J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.3.337 ISSN(Online) 2287-6081
Volume 30, Number 3 (August 2023), Pages 337–345

APPLICATIONS OF THE SCHWARZ LEMMA RELATED TO BOUNDARY POINTS

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Abstract. Different versions of the boundary Schwarz lemma for the $\mathcal{N}(\rho)$ class are discussed in this study. Also, for the function $g(z) = z+b_2z^2+b_3z^3+...$ defined in the unit disc D such that $g \in \mathcal{N}(\rho)$, we estimate a modulus of the angular derivative of $g(z)$ function at the boundary point $1 \in \partial D$ with $g'(1) = 1 + \sigma(1-\rho)$, where $\rho = \frac{1}{n}\sum_{i=1}^{n} g(c_i) = \frac{g'(c_1)+g'(c_2)+...+g'(c_n)}{n} \in g'(D)$ and $\rho \neq 1$, $\sigma > 1$ and $c_1, c_2, ..., c_n \in \partial D$. That is, we shall give an estimate below $|g''(1)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$. Estimating is made by using the arithmetic average of n different derivatives $g'(c_1), g'(c_2), ..., g'(c_n)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $g(z) = z + b_2z^2 + b_3z^3 + ...$ which are analytic in $D = \{z : |z| < 1\}$ and for $c_1, c_2, ..., c_n \in \partial D = \{z : |z| = 1\}$, $g(z)$ has an angular limit $g(c_1)$, $g(c_2)$, ..., $g(c_n)$ at $c_1, c_2, ..., c_n$. Also, $\mathcal{N}(\rho)$ be the subclass of $\mathcal{A}$ consisting of all functions g which satisfy

$$\left| zg''(z) - \frac{zg''(z)}{g'(z)} \right| < \frac{|1-\rho|^2 \sigma^2}{1+|1-\rho|\sigma}, \ z \in D \tag{1.1}$$

for some complex $\rho = \frac{1}{n}\sum_{i=1}^{n} g(c_i) = \frac{g'(c_1)+g'(c_2)+...+g'(c_n)}{n} \in g'(D)$ and $\rho \neq 1$, $\sigma > 1$.

First of all, in this study, we will examine the modulus of the derivative of the analytical function $g(z)$ at the zero point. In this study, an upper bound will be found for the modulus of b_2, one of the coefficients in the Taylor expansion of the function. In particular, this upper bound will show us the relationship between the coefficient b_2 and ρ. That is, the subject of the present paper is to discuss some properties of the function $g(z)$ which belongs to the class of $\mathcal{N}(\rho)$ by applying

Received by the editors October 18, 2022. Accepted June 05, 2023.
2020 *Mathematics Subject Classification.* 30C80.
Key words and phrases. Julia-Wolff lemma, analytic function, Schwarz lemma, angular derivative.

Schwarz lemma. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows ([5], p.329):

Lemma 1.1 (Schwarz lemma)**.** *Suppose that* h *is analytic in the unit disc,* $|h(z)| < 1$ *for* $|z| < 1$ *and* $h(0) = 0$*. Then*

$$i-)\; |h(z)| \leq |z|$$

$$ii-)\; \left|h'(0)\right| \leq 1$$

with equality in either of the above if and only if $h(z) = ze^{i\theta}$*,* θ *real.*

In the proofs of our results, we shall need the following lemma due to Jack's lemma [6] and due to Miller-Mocanu [11]

Lemma 1.2. *Let* $h(z)$ *be a non-constant anaytic function in* D *with* $h(0) = 0$*. If*

$$|h(z_0)| = \max\left\{|h(z)| : |z| \leq |z_0|\right\},$$

then there exists a real number $k \geq 1$ *such that*

$$\frac{z_0 h'(z_0)}{h(z_0)} = k$$

and

$$1 + \Re\left(\frac{z_0 h''(z_0)}{h'(z_0)}\right) \geq k.$$

In this work, an application of Schwarz Lemma for the class $\mathcal{N}(\rho)$ will be given. The modulus of the second derivative of the function $g(z)$ at the point zero will be estimated from above by arithmetic averaging of images of the angular derivatives at different boundary points. Let us define the following function for this consideration:

$$\varphi(z) = \frac{g'(z) - 1}{\sigma(1 - \rho)},\; g(z) \in \mathcal{N}(\rho). \tag{1.2}$$

Here, $\varphi(z)$ is analytic and $\varphi(0) = 0$. Now, let us show that $|\varphi(z)| < 1$ for $z \in D$. From (1.2), we have

$$g'(z) = 1 + \sigma(1 - \rho)\varphi(z),$$

$$g''(z) = \sigma(1 - \rho)\varphi'(z)$$

and

$$\begin{aligned} zg''(z)-\frac{zg''(z)}{g'(z)} &= \sigma(1-\rho)\,z\varphi'(z)-\frac{\sigma(1-\rho)\,z\varphi'(z)}{1+\sigma(1-\rho)\,\varphi(z)} \\ &= \sigma(1-\rho)\,z\varphi'(z)\left(1-\frac{1}{1+\sigma(1-\rho)\,\varphi(z)}\right) \\ &= \sigma(1-\rho)\,z\varphi'(z)\frac{\sigma(1-\rho)\,\varphi(z)}{1+\sigma(1-\rho)\,\varphi(z)} \\ &= \frac{\sigma^2(1-\rho)^2\,z\varphi'(z)\varphi(z)}{1+\sigma(1-\rho)\,\varphi(z)}. \end{aligned}$$

We suppose that there exists a $z_0 \in D$ such that

$$\max_{|z|\leq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1.$$

From Jack's lemma, we take $\varphi(z_0)=e^{i\theta}$and $z_0'\varphi(z_0)=k\varphi(z_0)$. Therefore, we have that

$$\begin{aligned} \left|z_0g''(z_0)-\frac{z_0g''(z_0)}{g'(z_0)}\right| &= \left|\frac{\sigma^2(1-\rho)^2\,z_0\varphi'(z_0)\varphi(z_0)}{1+\sigma(1-\rho)\,\varphi(z_0)}\right|=\left|\frac{\sigma^2(1-\rho)^2\,k\,(m(z_0))^2}{1+\sigma(1-\rho)\,m(z_0)}\right| \\ &= \left|\frac{\sigma^2(1-\rho)^2\,ke^{2i\theta}}{1+\sigma(1-\rho)\,e^{i\theta}}\right|=\frac{\sigma^2(1-\rho)^2\,k}{\left|1+\sigma(1-\rho)\,e^{i\theta}\right|}. \end{aligned}$$

Since $\left|1+\sigma(1-\rho)\,e^{i\theta}\right|\leq 1+\sigma|1-\rho|$ and $k\geq 1$, we get

$$\left|z_0g''(z_0)-\frac{z_0g''(z_0)}{g'(z_0)}\right|\geq\frac{\sigma^2|1-\rho|^2}{1+\sigma|1-\rho|}.$$

This contradicts $g(z)\in\mathcal{N}(\rho)$. This mean that there is no point $z_0\in D$ such that

$$\max_{|z|\leq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1.$$

Hence, we obtain $|\varphi(z)|<1$ for $|z|<1$. Thus, from the Schwarz lemma, we take

$$\begin{aligned} \varphi(z) &= \frac{g'(z)-1}{\sigma(1-\rho)}=\frac{1}{\sigma(1-\rho)}\left(1+2b_2z+3b_3z^2+....-1\right) \\ &= \frac{2b_2z+3b_3z^2+...}{\sigma(1-\rho)}. \end{aligned}$$

If we take the derivative of both sides of the equation and substitute zero for z, then

$$\varphi'(0)=\frac{2b_2}{\sigma(1-\rho)},$$

$$\left|\varphi'(0)\right|=\frac{2|b_2|}{\sigma|1-\rho|}\leq 1$$

and

$$|b_2| \leq \frac{\sigma |1-\rho|}{2}.$$

Therefore, we obtain the following lemma.

Lemma 1.3. *Let $g(z) \in \mathcal{N}(\rho)$. Then we have the inequality*

$$\left|g''(0)\right| \leq \sigma |1-\rho|. \tag{1.3}$$

Additionally, there are numerous research about the Schwarz lemma because of how broadly applicable it is. Electrical and electronic engineering uses it in a variety of ways. One example of the Schwarz Lemma's use in electrical engineering is the synthesis of circuits using positive real functions and boundary analysis of these functions. Examples of how the Schwarz Lemma has been used to engineering research include transfer functions in control engineering and multi-notch filters in signal processing [14]. Another one of these studies, known as the boundary variant of the Schwarz lemma, talks about estimating the derivative of the function at a boundary point of the unit disc from below its modulus. The boundary version of Schwarz lemma is given as follows [10, 16]:

Lemma 1.4. *If $h(z)$ extends continuously to some boundary point c with $|c| = 1$, $h(0) = 0$, $|h(z)| < 1$ for $z \in D$ and if $|h(c)| = 1$ and $h'(c)$ exists, then*

$$\left|h'(c)\right| \geq \frac{2}{1+|h'(0)|} \tag{1.4}$$

and

$$\left|h'(c)\right| \geq 1. \tag{1.5}$$

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 7, 9, 10, 11, 13]. Mercer [8] proves a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtained a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [10].

We shall investigate how the second derivative of the function belonging to the $\mathcal{N}(\rho)$ class behaves in the unit disc's boundary. This examination will be made at some boundary point 1. With the use of arithmetic averaging of images of the angular derivatives at various boundary points, the modulus of the second derivative will be inferred from the following. That is, the inequalities including the modulus of $g''(1)$ with $g'(c_1)$, $g'(c_2), ..., g'(c_n)$ will be obtained.

For our main results, we shall need the following lemma due to Julia-Wolff [15].

Lemma 1.5 (Julia-Wolff lemma)**.** *Let h be an analytic function in D, $h(0) = 0$ and $h(D) \subset D$. If, in addition, the function h has an angular limit $h(c)$ at $c \in \partial D$, $|h(c)| = 1$, then the angular derivative $h'(c)$ exists and $1 \leq |h'(c)| \leq \infty$.*

Corollary 1.6. *The analytic function h has a finite angular derivative $h'(c)$ if and only if h' has the finite angular limit $h'(c)$ at $c \in \partial D$.*

2. Main Results

In this section, we discuss a different version of the boundary Schwarz Lemma for the class $\mathcal{N}(\rho)$. The modulus of the second derivative of the analytic function $g(z)$ is estimated from below by associating with the derivatives at the different boundary points. Estimating is made by using the arithmetic average of different derivatives $g'(c_1)$, $g'(c_2), ..., g'(c_n)$. Motivated by the results of the work presented in [2], the following result has been obtained.

Theorem 2.1. *Let $g(z) \in \mathcal{N}(\rho)$ and $g'(z_1) = 1$ for $0 < |z_1| < 1$. Assume that, for $1 \in \partial D$, g has anagular limit $g(1)$ at 1, $g'(1) = 1 + \sigma(1-\rho)$. Then we have the inequality*

$$(2.1)\quad |g''(1)| \geq \sigma|1-\rho| \left(1 + \frac{1-|z_1|^2}{|1-z_1|^2} + \frac{\sigma|1-\rho||z_1| - |g''(0)|}{\sigma|1-\rho||z_1| + |g''(0)|} \right.$$
$$\left. \times \left[1 + \frac{\sigma^2|1-\rho|^2|z_1|^2 + g''(z_1)\left(1-|z_1|^2\right)|g''(0)| - \sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right) - \sigma|1-\rho|g''(0)}{\sigma^2|1-\rho|^2|z_1|^2 + g''(z_1)\left(1-|z_1|^2\right)|g''(0)| + \sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right) + \sigma|1-\rho|g''(0)} \frac{1-|z_1|^2}{|1-z_1|^2}\right]\right).$$

Proof. Let

$$\theta(z) = \frac{z - z_1}{1 - \overline{z_1}z}.$$

Also, let $\lambda : D \to D$ be an analytic function and a point $z_1 \in D$ in order to satisfy

$$\left|\frac{\lambda(z) - \lambda(z_1)}{1 - \overline{\lambda(z_1)}\lambda(z)}\right| \leq \left|\frac{z - z_1}{1 - \overline{z_1}z}\right| = |\theta(z)|$$

and

$$(2.2)\qquad |\lambda(z)| \leq \frac{|\lambda(z_1)| + |\theta(z)|}{1 + |\lambda(z_1)|\,|\theta(z)|}$$

by Schwarz-pick lemma [5]. If $\vartheta : D \to D$ is an analytic function and $0 < |z_1| < 1$, letting

$$\lambda(z) = \frac{\vartheta(z) - \vartheta(0)}{z\left(1 - \overline{\vartheta(0)}\vartheta(z)\right)}$$

in (2.2), we obtain

$$\left|\frac{\vartheta(z) - \vartheta(0)}{z\left(1 - \overline{\vartheta(0)}\vartheta(z)\right)}\right| \leq \frac{\left|\frac{\vartheta(z_1)-\vartheta(0)}{z_1\left(1-\overline{\vartheta(0)}\vartheta(z_1)\right)}\right| + |\theta(z)|}{1 + \left|\frac{\vartheta(z_1)-\vartheta(0)}{z_1\left(1-\overline{\vartheta(0)}\vartheta(z_1)\right)}\right| |\theta(z)|}$$

and

$$|\vartheta(z)| \leq \frac{|\vartheta(0)| + |z| \frac{|C|+|\theta(z)|}{1+|C||\theta(z)|}}{1 + |\vartheta(0)| |z| \frac{|C|+|\theta(z)|}{1+|C||\theta(z)|}}, \tag{2.3}$$

where

$$C = \frac{\vartheta(z_1) - \vartheta(0)}{z_1\left(1 - \overline{\vartheta(0)}\vartheta(z_1)\right)}.$$

Let

$$\varphi(z) = \frac{g'(z) - 1}{\sigma(1-\rho)}, \; g(z) \in \mathcal{N}(\rho).$$

The function $\varphi(z)$ is an analytic function in the unit disc D, $|\varphi(z)| < 1$ for $z \in D$.

If we take

$$\vartheta(z) = \frac{\varphi(z)}{z\frac{z-z_1}{1-\overline{z_1}z}},$$

where then

$$\vartheta(z_1) = \frac{\varphi'(z_1)\left(1 - |z_1|^2\right)}{z_1}, \; \vartheta(0) = \frac{\varphi'(0)}{-z_1}$$

and

$$C = \frac{\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1} + \frac{\varphi'(0)}{z_1}}{z_1\left(1 + \frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\frac{\varphi'(0)}{z_1}\right)},$$

where $|C| \leq 1$. Let $|\vartheta(0)| = \alpha$ and

$$\mathrm{K} = \frac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1 + \left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}.$$

From (2.3), we get

$$|\varphi(z)| \leq |z| \, |q(z)| \, \frac{\alpha + |z| \, \frac{\mathrm{K}+|q(z)|}{1+\mathrm{K}|q(z)|}}{1 + \alpha \, |z| \, \frac{\mathrm{K}+|q(z)|}{1+\mathrm{K}|q(z)|}}$$

and

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\alpha \, |z| \, \frac{\mathrm{K}+|\theta(z)|}{1+\mathrm{K}|\theta(z)|} - \alpha \, |z| \, |\theta(z)| - |\theta(z)| \, |z|^2 \, \frac{\mathrm{K}+|\theta(z)|}{1+\mathrm{K}|\theta(z)|}}{(1-|z|) \left(1+\alpha \, |z| \, \frac{\mathrm{K}+|\theta(z)|}{1+\mathrm{K}|\theta(z)|}\right)} = \phi(z). \tag{2.4}$$

Let $r(z) = 1 + \alpha \, |z| \, \frac{\mathrm{K}+|\theta(z)|}{1+\mathrm{K}|\theta(z)|}$ and $\tau(z) = 1 + \mathrm{K} \, |\theta(z)|$. Then

$$\phi(z) = \frac{1-|z|^2 \, |\theta(z)|^2}{(1-|z|) \, r(z)\tau(z)} + \mathrm{K} \, |\theta(z)| \, \frac{1-|z|^2}{(1-|z|) \, r(z)\tau(z)} + |z| \, \mathrm{K}\alpha \frac{1-|\theta(z)|^2}{(1-|z|) \, r(z)\tau(z)}.$$

Since

$$\lim_{z\to 1} r(z) = \lim_{z\to 1} \left(1 + \alpha \, |z| \, \frac{\mathrm{K}+|\theta(z)|}{1+\mathrm{K} \, |\theta(z)|}\right) = 1 + \alpha,$$

$$\lim_{z\to 1} \tau(z) = \lim_{z\to 1} 1 + \mathrm{K} \, |\theta(z)| = 1 + \mathrm{K}$$

and

$$1 - |\theta(z)|^2 = 1 - \left|\frac{z - z_1}{1 - \overline{z_1} z}\right|^2 = \frac{\left(1-|z_1|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_1}z|^2}, \tag{2.5}$$

passing to the angular limit in (2.4) gives

$$\begin{aligned} \left|\varphi'(1)\right| &\geq \frac{2}{(1+\alpha)(1+\mathrm{K})} \left(1 + \frac{1-|z_1|^2}{|1-z_1|^2} + \mathrm{K} + \alpha\mathrm{K} \frac{1-|z_1|^2}{|1-z_1|^2}\right) \\ &= 1 + \frac{1-|z_1|^2}{|1-z_1|^2} + \frac{1-\alpha}{1+\alpha} \left(1 + \frac{1-\mathrm{K}}{1+\mathrm{K}} \frac{1-|z_1|^2}{|1-z_1|^2}\right). \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{1-\alpha}{1+\alpha} &= \frac{1-|\vartheta(0)|}{1+|\vartheta(0)|} = \frac{1-\left|\frac{\varphi'(0)}{z_1}\right|}{1+\left|\frac{\varphi'(0)}{z_1}\right|} = \frac{|z_1| - |\varphi'(0)|}{|z_1| + |\varphi'(0)|} \\ &= \frac{|z_1| - \left|\frac{2b_2}{\sigma(1-\rho)}\right|}{|z_1| + \left|\frac{2b_2}{\sigma(1-\rho)}\right|} = \frac{\sigma \, |1-\rho| \, |z_1| - |g''(0)|}{\sigma \, |1-\rho| \, |z_1| + |g''(0)|}, \end{aligned}$$

$$\frac{1-\mathrm{K}}{1+\mathrm{K}} = \frac{1-\frac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\varphi'(z_1)(1-|z_1|^2)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}}{1+\frac{\left|\frac{\varphi'(z_1)(1-|z_1|^2)}{z_1}\right|+\left|\frac{\varphi'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\varphi'(z_1)(1-|z_1|^2)}{z_1}\right|\left|\frac{\varphi'(0)}{z_1}\right|\right)}}$$

$$= \frac{1-\frac{\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}(1-|z_1|^2)}{z_1}\right|\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|\right)}}{1+\frac{\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}(1-|z_1|^2)}{z_1}\right|+\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}(1-|z_1|^2)}{z_1}\right|\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|\right)}}$$

and

$$\frac{1-\mathrm{K}}{1+\mathrm{K}} = \frac{|z_1|\left(1+\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|\right)-\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}\left(1-|z_1|^2\right)}{z_1}\right|-\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}(1-|z_1|^2)}{z_1}\right|\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|\right)+\left|\frac{\frac{g''(z_1)}{\sigma(1-\rho)}(1-|z_1|^2)}{z_1}\right|+\left|\frac{\frac{g''(0)}{\sigma(1-\rho)}}{z_1}\right|}$$

$$=\frac{\sigma^2|1-\rho|^2|z_1|^2+g''(z_1)\left(1-|z_1|^2\right)|g''(0)|-\sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right)-\sigma|1-\rho|g''(0)}{\sigma^2|1-\rho|^2|z_1|^2+g''(z_1)\left(1-|z_1|^2\right)|g''(0)|+\sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right)+\sigma|1-\rho|g''(0)},$$

we obtain

$$|\varphi'(1)| \geq 1+\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{\sigma|1-\rho||z_1|-|g''(0)|}{\sigma|1-\rho||z_1|+|g''(0)|}$$
$$\times\left[1+\frac{\sigma^2|1-\rho|^2|z_1|^2+g''(z_1)\left(1-|z_1|^2\right)|g''(0)|-\sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right)-\sigma|1-\rho|g''(0)}{\sigma^2|1-\rho|^2|z_1|^2+g''(z_1)\left(1-|z_1|^2\right)|g''(0)|+\sigma|1-\rho|g''(z_1)\left(1-|z_1|^2\right)+\sigma|1-\rho|g''(0)}\frac{1-|z_1|^2}{|1-z_1|^2}\right].$$

From definition of $\varphi(z)$, we have

$$\left|\varphi'(1)\right| = \frac{g''(1)}{\sigma\,|1-\rho|}.$$

Thus, we obtain the inequality (2.1). □

REFERENCES

1. T. Akyel & B. N. Örnek: Some Remarks on Schwarz lemma at the boundary. Filomat **31** (2017), no. 13, 4139-4151. https://doi.org/10.2298/FIL1713139A
2. T. A. Azeroğlu & B. N. Örnek: A refined Schwarz inequality on the boundary. Complex Variab. Elliptic Equa. **58** (2013) 571-577. https://doi.org/10.1080/17476933.2012.718338

3. H.P. Boas: Julius & Julia: Mastering the Art of the Schwarz lemma. Amer. Math. Monthly **117** (2010) 770-785. https://doi.org/10.4169/000298910x521643
4. V.N. Dubinin: The Schwarz inequality on the boundary for functions regular in the disc. J. Math. Sci. **122** (2004) 3623-3629. https://doi.org/10.1023/B:JOTH.0000035237.43977.39
5. G.M. Golusin: Geometric Theory of Functions of Complex Variable [in Russian]. 2nd edn., Moscow 1966.
6. I.S. Jack: Functions starlike and convex of order α. J. London Math. Soc. **3** (1971) 469-474. https://doi.org/10.1112/jlms/s2-3.3.469
7. M. Mateljević, N. Mutavdzc & B.N. Örnek: Note on Some Classes of Holomorphic Functions Related to Jack's and Schwarz's Lemma. Appl. Anal. Discrete Math. **16** (2022), 111-131. https://doi.org/10.2298/AADM200319006M
8. P.R. Mercer: Sharpened Versions of the Schwarz Lemma. Journal of Mathematical Analysis and Applications. **205** (1997) 508-511. https://doi.org/10.1006/jmaa.1997.5217
9. P.R. Mercer: Boundary Schwarz inequalities arising from Rogosinski's lemma. Journal of Classical Analysis **12** (2018) 93-97. https://doi.org/10.7153/jca-2018-12-08
10. P.R. Mercer: An improved Schwarz Lemma at the boundary. Open Mathematics **16** (2018) 1140-1144. https://doi.org/10.1515/math-2018-0096
11. S.S. Miller & P.T. Mocanu: Second-order differential inequalities in the complex plane. J. Math. Anal. Appl. **65**(1978), 289-305. https://doi.org/10.10160022-247X(78)90181-6
12. R. Osserman: A sharp Schwarz inequality on the boundary. Proc. Amer. Math. Soc. **128** (2000) 3513-3517. https://doi.org/10.48550/arXiv.math/9712280
13. B.N. Örnek: On a class of analytic function related to Schwarz lemma. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **29** (2022), no. 1, 113-124. https://doi.org/10.7468/jksmeb.2022.29.1.113
14. B.N. Örnek & T. Düzenli: Boundary Analysis for the Derivative of Driving Point Impedance Functions. IEEE Transactions on Circuits and Systems II: Express Briefs **65** (2018), no. 9, 1149-1153. https://doi.org/10.1109/TCSII.2018.2809539
15. Ch. Pommerenke: Boundary Behaviour of Conformal Maps. Springer-Verlag, Berlin. 1992.
16. H. Unkelbach: Über die Randverzerrung bei konformer Abbildung. Math. Z. **43** (1938), 739-742. https://doi.org/10.1007/BF01181115

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