JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **36**, No. 3, August 2023
http://dx.doi.org/10.14403/jcms.2023.36.3.149

A NOTE ON THE DERIVATIONS OF TWO KNOWN COMBINATORIAL IDENTITIES VIA A HYPERGEOMETRIC SERIES APPROACH

Arjun Kumar Rathie and Dongkyu Lim

Abstract. The aim of this note is to derive two known combinatorial identities via a hypergeometric series approach using Saalschitz's classical summation theorem.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by [8]:

$$
{}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n \cdots (a_p)_n}{(b_1)_n\,(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.1}
$$

where $(a)_n$ denotes the well-known pochhammer symbols (or the raised or the shifted factorial, since $(1)_n = n!$) defined for any complex number $a(\neq 0)$ by

$$
(a)_n = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}
$$

Also, in terms of well-known Gamma functions, Pochhammer symbol $(a)_n$ is given by

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.
$$

Here, as usual, p and q are non-negative integers and the parameters a_j $(1 \leq j \leq p)$ and b_j $(1 \leq j \leq q)$ can take arbitrary values (real of

Received June 09, 2023; Accpted August 29, 2023;

2020 Mathematics Subject Classification: Primary 12A34, 56B34; Secondary 78C34.

Key words and phrases: generalized hypergeometric function, Saalschiitz theorem, central binomial coefficent.

Corresponding author.

This work was supported by a Research Grant of Andong National University.

complex) with one exception that b_j $(1 \leq j \leq q)$ should not be zero or a negative integer.

By well-known ratio tests, it can be easily verified that the generalized hypergeometric function ${}_pF_q(z)$ converges for all $|z| < \infty$ $(p \leq q)$, $|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1)$ with $\Re(s) > 0$, where s is the parametric excess defined by

$$s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

For $p = 2$ and $q = 1$, we have the well-known hypergeometric function ${}_2F_1(z)$ popularly known in the literature as the Gauss's hypergeometric function.

It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications in the areas of mathematics, theoretical physics, mathematical physics, statistics, and engineering mathematics.

For more details about the generalized hypergeometric functions ${}_pF_q(z)$ about its convergence conditions, properties and connection with several elementary functions, we refer the standard texts [1, 2, 6].

It is interesting to mention here that whenever a generalized hypergeometric function reduces to the gamma function, the results are very important from the applications point of view. Thus the classical summation theorems such as those of Gauss, Gauss-second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschiitz for the series ${}_3F_2$ and others play an important role. However, in our present investigation, we would like to mention here the following classical Saalschiitz [8] summation theorem for the series ${}_3F_2$ viz.

$$(1.2) \qquad {}_3F_2\left[\begin{matrix} -n,\ a,\ b \\ c,\ 1+a+b-c-n \end{matrix};1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},$$

Remark 1.1. For generalizations and extensions of the above-mentioned classical summation theorems, we refer research papers by Rakha and Rathie [7] and Kim et al. [4].

On the other hand, we mention here the following two well-known combinatorial identities recorded in Gould [3, Equ. (3.96) and (3.95), p.33] and Sprugnoli [9, Equ. (3.95) and (3.96), p.77] viz.

$$(1.3) \qquad \sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{2^{4n}}{(2n+1)}\binom{2n}{n} = \frac{2^{2n}(1)_n}{\left(\frac{3}{2}\right)_n}$$

and

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \binom{2n+1}{n} = \frac{2^{2n}\left(\frac{3}{2}\right)_n}{(2)_n}. \tag{1.4}$$

Very recently Qi et al. [5] established the identity (1.3) by two different methods using series expansion of powers of arcsine and also discussed the identity (1.4).

The aim of this short note is to establish the combinatorial identities (1.3) and (1.4) by using classical Saalschiitz summation theorem (1.2).

2. Derivations of two combinatorial identities (1.3) and (1.4) using Saalschiitz summations Theorem (1.2)

2.1. Derivation of the identity (1.3)

In order to derive the identity (1.3), we proceed as follows. Denoting the left-hand side of the identity (1.3) by S_1, we have

$$S_1 = \sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\binom{2n-2k}{n-k}. \tag{2.1}$$

Now making use of the following elementary results and identities:

$$(i)\quad a = \frac{\Gamma(a+1)}{\Gamma(a)}$$

$$(ii)\quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$(iii)\quad \Gamma(a-n) = \frac{(-1)^n\Gamma(a)}{(1-a)_n}$$

$$(iv)\quad \Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\sqrt{\pi}}$$

$$\text{and}\quad (v)\quad \binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

It is not difficult to see that

$$(a)\quad 2k+1=\frac{\left(\frac{3}{2}\right)_k}{\left(\frac{1}{2}\right)_k}$$

$$(b)\quad \binom{2k}{k}=\frac{2^{2k}\left(\frac{1}{2}\right)_k}{k!}$$

$$\text{and}\quad (c)\quad \binom{2n-2k}{n-k}=\frac{2^{2n-2k}(-n)_k\left(\frac{1}{2}\right)_k}{(1)_n\left(\frac{1}{2}-n\right)_k}.$$

Substituting all the expressions in (2.1), we have after some simplification

$$S_1=\frac{2^{2n}\left(\frac{1}{2}\right)_n}{(1)_n}\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k\left(\frac{1}{2}-n\right)_k k!}.$$

Finally, summing up the series, we have

$$S_1=\frac{2^{2n}\left(\frac{1}{2}\right)_n}{(1)_n}{}_3F_2\left[\begin{matrix}-n,\ \frac{1}{2},\ \frac{1}{2}\\ \frac{3}{2},\ \frac{1}{2}-n\end{matrix};1\right].$$

We now observe that the ${}_3F_2$ can be evaluated with the help of the classical Saalschiitz summation theorem (1.2) by letting $a=b=\frac{1}{2}$ and $c=\frac{3}{2}$. We thus at once arrive at the right-hand side of (1.3). this completes the proof of combinatorial identity.

2.2. Derivation of the identity (1.4)

Denoting the left-hand side of (1.4) by S_2 and proceeding exactly on the similar lines as give in (2.1), we, after some simplification arrive at the following expression

$$S_2=\frac{2^{2n}\left(\frac{1}{2}\right)_n}{(1)_n}{}_3F_2\left[\begin{matrix}-n,\ \frac{1}{2},\ 1\\ 2,\ \frac{1}{2}-n\end{matrix};1\right]..$$

The ${}_3F_2$ can be evaluated by Saalschiitz summation theorem (1.2) by letting $a=\frac{1}{2}$, $b=1$ and $c=2$. So we left this as an exercise to the interested reader.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 2000.

[2] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner, New York, 1964.

[3] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., Morgantown, WV. 1972.
[4] Y. S. Kim, M. A. Rakha and A. K. Rathie, *Extensions of Certain Classical Summation Theorems for the Series ${}_2F_1$, ${}_3F_2$, and ${}_4F_3$ with Applications in Ramanujan's Summations*, Int. J. Math. Sci., **2010** (2010), Article ID 309503, 26 pages.
[5] F. Qi, C.-P. Chen and D. Lim, *Several identities containing central binomial coefficients and derived from series expansions of powers of the arcsine function*, Results in Nonlinear Analysis, **4** (2021), no. 1, 57–64.
[6] E.D. Rainville, *Special Functions*, The Macmillan Company, New York. 1960; Reprinted by chelsea Publishing Company, Bronx, NY, 1971.
[7] M. A. Rakha and A. K. Rathie, *Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ with applications*, Integral Transforms Spec. Funct., **22** (2011), no. 11, 823–840.
[8] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1960.
[9] R. Sprugnoli, *Riordan Array Proofs of Identities in Gould's Book*, University of Florence, Italy, 2008.

Arjun Kumar Rathie
Department of Mathematics
Vedant College of Engineering & Technology,
(Rajasthan Technical University)
Bundi-323021, Rajasthan state, India
E-mail: arjunkumarrathie@gmail.com

Dongkyu Lim
Department of Mathematics Education
Andong National University
Andong 36729, Republic of Korea
E-mail: dklim@anu.ac.kr