 3.26. *Let R the ring of polynomials over a field F in countably many commuting indeterminates $x_1, x_2, ...$ modulo the ideal generated by $\{x_1^2, x_2^2 - x_1, x_3^2 - x_2, ...\}$. As observed in [[21], Example 4.4] R is a local ring with the unique maximal ideal $Jac(R) = J = J^2 = < x_1, x_2, ... > / < x_1^2, x_2^2 - x_1, x_3^2 - x_2, ... >$. Let $M = J^{(\mathbb{N})}$. Since M does not have a δ-supplement in $R^{(\mathbb{N})}$ (see [[12], Example 2.3] and [[16], Example 2.8] for details), then M is not with the property (δ-E). On the other hand M is a module with the property (δ-CE) as R is a δ-semiperfect ring.*

Example 3.27. *Let $Q = \Pi_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$ and R be the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q. It can be seen in [21], the ring R is a δ-semiperfect ring which is not semipefect. So there exists an R-module M that has not the property (CE) by [[6],Theorem 2.12]. However, by Theorem 3.25, M has the property (δ-CE).*

Corollary 3.28. *Every submodule of a module that has the property (δ-CEE) also has the property (δ-CEE).*

Proof. Let M be a module that has the property (δ-CEE) and K be any submodule of M. By Proposition 3.6, it is enough to show that every submodule of K has the property (δ-CE). For any submodule T of K it is possible to say that T is also a submodule of M and so by virtue of hypothesis T has the property (δ-CE) from Proposition 3.6. Hence K has the property (δ-CEE). □

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Esra ÖZTÜRK SÖZEN
Faculty of Sciences and Arts, Department of Mathematics,
Sinop University, Sinop, Turkey.
E-mail : esozen@sinop.edu.tr