INFINITE FAMILIES OF CONGRUENCES MODULO 2 FOR
2-CORE AND 13-CORE PARTITIONS

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Abstract. A partition of $n$ is called a $t$-core partition if none of its hook number is divisible by $t$. In 2019, Hirschhorn and Sellers [5] obtained a parity result for 3-core partition function $a_3(n)$. Motivated by this result, both the authors [8] recently proved that for a non-negative integer $\alpha$, $a_3^\alpha m(n)$ is almost always divisible by an arbitrary power of 2 and 3 and $a_t(n)$ is almost always divisible by an arbitrary power of $p_j$, where $j$ is a fixed positive integer and $t = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with primes $p_i \geq 5$. In this article, by using Hecke eigenform theory, we obtain infinite families of congruences and multiplicative identities for $a_2(n)$ and $a_{13}(n)$ modulo 2 which generalizes some results of Das [2].

1. Introduction

A partition $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ of $n$ is a non-increasing sequence of positive integers whose sum is $n$ and the positive integers $\beta_i$ are called parts of the partition $\beta$. A partition $\beta$ of $n$ can be represented by the Young diagram $[\beta]$ (also known as the Ferrers graph) which consists of the $s$ number of rows such that the $i^{th}$ row has $\beta_i$ number of dots • and all the rows start in the first column. An illustration of the Young diagram for $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ is as follows.

\[
[\beta] := \begin{array}{c}
\bullet \quad \bullet \quad \cdots \quad \cdots \quad \beta_1 \text{ dots} \\
\bullet \quad \bullet \quad \cdots \quad \beta_2 \text{ dots} \\
\vdots \\
\bullet \quad \cdots \quad \beta_r \text{ dots}
\end{array}
\]

For $1 \leq i \leq r$ and $1 \leq j \leq \beta_i$, the dot of $[\beta]$ which lies in the $i^{th}$ row and $j^{th}$ column is denoted by $(i, j)^{th}$-dot of $\beta$. Let $\beta_j$ denote the number of dots in $j^{th}$ column. The hook number $H_{i,j}$ of $(i, j)^{th}$-dot is defined by $\beta_i + \beta_j - i - j + 1$. In other words, $H_{i,j} = 1 + h_0$, where $h_0$ is the sum of the number of dots lying right to the $(i, j)^{th}$-dot in the $i^{th}$ row and the number of dots lying below the...
(i, j)\textsuperscript{th} dot in the \(j\)\textsuperscript{th} column. Given a partition \(\beta\) of \(n\), we say that it is a \(t\)-core partition if none of its hook number is divisible by \(t\).

**Example 1.** The Young diagram of the partition \(\beta = (6, 3, 1)\) of 10 is

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

where the superscript on each dot represents its hook number. It can be easily observed that this is a \(t\)-core partition of 10 for \(t = 7\) and \(t \geq 9\).

**Example 2.** There are no 3-core partitions of 7. This can be easily verified by looking at the Young diagram of each partition of 7.

For a positive integer \(n\), let \(a_t(n)\) denote the number of \(t\)-core partitions of \(n\). Its generating function is given by

\[
\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^t}{(1 - q^n)} = \left(\frac{q^t; q^t}{q; q}\right)_\infty,
\]

where \(\left(\frac{a}{q}\right)_{\infty} = (1 - a)(1 - aq)(1 - aq^2)\cdots\).

In [3, Corollary 1], Garvan, Kim, Stanton obtained the congruence

\[
a_p(p^n - \delta_p) \equiv 0 \pmod{p^j},
\]

where \(p \in \{5, 7, 11\}\), \(n, j\) are positive integers and \(\delta_p = \frac{p^2 - 1}{24}\). In [4, Proposition 3], Granville and Ono proved similar congruences, namely

\[
a_{5j}(5^jn - \delta_{5,j}) \equiv 0 \pmod{5^j}, \quad a_{7j}(7^jn - \delta_{7,j}) \equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \quad a_{11j}(11^jn - \delta_{11,j}) \equiv 0 \pmod{11^j},
\]

where \(n, j\) are positive integers and \(\delta_{p,j} \equiv \frac{1}{24} \pmod{p^j}\) for \(p \in \{5, 7, 11\}\).

In 2019, Hirschhorn and Sellers [5] proved a parity result for \(a_3(n)\), i.e., for all \(n \geq 0\),

\[
a_3(n) = \begin{cases} 
1 & \text{mod 2} \quad \text{if } n = 3r^2 + 2r \text{ for some integer } r, \\
0 & \text{mod 2} \quad \text{otherwise.}
\end{cases}
\]

Motivated by this result, both the authors proved that for a non-negative integer \(\alpha\), \(a_{3^m}(n)\) is almost always divisible by an arbitrary power of 2 and 3. Moreover, they also proved that \(a_t(n)\) is almost always divisible by an arbitrary power of \(p_i^j\), where \(j\) is a fixed positive integer and \(t = p_1^{a_1}p_2^{a_2} \cdots p_m^{a_m}\) with primes \(p_i \geq 5\). In the following theorem, we obtain infinite families of congruences modulo 2 for \(a_2(n)\) and \(a_{13}(n)\) by using Hecke eigen form theory.

**Theorem 1.1.** Let \(k\) and \(n\) be non-negative integers. For each \(1 \leq i \leq k + 1\), let \(p_1, p_2, \ldots, p_{k+1}\) be prime numbers such that \(p_i \geq 5\). Then for any integer \(j \not\equiv 0 \pmod{p_{k+1}}\), we have
(i) \(a_2 \left(p_1^2 p_2^2 \cdots p_k^2 + p_1^2 p_2^2 \cdots p_k^2 \frac{p_{k+1}(8j+p_{k+1})-1}{8}\right) \equiv 0 \pmod{2},
\)
(ii) \(a_{13} \left(104p_1^2 p_2^2 \cdots p_k^2 + 13p_1^2 p_2^2 \cdots p_k^2 (\epsilon_p j + p_{k+1}) - 7\right) \equiv 0 \pmod{2},
\)
where
\[
\epsilon_p = \begin{cases} 
1 & \text{if } p \not\equiv 1 \pmod{8}, \\
8 & \text{if } p \equiv 1 \pmod{8}.
\end{cases}
\]

**Corollary 1.1.** Let \(n\) and \(k\) be non-negative integers. For a prime \(p \geq 5\) and an integer \(j \not\equiv 0 \pmod{p}\), we have
(i) \(a_2 \left(p^2(k+1)n + p^{2k+1} + \frac{p^{2k+1}-1}{8}\right) \equiv 0 \pmod{2},
\)
(ii) \(a_{13} \left(104p^{2k+2}n + 13\epsilon_p p^{2k+1} + 13p^{2k+2} - 7\right) \equiv 0 \pmod{2}.
\)
Furthermore, we prove the following multiplicative formulae for 2-core partitions and 13-core partitions modulo 2.

**Theorem 1.2.** Let \(k\) be a positive integer and \(p\) be a prime number such that \(p \equiv 7 \pmod{8}\). Let \(r\) be a non-negative integer such that \(p\) divides \(8r + 7\). Then
(i) \(a_2 \left(p^{k+1}n + pr + \frac{7p-1}{8}\right) \equiv \left(-\frac{2}{p}\right) a_2 \left(p^{k-1}n + \frac{8r+7-p}{8p}\right) \pmod{2},
\)
(ii) \(a_{13} \left(104p^{k+1}n + 104pr + 91p - 7\right) \equiv \left(-\frac{2}{p}\right) a_{13} \left(104p^{k-1}n + \frac{104r + 91}{p} - 7\right) \pmod{2}.
\)

**Corollary 1.2.** Let \(k\) be a positive integer and \(p\) be a prime number such that \(p \equiv 7 \pmod{8}\). Then
(i) \(a_2 \left(p^{2k}n + \frac{p^{2k}-1}{8}\right) \equiv \left(-\frac{2}{p}\right)^k a_2(n) \pmod{2}.
\)
(ii) \(a_{13} \left(104p^{2k}n + 13p^{2k} - 7\right) \equiv \left(-\frac{2}{p}\right)^k a_{13}(104n + 6) \pmod{2}.
\)

### 2. Preliminaries

We recall some basic facts and definition on modular forms. For more details, we refer to [6,9]. We start with some matrix groups. We define
\[
\Gamma := \text{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\
c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},
\]
\[
\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\
0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.
\]

For a positive integer \(N\), we define
\[
\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\
c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},
\]
\[
\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\
c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},
\]
and
\[
\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \ b \equiv c \equiv 0 \pmod{N} \right\}.
\]

A subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) is called a congruence subgroup if it contains \( \Gamma(N) \) for some positive integer \( N \) and the smallest \( N \) with this property is called its level. Note that \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are congruence subgroups of level \( N \), whereas \( \text{SL}_2(\mathbb{Z}) \) and \( \Gamma_\infty \) are congruence subgroups of level 1. The index of \( \Gamma_0(N) \) in \( \Gamma \) is
\[
[\Gamma : \Gamma_0(N)] = \prod_{p|N} \left(1 + \frac{1}{p}\right),
\]
where \( p \) runs over prime divisors of \( N \).

Let \( \mathbb{H} \) denote the upper half of the complex plane. The group \( \text{GL}_2^+(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\} \) acts on \( \mathbb{H} \) by
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = az + b \quad \text{and} \quad \begin{bmatrix} \frac{a + \xi b}{c + d} \\ \frac{c + \xi d} \end{bmatrix} = \frac{ar + bs}{cr + ds},
\]
where \( \xi \in \mathbb{Q} \cup \{\infty\} \). This gives an action of \( \text{GL}_2^+(\mathbb{R}) \) on the extended half plane \( \mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \). Suppose that \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). A cusp of \( \Gamma \) is an equivalence class in \( \mathbb{P}^1 = \mathbb{Q} \cup \{\infty\} \) under the action of \( \Gamma \).

The group \( \text{GL}_2^+(\mathbb{R}) \) also acts on functions \( f : \mathbb{H} \to \mathbb{C} \). In particular, suppose that \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2^+(\mathbb{R}) \). If \( f(z) \) is a meromorphic function on \( \mathbb{H} \) and \( k \) is an integer, then define the slash operator \( |_k \) by
\[
(f|_k \gamma)(z) := (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z).
\]

**Definition 2.1.** Let \( \Gamma \) be a congruence subgroup of level \( N \). A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a modular form of integer weight \( k \) on \( \Gamma \) if the following hold:

1. For all \( z \in \mathbb{H} \) and \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \),
\[
f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z).
\]
2. If \( \gamma \in \text{SL}_2(\mathbb{Z}) \), then \( (f|_k \gamma)(z) \) has a Fourier expansion of the form
\[
(f|_k \gamma)(z) := \sum_{n \geq 0} a_n(q_N^n),
\]
where \( q_N := e^{2\pi i z/N} \).

For a positive integer \( k \), the complex vector space of modular forms of weight \( k \) with respect to a congruence subgroup \( \Gamma \) is denoted by \( M_k(\Gamma) \).
Definition 2.2 ([9, Definition 1.15]). Let $\chi$ be a Dirichlet character modulo $N$. We say that a modular form $f \in M_k(\Gamma_1(N))$ has Nebentypus character $\chi$ if

$$f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k f(z)$$

for all $z \in \mathbb{H}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_k(\Gamma_0(N), \chi)$.

The relevant modular forms for the results obtained in this article arise from eta-quotients. We recall the Dedekind eta-function $\eta(z)$ which is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) := \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where $N$ and $r_\delta$ are integers with $N > 0$.

Theorem 2.1 ([9, Theorem 1.64]). If $f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $k = \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}$,

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}\quad\text{and}\quad\sum_{\delta | N} \frac{N}{\delta} \delta r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^k f(z)$$

for each $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character $\chi$ is defined by $\chi(d) := \left( -1 \right)^{s_\delta} d^{s_\delta}$, where $s_\delta = \prod_{\delta | N} \delta^{r_\delta}$.

Theorem 2.2 ([9, Theorem 1.65]). Let $c, d$ and $N$ be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f$ is an eta-quotient satisfying the conditions of Theorem 2.1 for $N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\frac{N}{24} \sum_{\delta | N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{\delta}) d^s}. $$

Suppose that $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.1 and that the associated weight $k$ is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$. Theorem 2.2 gives the necessary criterion for determining orders of an eta-quotient at cusps. In the proofs of our results, we use Theorems 2.1 and 2.2 to prove that $f(z) \in M_k(\Gamma_0(N), \chi)$ for certain eta-quotients $f(z)$ we consider in the sequel.
We recall the definition of Hecke operators and a few relevant results. Let \( m \) be a positive integer and \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi) \). Then the action of Hecke operator \( T_m \) on \( f(z) \) is defined by
\[
f(z)|T_m := \sum_{n=0}^{\infty} \left( \sum_{d|\gcd(n,m)} \chi(d)d^{k-1}a\left(\frac{mn}{d^2}\right) \right) q^n.
\]
In particular, if \( m = p \) is a prime, we have
\[
f(z)|T_p := \sum_{n=0}^{\infty} \left( a(pm) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right) \right) q^n.
\]
We note that \( a(n) = 0 \) unless \( n \) is a non-negative integer.

3. Proofs of Theorems 1.1 and 1.2

3.1. Prelude to the proofs

We define
\[
\sum_{n=1}^{\infty} b(n)q^n = q(q^8;q^8)_\infty(q^{16};q^{16})_\infty \quad \text{and} \quad \sum_{n=0}^{\infty} c(n)q^n := (q;q)_\infty^3.
\]
If \( p \nmid n \), then we set \( b\left(\frac{n}{p}\right) = 0 \) and \( c\left(\frac{n}{p}\right) = 0 \). We have the following result.

**Lemma 3.2.** For \( n \geq 0 \) and for a prime \( p \not\equiv 1 \pmod{8} \), we have
\[
b(pn) = (-1)^{\frac{p-1}{2}} b\left(\frac{n}{p}\right).
\]
Further if \( j \not\equiv 0 \pmod{p} \), then
\[
b(p^2n + pj) = 0.
\]

**Proof.** Let \( p \) be a prime with \( p \not\equiv 1 \pmod{8} \). Using (2.1), we note that
\[
\sum_{n=1}^{\infty} b(n)q^n = \eta(8z)\eta(16z).
\]
By using Theorem 2.1, we obtain that \( \eta(8z)\eta(16z) \in S_1(\Gamma_0(128), (\frac{-128}{\cdot}) \). Thus \( \eta(8z)\eta(16z) \) has the Fourier expansion given by
\[
\sum_{n=1}^{\infty} b(n)q^n = \eta(8z)\eta(16z) = q - q^9 - 2q^{17} + \cdots.
\]
Therefore, \( b(n) = 0 \) for all \( n \geq 0 \) with \( n \not\equiv 1 \pmod{8} \). Since \( \eta(8z)\eta(16z) \) is a Hecke eigenform, we obtain from [7, Table 1] that
\[
\eta(8z)\eta(16z) | T_p = \sum_{n=1}^{\infty} \left( b(pn) + \left(-\frac{128}{p}\right) b\left(\frac{n}{p}\right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} b(n)q^n.
\]
Note that \( \left( \frac{-128}{p} \right) = \left( \frac{-2}{p} \right) \). Comparing the coefficients of \( q^n \) on both sides of the above equation, we get

\[
(3.4) \quad b(pn) + \left( \frac{-2}{p} \right) b \left( \frac{n}{p} \right) = \lambda(p)b(n).
\]

Since \( b(1) = 1 \) and \( b\left( \frac{1}{p} \right) = 0 \), by substituting \( n = 1 \) in the above expression, we get \( \lambda(p) = 0 \). Hence, we conclude from (3.4) that

\[
(3.5) \quad b(pn) + \left( \frac{-2}{p} \right) b \left( \frac{n}{p} \right) = 0,
\]

which proves (3.2). For \( j \not\equiv 0 \pmod{p} \), replacing \( n \) by \( pn + j \) in (3.5), we get \( b(p^n + pj) = 0 \) which proves (3.3). \( \square \)

Lemma 3.3. For \( n \geq 0 \) and for a prime \( p \equiv 1 \pmod{4} \), we have

\[
(3.6) \quad c \left( p^2n + \frac{p^2 - 1}{8} \right) = pc(n).
\]

If \( p \nmid n \), then

\[
(3.7) \quad c \left( pn + \frac{p^2 - 1}{8} \right) = 0.
\]

Proof. From [1, Page 39, Entry 24(ii)], we have

\[
(q; q)_{\infty}^3 = \sum_{n=0}^{\infty} (-1)^n(2n + 1)q^{\frac{n(n+1)}{2}}.
\]

Thus

\[
c(n) = \sum_{k=0}^{\infty} (-1)^k(2k + 1) = \sum_{k=0}^{\infty} (-1)^k(2k + 1).
\]

This implies that

\[
c \left( pn + \frac{p^2 - 1}{8} \right) = \sum_{k=0}^{\infty} (-1)^k(2k + 1).
\]

Note that if \( (2k + 1)^2 = 8pn + p^2 \), then \( p \mid (2k + 1) \) and therefore, we can write \( 2k + 1 = p(2k' + 1) \) for some positive integer \( k' \). Further for such \( k \), we have

\[
k = \frac{2k + 1}{2} - \frac{1}{2} = \frac{p(2k' + 1)}{2} + \frac{p - 1}{2} \quad \text{which gives} \quad (-1)^k = (-1)^{k'}.
\]

Hence

\[
c \left( pn + \frac{p^2 - 1}{8} \right) = p \sum_{k=0}^{\infty} (-1)^k(2k' + 1) = pc \left( \frac{n}{p} \right)\) for \( (2k' + 1)^2 = \frac{8p}{p} + 1 \).\]
Replacing $n$ by $pn$, we obtain (3.6). Also, (3.7) follows since $c\left(\frac{n}{p}\right) = 0$ if $p \nmid n$. This completes the proof.

We recall the following identity for 13-core partitions obtained by Kuwali Das.

**Lemma 3.4** ([2, Theorem 1]). We have
\[
\sum_{n=0}^{\infty} a_{13}(104n+6)q^n \equiv (q;q)_{\infty} (q^2;q^2)_{\infty} \pmod{2}.
\]

**Lemma 3.5.** For $j \not\equiv 0 \pmod{p}$ and $n \geq 0$, we have
\[
a_{2p^2n} \equiv 0 \pmod{2},
\]
(3.8)
\[
a_{2p^2n+\frac{p^2-1}{8}} \equiv \delta_p a_2(n) \pmod{2},
\]
(3.9)
\[
a_{13}(104p^2n+13p(\epsilon_p j + p) - 7) \equiv 0 \pmod{2},
\]
(3.10)
\[
a_{13}(104p^2n+13p^2 - 7) \equiv \delta_p a_{13}(104n+6) \pmod{2},
\]
(3.11)
where
\[
\epsilon_p = \begin{cases} 1 & \text{if } p \not\equiv 1 \pmod{8}, \\ 8 & \text{if } p \equiv 1 \pmod{8} \end{cases}
\]
and \(\delta_p = \begin{cases} (-1)\left(\frac{-2}{p}\right) & \text{if } p \not\equiv 1 \pmod{8}, \\ p & \text{if } p \equiv 1 \pmod{8} \end{cases}\).

**Proof.** We consider the two cases $p \not\equiv 1 \pmod{8}$ and $p \equiv 1 \pmod{8}$ separately as follows.

**Case 1:** $p \not\equiv 1 \pmod{8}$.

From (1.1), we have
\[
\sum_{n=0}^{\infty} a_2(n)q^n \equiv (q^8;q^8)_{\infty} \equiv (q;q)_{\infty} (q^2;q^2)_{\infty} \pmod{2}.
\]
Thus using Lemma 3.4, we have
\[
\sum_{n=0}^{\infty} a_2(n)q^n \equiv (q^8;q^8)_{\infty} \equiv (q;q)_{\infty} (q^2;q^2)_{\infty} \pmod{2}.
\]
(3.12)
\[
\sum_{n=0}^{\infty} a_2(n)q^{8n+1} \equiv \sum_{n=0}^{\infty} a_{13}(104n+6)q^{8n+1} \equiv q(q^8;q^8)_{\infty} (q^{16};q^{16})_{\infty} \pmod{2}.
\]

From (3.1) and (3.12), we get
\[
a_2(n) \equiv a_{13}(104n+6) \equiv b(8n+1) \pmod{2}.
\]
(3.13)
Let $r \not\equiv 0 \pmod{p}$. From (3.3), we have
\[
b(p^2n + pr) = 0.
\]
Replacing $n$ by $8n - pr + 1$, we obtain
\[
b(8p^2n - p^3r + p^2 + pr) = 0.
\]
Note that $8p^2n - p^3r + p^2 + pr = 8(p^2n - pr \frac{p^2 - 1}{8} + \frac{p^2 - 1}{8}) + 1$. Therefore, using (3.13), we obtain

\[
\begin{align*}
(3.14) & \quad a_2 \left( p^2n - pr \frac{p^2 - 1}{8} + \frac{p^2 - 1}{8} \right) \\
& \equiv a_{13} (104p^2n - 13pr(p^2 - 1) + 13p^2 - 7) \\
& \equiv 0 \pmod{2}.
\end{align*}
\]

Since $\gcd(\frac{p^2 - 1}{8}, p) = 1$ and $\gcd(p^2 - 1, p) = 1$, when $r$ runs over a residue system excluding the multiples of $p$, so do $-r(p^2 - 1)$ and $-r(p^2 - 1)$. Thus for $j \not\equiv 0 \pmod{p}$, (3.14) can be written as

\[
\begin{align*}
a_2 \left( p^2n +pj + \frac{p^2 - 1}{8} \right) & \equiv 0 \pmod{2}
\end{align*}
\]

and

\[
\begin{align*}
a_{13} (104p^2n + 13pj + 13p^2 - 7) & \equiv 0 \pmod{2}.
\end{align*}
\]

This proves (3.8) and (3.10) in the case of $p \not\equiv 1 \pmod{8}$.

Next, replacing $n$ by $8pn + p$ in (3.2), we obtain

\[
\begin{align*}
(3.15) & \quad b(8p^2n + p^2) = (-1) \left( \frac{-2}{p} \right) b(8n + 1).
\end{align*}
\]

Note that $8p^2n + p^2 = 8(p^2n + \frac{p^2 - 1}{8}) + 1$. Therefore, using (3.13) in (3.15), we get

\[
\begin{align*}
& \quad a_2 \left( p^2n + \frac{p^2 - 1}{8} \right) \equiv (-1) \left( \frac{-2}{p} \right) a_2(n) \pmod{2}
\end{align*}
\]

and

\[
\begin{align*}
& \quad a_{13} (104p^2n + 13p^2 - 7) \equiv (-1) \left( \frac{-2}{p} \right) a_{13} (104n + 6) \pmod{2},
\end{align*}
\]

which proves (3.9) and (3.11) in the case of $p \not\equiv 1 \pmod{8}$.

**Case 2:** $p \equiv 1 \pmod{8}$.

From (1.1), we have

\[
\begin{align*}
(3.16) & \quad \sum_{n=0}^{\infty} a_2(n)q^n \equiv (q^2;q^2)_\infty (q;q)_\infty^2 \equiv (q;q^3)_\infty^3 \pmod{2}.
\end{align*}
\]

From Lemma 3.4, we have

\[
\begin{align*}
(3.17) & \quad \sum_{n=0}^{\infty} a_{13}(104n + 6)q^n \equiv (q;q)_\infty (q^2;q^2)_\infty (q^3;q)_\infty^3 \equiv (q; q^3)_\infty^3 \pmod{2}.
\end{align*}
\]

Invoking (3.1), (3.16) and (3.17), we have

\[
\begin{align*}
(3.18) & \quad a_2(n) \equiv a_{13}(104n + 6) \equiv c(n) \pmod{2}.
\end{align*}
\]
If \( p \nmid n \), then from (3.7) and (3.18), we get
\[
a_2 \left( pn + \frac{p^2 - 1}{8} \right) \equiv a_{13} \left( 104pn + 13p^2 - 7 \right) \equiv c \left( pn + \frac{p^2 - 1}{8} \right) \equiv 0 \pmod{2}.
\]
Next replacing \( n \) by \( pn + j \) for \( j \neq 0 \pmod{p} \), we obtain
\[
a_2 \left( p^2n + pj + \frac{p^2 - 1}{8} \right) \equiv a_{13} \left( 104p^2n + 104pj + 13p^2 - 7 \right) \equiv 0 \pmod{2},
\]
which proves (3.8) and (3.10) in the case of \( p \equiv 1 \pmod{8} \).

Next using (3.6) and (3.18), we get
\[
a_2 \left( p^2n + \frac{p^2 - 1}{8} \right) \equiv a_{13} \left( 104p^2n + 13p^2 - 7 \right) \equiv c \left( p^2n + \frac{p^2 - 1}{8} \right) \equiv p\delta \equiv p\alpha (\text{mod } 2),
\]
which proves (3.9) and (3.11) in the case of \( p \equiv 1 \pmod{8} \).

\( \square \)

3.6. **Proof of Theorem 1.1(i)**

For \( 1 \leq i \leq k - 1 \), we note that
\[
p_i^2p_i^2p_{i+1}^2 \cdots p_k^2n + \frac{p_i^2p_{i+1}^2 \cdots p_k^2 - 1}{8}
= p_i^2 \left( p_{i+1}^2 \cdots p_k^2n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8} \right) + \frac{p_i^2 - 1}{8}.
\]
Thus, for \( 1 \leq i \leq k - 1 \), using (3.9) for \( p = p_i \), we have
\[
a_2 \left( p_i^2p_{i+1}^2 \cdots p_k^2n + \frac{p_i^2p_{i+1}^2 \cdots p_k^2 - 1}{8} \right)
\equiv \delta_{p_i} \alpha_{p_i} \left( p_{i+1}^2 \cdots p_k^2n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8} \right) \pmod{2}.
\]
Also from (3.9), we have
\[
a_2 \left( p_i^2n + \frac{p_i^2 - 1}{8} \right) \equiv \delta_{p_i} \alpha_{p_i} (\text{mod } 2).
\]
Therefore, from the congruences in the above two displays, we get
\[
a_2 \left( p_1^2p_2^2 \cdots p_k^2n + \frac{p_1^2p_2^2 \cdots p_k^2 - 1}{8} \right) \equiv \delta_{p_1} \delta_{p_2} \cdots \delta_{p_k} \alpha_{p_k} (\text{mod } 2).
\]
Replacing \( n \) by \( p_{k+1}^2n + \frac{p_{k+1}(8j + p_{k+1}) - 1}{8} \) in the above expression and then using (3.8) for \( p = p_{k+1} \), we get
\[
a_2 \left( p_1^2p_2^2 \cdots p_k^2p_{k+1}^2n + \frac{p_1^2p_2^2 \cdots p_k^2p_{k+1}(8j + p_{k+1}) - 1}{8} \right)
\]

Also from (3.11), we have

\[ \equiv \delta_p \delta_{p_2} \cdots \delta_{p_k} a_2 \left( p_{k+1}^2 n + p_{k+1} j + \frac{p_{k+1}^2 - 1}{8} \right) \]
\[ \equiv 0 \pmod{p_{k+1}}, \]
when \( j \not\equiv 0 \pmod{p_{k+1}} \). This completes the proof of Theorem 1.1(i).

3.7. Proof of Theorem 1.1(ii)

The proof is similar to the proof of Theorem 1.1(i). For \( 1 \leq i \leq k - 1 \), we note that

\[ 104 p_i^2 p_{i+1}^2 \cdots p_k^2 n + 13 p_i^2 p_{i+1}^2 \cdots p_k^2 = 7 \]
\[ = 104 p_i^{2} \left( p_{i+1}^2 \cdots p_k^2 n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8} \right) + 13 p_i^2 - 7. \]
Thus, for \( 1 \leq i \leq k - 1 \), (3.11) implies

\[ a_{13}(104 p_i^2 p_{i+1}^2 \cdots p_k^2 n + 13 p_i^2 p_{i+1}^2 \cdots p_k^2 - 7) \]
\[ \equiv \delta_p a_{13} \left( 104 \left( p_{i+1}^2 \cdots p_k^2 n + \frac{p_{i+1}^2 \cdots p_k^2 - 1}{8} \right) + 6 \right) \]
\[ \equiv \delta_p a_{13}(104 p_i^2 p_{i+1}^2 \cdots p_k^2 n + 13 p_i^2 p_{i+1}^2 \cdots p_k^2 - 7) \pmod{2}. \]
Also from (3.11), we have

\[ a_{13}(104 p_i^2 n + 13 p_i^2 - 7) \equiv \delta_p a_{13}(104 n + 6) \pmod{2}. \]
Therefore, from the above two congruences, we get

\[ a_{13}(104 p_i^2 p_{i+1}^2 \cdots p_k^2 n + 13 p_i^2 p_{i+1}^2 \cdots p_k^2 - 7) \equiv \delta_p \delta_{p_2} \cdots \delta_{p_k} a_{13}(104 n + 6) \pmod{2}. \]
Replacing \( n \) by \( p_{k+1}^2 n + \frac{p_{k+1}(\epsilon_{p_{k+1}} j + p_{k+1}) - 1}{8} \) in the above expression and then using (3.10), we get

\[ a_{13}(104 p_i^2 p_{i+1}^2 \cdots p_k^2 p_{k+1}^2 n + 13 p_i^2 p_{i+1}^2 \cdots p_k^2 p_{k+1}(\epsilon_{p_{k+1}} j + p_{k+1}) - 7) \]
\[ \equiv \delta_p \delta_{p_2} \cdots \delta_{p_k} a_{13}(104 p_{k+1}^2 n + 13 p_{k+1}(\epsilon_{p_{k+1}} j + p_{k+1}) - 1) + 6) \]
\[ \equiv \delta_p \delta_{p_2} \cdots \delta_{p_k} a_{13}(104 p_{k+1}^2 n + 13 p_{k+1}(\epsilon_{p_{k+1}} j + p_{k+1})) - 7) \]
\[ \equiv 0 \pmod{2}, \]
when \( j \not\equiv 0 \pmod{p_{k+1}} \). This completes the proof of Theorem 1.1(ii).

3.8. Proof of Theorem 1.2

For any prime \( p \equiv 7 \pmod{8} \), we get from (3.2) that

\[ b(pn) = (-1) \left( \frac{-2}{p} \right) b \left( \frac{n}{p} \right). \]
Let \( r \not\equiv 0 \pmod{p} \). Replacing \( n \) by \( 8(p^k n + r) + 7 \), we obtain

\[ b(8(p^{k+1} n + pr) + 7) = (-1) \left( \frac{-2}{p} \right) b \left( \frac{8(p^k n + r) + 7}{p} \right). \]
which can be rewritten as
\begin{equation}
\frac{b}{8} \left( p^{k+1}n + pr + \frac{7p - 1}{8} \right) + 1
= (-1) \left( \frac{-2}{p} \right) b \left( \frac{p^{k-1}n + \frac{8r + 7 - p}{8p}}{8p} \right) + 1.
\end{equation}

We note here that $\frac{7p - 1}{8}$ and $\frac{8r + 7 - p}{8p}$ are integers. Therefore, using (3.13) and (3.19), we get
\begin{equation}
\frac{a_2}{p} \left( p^{k+1}n + pr + \frac{7p - 1}{8} \right)
\equiv (-1) \left( \frac{-2}{p} \right) a_2 \left( p^{k-1}n + \frac{8r + 7 - p}{8p} \right) \pmod{2},
\end{equation}
and
\begin{equation}
\frac{a_{13}}{p} \left( 104p^{k+1}n + 104pr + 91p - 7 \right)
\equiv (-1) \left( \frac{-2}{p} \right) a_{13} \left( 104p^{k-1}n + \frac{104r + 91}{p} - 7 \right) \pmod{2}.
\end{equation}

### 3.9. Proof of Corollary 1.2

Let $p$ be a prime such that $p \equiv 7 \pmod{8}$. Choose a non negative integer $r$ such that $8r + 7 = p^{2k-1}$. Replacing $k$ by $2k - 1$ in (3.20), we obtain
\begin{align*}
a_2 \left( p^{2k}n + \frac{p^{2k} - 1}{8} \right)
&\equiv (-1) \left( \frac{-2}{p} \right) a_2 \left( p^{2k-2}n + \frac{p^{2k-2} - 1}{8} \right) \\
&\equiv \cdots \equiv (-1)^k \left( \frac{-2}{p} \right)^k a_2(n) \pmod{2}.
\end{align*}

Replacing $k$ by $2k - 1$ in (3.21), we obtain
\begin{align*}
a_{13} \left( 104p^{2k}n + 13p^{2k} - 7 \right)
&\equiv (-1) \left( \frac{-2}{p} \right) a_{13} \left( 104p^{2k-2}n + 13p^{2k-2} - 7 \right) \\
&\equiv \cdots \equiv (-1)^k \left( \frac{-2}{p} \right)^k a_{13}(104n + 6) \pmod{2}.
\end{align*}

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