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APPLICATION OF CONTRACTION MAPPING PRINCIPLE IN PERIODIC BOUNDARY VALUE PROBLEMS

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Abstract. We prove some common fixed point theorems for β-non-decreasing mappings under contraction mapping principle on partially ordered metric spaces. We study the existence of solution for periodic boundary value problems and also give an example to show the degree of validity of our hypothesis. Our results improve and generalize various known results.

1. Introduction

Fixed point theorems in metric spaces play a major role for solving problems in applied mathematics and science. The Banach contraction mapping principle is a classical and powerful tool in nonlinear analysis, it guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces.

Ran and Reurings [24] extended the Banach contraction principle in partially ordered sets with applications, while Nieto and López [23] extended the results of Ran and Reurings [24] and applied it in periodic boundary value problems. Some of our basic references are [1, $7-11$, $13-16$, $26-28$].

In this paper, we establish a unique common fixed point theorem for β-non-decreasing mappings under contraction mapping principle on partially ordered metric spaces. With the help of the obtain results, we indicate the formation of coupled fixed point results. We apply our result to obtain the solution for periodic boundary value problems and also give an example to show the degree of validity of our hypothesis. We modify, improve, sharpen, enrich and generalize the results of Alotaibi and Alsulami [2], Alsulami [3], Gnana-Bhaskar and Lakshmikantham [4], Harjani et

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al. [17], Harjani and Sadarangani [18], Lakshmikantham and Ciric [21], Luong and Thuan [22], Nieto and Rodriguez-Lopez [23], Ran and Reurings [24], Razani and Parvaneh [25], Su [28] and many other famous results in the existing literature.

2. Fixed Point Results

Throughout the paper, we denote by X a non-empty set and $\preceq$ will represent a partial order on X. Given $n \in \mathbb{N}$ with $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). For simplicity, if $x \in X$, we denote $\beta(x)$ by βx.

Definition 2.1 ([4, 11])**.** A partially ordered metric space $(X, d, \preceq)$ is a metric space (X, d) provided with a partial order $\preceq$. A partially ordered metric space $(X, d, \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X, d, \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

We say that α is $(\beta, \preceq)$-*non-decreasing* if $\alpha x \preceq \alpha y$ for all $x, y \in X$ such that $\beta x \preceq \beta y$. If β is the identity mapping on X, we say that α is $\preceq$-*non-decreasing*.

Definition 2.2 ([6])**.** Two self-mappings α and β of a non-empty set X are said to be *commutative* if $\alpha\beta x = \beta\alpha x$ for all $x \in X$.

Definition 2.3 ([19])**.** Let $(X, d, \preceq)$ be a partially ordered metric space. Two mappings $\alpha, \beta : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} d(\alpha\beta x_n,\ \beta\alpha x_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} \alpha x_n = \lim_{n\to\infty} \beta x_n \in X.$$

Definition 2.4 ([20])**.** Two self-mappings α and β of a non-empty set X are said to be *weakly compatible* if they commute at their coincidence points, that is, if $\alpha x = \beta x$ for some $x \in X$, then $\alpha\beta x = \beta\alpha x$.

Definition 2.5 ([28])**.** An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfied the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and $\alpha, \beta : X \to X$ be two mappings satisfying*

(i) α is $(\beta, \preceq)-$non-decreasing and $\alpha(X) \subseteq \beta(X)$,

(ii) there exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$,

(iii) there exist an altering distance function ψ and a right upper semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\psi(d(\alpha x,\ \alpha y)) \leq \varphi(d(\beta x,\ \beta y)),$$

for all $x, y \in X$ with $\beta x \preceq \beta y$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Also assume that one of the following conditions holds.

(a) (X, d) is complete, α and β are continuous and the pair (α, β) is compatible,

(b) $(\beta(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular,

(c) (X, d) is complete, β is continuous and monotone non-decreasing, the pair (α, β) is compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then α and β have a coincidence point. Furthermore,

(iv) for each $x, y \in X$ there exists $u \in X$ such that αu is comparable to αx and αy and also the pair (α, β) is weakly compatible.

Then α and β have a unique common fixed point.

Proof. Since $x_0 \in X$ and by (i), we have $\alpha x_0 \in \alpha(X) \subseteq \beta(X)$, there exists $x_1 \in X$ such that $\alpha x_0 = \beta x_1$. Then, by (ii), we have $\beta x_0 \preceq \alpha x_0 = \beta x_1$. Since α is $(\beta, \preceq)$-non-decreasing, $\alpha x_0 \preceq \alpha x_1$. Now $\alpha x_1 \in \alpha(X) \subseteq \beta(X)$, so there exists $x_2 \in X$ such that $\alpha x_1 = \beta x_2$. Then $\beta x_1 = \alpha x_0 \preceq \alpha x_1 = \beta x_2$. Since α is $(\beta, \preceq)-$non-decreasing, $\alpha x_1 \preceq \alpha x_2$. Repeating this argument, we get a sequence $\{x_n\}_{n\geq 0}$ such that $\{\beta x_n\}$ is $\preceq$-non-decreasing, $\beta x_{n+1} = \alpha x_n \preceq \alpha x_{n+1} = \beta x_{n+2}$ and

$$\beta x_{n+1} = \alpha x_n \text{ for all } n \geq 0. \tag{2.1}$$

Let $\zeta_n = d(\beta x_n, \beta x_{n+1})$ for all $n \geq 0$. Now, by using contractive condition (iii), we have

$$\psi(d(\beta x_{n+1},\ \beta x_{n+2})) = \psi(d(\alpha x_n,\ \alpha x_{n+1})) \leq \varphi(d(\beta x_n,\ \beta x_{n+1})), \tag{2.2}$$

which, by the fact $\psi(t) > \varphi(t)$ for all $t > 0$, implies that

$$\psi(d(\beta x_{n+1},\ \beta x_{n+2})) < \psi(d(\beta x_n,\ \beta x_{n+1})).$$

It follows, from the monotonicity of ψ, that

$$d(\beta x_{n+1},\ \beta x_{n+2}) < d(\beta x_n,\ \beta x_{n+1}), \text{ that is, } \zeta_{n+1} < \zeta_n.$$

This shows that the sequence $\{\zeta_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = \zeta. \tag{2.3}$$

We claim that $\zeta = 0$. If possible, suppose $\zeta > 0$. Taking $n \to \infty$ in (2.2), by using the property of ψ, φ and (2.3), we obtain

$$\psi(\zeta) = \lim_{n\to\infty} \psi(d(\beta x_{n+1},\ \beta x_{n+2})) \leq \lim_{n\to\infty} \varphi(d(\beta x_n,\ \beta x_{n+1})) \leq \varphi(\zeta),$$

which contradicts the fact that $\psi(t) > \varphi(t)$ for all $t > 0$. Hence, by (2.3), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\beta x_n,\ \beta x_{n+1}) = 0. \tag{2.4}$$

Now we shall show that $\{\beta x_n\}_{n\geq 0}$ is a Cauchy sequence in X. If possible, suppose that $\{\beta x_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ exist such that for all positive integers k, and

$$d(\beta x_{n(k)},\ \beta x_{m(k)}) \geq \varepsilon,\ \text{ for } n(k) > m(k) > k.$$

Suppose $n(k)$ is the smallest such positive integer, then

$$d(\beta x_{n(k)-1},\ \beta x_{m(k)}) < \varepsilon.$$

By using triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq d(\beta x_{n(k)},\ \beta x_{m(k)}) \\ &\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + d(\beta x_{n(k)-1},\ \beta x_{m(k)}) \\ &\leq d(\beta x_{n(k)},\ \beta x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (2.4), we have

$$\lim_{k\to\infty} d(\beta x_{n(k)},\ \beta x_{m(k)}) = \varepsilon. \tag{2.5}$$

Again, by using triangle inequality, we have

$$\begin{aligned} & d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) \\ \leq\ & d(\beta x_{n(k)+1},\ \beta x_{n(k)}) + d(\beta x_{n(k)},\ \beta x_{m(k)}) + d(\beta x_{m(k)},\ \beta x_{m(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.4) and (2.5), we have

$$\lim_{k\to\infty} d(\beta x_{n(k)+1},\ \beta x_{m(k)+1}) = \varepsilon. \tag{2.6}$$

As $n(k) > m(k)$, so $\beta x_{n(k)} \succeq \beta x_{m(k)}$, and hence by using contractive condition (iii), we have

$$\psi(d(\beta x_{n(k)+1},\ \beta x_{m(k)+1})) = \psi(d(\alpha x_{n(k)},\ \alpha x_{m(k)})) \leq \varphi(d(\beta x_{n(k)},\ \beta x_{m(k)})).$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, φ and (2.5), (2.6), we have

$$\psi(\varepsilon) \leq \varphi(\varepsilon),$$

which contradicts the fact that $\varepsilon > 0$. Consequently $\{\beta x_n\}_{n\geq 0}$ is a Cauchy sequence in X.

Suppose now that (a) holds, that is, $(X,\, d)$ is complete, α and β are continuous and the pair $(\alpha,\, \beta)$ is compatible. Since $(X,\, d)$ is complete, therefore there exists $z \in X$ such that $\{\beta x_n\} \to z$. It follows, from (2.1), that $\{\alpha x_n\} \to z$. As α and β are continuous, then $\{\alpha\beta x_n\} \to \alpha z$ and $\{\beta\beta x_n\} \to \beta z$. Since the pair $(\alpha,\, \beta)$ is compatible, we conclude that

$$d(\beta z,\ \alpha z) = \lim_{n\to\infty} d(\beta\beta x_{n+1},\ \alpha\beta x_n) = \lim_{n\to\infty} d(\beta\alpha x_n,\ \alpha\beta x_n) = 0,$$

that is, z is a coincidence point of α and β.

Suppose now that (b) holds, that is, $(\beta(X),\, d)$ is complete and $(X,\, d,\, \preceq)$ is non-decreasing-regular. Since $\{\beta x_n\}$ is a Cauchy sequence in the complete space $(\beta(X),\, d)$, therefore there exist $y \in \beta(X)$ such that $\{\beta x_n\} \to y$. Let $z \in X$ be any point such that $y = \beta z$, then in such case $\{\beta x_n\} \to \beta z$. Also, as $(X,\, d,\, \preceq)$ is non-decreasing-regular, $\{\beta x_n\}$ is $\preceq$-non-decreasing and converging to βz, so we get $\beta x_n \preceq \beta z$ for all $n \geq 0$. Applying the contractive condition (iii), we have

$$\psi(d(\beta x_{n+1},\ \alpha z)) = \psi(d(\alpha x_n,\ \alpha z)) \leq \varphi(d(\beta x_n,\ \beta z)).$$

Taking $n \to \infty$ in the above inequality, by using (ii_ψ) and the fact that $\{\beta x_n\} \to \beta z$, we get $d(\beta z,\, \alpha z) = 0$, that is, z is a coincidence point of α and β.

Suppose now that (c) holds, that is, $(X,\, d)$ is complete, β is continuous and monotone non-decreasing, the pair $(\alpha,\, \beta)$ is compatible and $(X,\, d,\, \preceq)$ is non-decreasing-regular. As $(X,\, d)$ is complete, so there exists $z \in X$ such that $\{\beta x_n\} \to z$. It follows, from (2.1), that $\{\alpha x_n\} \to z$. Since β is continuous, therefore $\{\beta\beta x_n\} \to \beta z$. Furthermore, as the pair $(\alpha,\, \beta)$ is compatible, so we have

$$\lim_{n\to\infty} d(\beta\beta x_{n+1},\ \alpha\beta x_n) = \lim_{n\to\infty} d(\beta\alpha x_n,\ \alpha\beta x_n) = 0$$

and $\{\beta\beta x_n\} \to \beta z$. These facts together imply that $\{\alpha\beta x_n\} \to \beta z$.

Since $(X,\ d,\ \preceq)$ is non-decreasing-regular and $\{\beta x_n\}$ is $\preceq$-non-decreasing and converging to z, we obtain $\beta x_n \preceq z$ for all $n \geq 0$, which, by the monotonicity of β, implies $\beta\beta x_n \preceq \beta z$. Applying the contractive condition (iii), we get

$$\psi(d(\alpha\beta x_n,\ \alpha z)) \leq \varphi(d(\beta\beta x_n,\ \beta z)).$$

Taking $n \to \infty$ in the above inequality, by using (ii_ψ), the fact that $\{\alpha\beta x_n\} \to \beta z$ and $\{\beta\beta x_n\} \to \beta z$, we get $d(\beta z,\ \alpha z) = 0$, that is, z is a coincidence point of α and β.

Since the set of coincidence points of α and β is non-empty, suppose x and y are coincidence points of α and β, that is, $\alpha x = \beta x$ and $\alpha y = \beta y$. Now, we shall show that $\beta x = \beta y$. By the assumption, there exists $u \in X$ such that αu is comparable with αx and αy. Put $u_0 = u$ and choose $u_1 \in X$ so that $\beta u_0 = \alpha u_1$. Then, we can inductively define sequences $\{\beta u_n\}$ where $\beta u_{n+1} = \alpha u_n$ for all $n \geq 0$. Hence $\alpha x = \beta x$ and $\alpha u = \alpha u_0 = \beta u_1$ are comparable. Suppose that $\beta u_1 \preceq \beta x$. We claim that $\beta u_n \preceq \beta x$ for each $n \in \mathbb{N}$. For this, we shall use mathematical induction. As $\beta u_1 \preceq \beta x$ and so our claim is true for $n = 1$.

Now, suppose that $\beta u_n \preceq \beta x$ holds for some $n > 1$. Since α is β-nondecreasing with respect to $\preceq$, we get $\beta u_{n+1} = \alpha u_n \preceq \alpha x = \beta x$. Thus our claim is proved.

Let $\xi_n = d(\beta u_n,\ \beta x)$ for all $n \geq 0$. Since $\beta u_n \preceq \beta x$, by using the contractive condition (iii), we have

$$\psi(d(\beta u_{n+1},\ \beta x)) = \psi(d(\alpha u_n,\ \alpha x)) \leq \varphi(d(\beta u_n,\ \beta x)), \tag{2.7}$$

which, by the fact that $\psi(t) > \varphi(t)$ for all $t > 0$, implies

$$\psi(d(\beta u_{n+1},\ \beta x)) < \psi(d(\beta u_n,\ \beta x)).$$

It follows, from the monotonicity of ψ, that

$$d(\beta u_{n+1},\ \beta x) < d(\beta u_n,\ \beta x),\ \text{that is, } \xi_{n+1} < \xi_n.$$

This shows that the sequence $\{\xi_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers. Then there exists $\xi \geq 0$ such that

$$\xi_n = \lim_{n\to\infty} d(\beta u_n,\ \beta x) = \xi. \tag{2.8}$$

We claim that $\xi = 0$. If possible, suppose $\xi > 0$. Taking $n \to \infty$ in (2.7), by using the property of ψ, φ and (2.8), we obtain

$$\psi(\xi) = \lim_{n\to\infty} \psi(d(\beta u_{n+1},\ \beta x)) \leq \lim_{n\to\infty} \varphi(d(\beta u_n,\ \beta x)) \leq \varphi(\xi),$$

which contradicts the fact that $\xi > 0$. Thus, by (2.8), we get

$$\lim_{n\to\infty} d(\beta u_n,\ \beta x) = 0. \tag{2.9}$$

Similarly, one can prove that

$$\lim_{n\to\infty} d(\beta u_n,\ \beta y) = 0. \tag{2.10}$$

Hence, by (2.9) and (2.10), we get

$$\beta x = \beta y. \tag{2.11}$$

Since $\alpha x = \beta x$, by weak compatibility of α and β, we have $\alpha\beta x = \beta\alpha x = \beta\beta x$. Let $z = \beta x$, then $\alpha z = \beta z$, that is, z is a coincidence point of α and β. Then from (2.11) with $y = z$, it follows that $\beta x = \beta z$, that is, $z = \beta z = \alpha z$. Hence z is a common fixed point of α and β. To prove the uniqueness, assume that w is another common fixed point of α and β. Then by (2.11) we have $w = \beta w = \beta z = z$. Hence the common fixed point of α and β is unique. □

If we take $\beta = I$ (the identity mapping) in Theorem 2.1, we get the following result:

Corollary 2.2. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space and $\alpha : X \to X$ be a mapping satisfying*

(i) α is $\preceq$ −non-decreasing,

(ii) there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$,

(iii) there exist an altering distance function ψ and a right upper semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that

$$\psi(d(\alpha x,\ \alpha y)) \leq \varphi(d(x,\ y)),$$

for all $x,\ y \in X$ with $x \preceq y$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Then α has a fixed point.

If we take $\psi(t) = t$ and $\varphi(t) = kt$ with $k < 1$ for all $t \geq 0$ in Theorem 2.1, we get the following result:

Corollary 2.3. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $\alpha,\ \beta : X \to X$ be two mappings satisfying (i) and (ii) of Theorem 2.1 and*

(iii) there exists $k < 1$ such that

$$d(\alpha x,\ \alpha y) \leq kd(\beta x,\ \beta y),$$

for all $x,\ y \in X$ with $\beta x \preceq \beta y$. Also assume that one of the conditions $(a) - (c)$ of Theorem 2.1 holds. Then α and β have a coincidence point. Furthermore, if condition (iv) of Theorem 2.1 holds. Then α and β have a unique common fixed point.

If we take $\beta = I$ (the identity mapping) in Corollary 2.3, we get the following result:

Corollary 2.4. *Let $(X,\ d,\ \preceq)$ be a partially ordered complete metric space and $\alpha : X \to X$ be a mapping satisfying (i) and (ii) of Corollary 2.2 and*

(iii) there exists $k < 1$ such that

$$d(\alpha x,\ \alpha y) \leq kd(x,\ y),$$

for all $x,\ y \in X$ with $x \preceq y$. Then α has a fixed point.

Example 2.1. Let $X = \mathbb{R}$ be a metric space with the usual metric $d : X^2 \to [0, +\infty)$ equipped with the natural ordering of real numbers $\leq$. Let $\alpha,\ \beta : X \to X$ be defined as

$$\alpha x = \ln(1 + x^2) \text{ and } \beta x = x^2, \text{ for all } x \in X.$$

Define

$$\psi(t) = t \text{ and } \varphi(t) = \ln(1 + t), \text{ for } t \geq 0$$

Now

$$\begin{aligned}
\psi(d(\alpha x,\ \alpha y)) &= d(\alpha x,\ \alpha y) \\
&= |\alpha x - \alpha y| \\
&= \left|\ln(1 + x^2) - \ln(1 + y^2)\right| \\
&= \left|\ln\left(\frac{1 + x^2}{1 + y^2}\right)\right| \\
&= \left|\ln\left(1 + \frac{x^2 - y^2}{1 + y^2}\right)\right| \\
&\leq \ln\left(1 + \left|x^2 - y^2\right|\right) \\
&\leq \ln\left(1 + |\beta x - \beta y|\right) \\
&\leq \ln\left(1 + d(\beta x,\ \beta y)\right) \\
&\leq \varphi(d(\beta x,\ \beta y)).
\end{aligned}$$

Thus the contractive condition of Theorem 2.1 is satisfied for all $x, y \in X$. Furthermore, all the other conditions of Theorem 2.1 are satisfied and $z = 0$ is a unique common fixed point of α and β.

3. Coupled Fixed Point Results

Consider the partially ordered metric space $(X^2, \delta, \sqsubseteq)$, where $\delta : X^2 \times X^2 \to [0, +\infty)$ defined by

$$\delta(V,\ W) = \max\{d(x,\ u),\ d(y,\ v)\},\ \forall V = (x,\ y) \text{ and } W = (u,\ v) \in X^2,$$

and $\sqsubseteq$ was introduced as follows

$$W \sqsubseteq V \Leftrightarrow x \succeq u \text{ and } y \preceq v,\ \text{ for all } W = (u,\ v) \text{ and } V = (x,\ y) \in X^2.$$

Let $f : X^2 \to X$ and $g : X \to X$ be two mappings, then we define mappings Φ, $\Theta : X^2 \to X^2$, for all $V = (x, y) \in X^2$, by

$$\Phi(V) = (f(x,\ y),\ f(y,\ x)) \text{ and } \Theta(V) = (gx,\ gy).$$

Definition 3.1 ([12])**.** Let $f : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a *coupled fixed point* of f if

$$f(x,\ y) = x \text{ and } f(y,\ x) = y.$$

Definition 3.2 ([4])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $f : X^2 \to X$ be a given mapping. We say that f has the *mixed monotone property* if for all x, $y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ x_1 \preceq x_2 \implies f(x_1,\ y) \preceq f(x_2,\ y), \\ y_1,\ y_2 &\in X,\ y_1 \preceq y_2 \implies f(x,\ y_1) \succeq f(x,\ y_2). \end{aligned}$$

Definition 3.3 ([21])**.** Let $f : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of the mappings f and g if

$$f(x,\ y) = gx \text{ and } f(y,\ x) = gy.$$

Definition 3.4 ([21])**.** Let $f : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *common coupled fixed point* of the mappings f and g if

$$x = f(x,\ y) = gx \text{ and } y = f(y,\ x) = gy.$$

Definition 3.5 ([21]). Mappings $f : X^2 \to X$ and $g : X \to X$ are said to be *commutative* if

$$gf(x,\ y) = f(gx,\ gy),\ \text{for all } (x,\ y) \in X^2.$$

Definition 3.6 ([21]). Let $(X,\ \preceq)$ be a partially ordered set. Suppose $f : X^2 \to X$ and $g : X \to X$ are given mappings. We say that f has the *mixed* $g-$ *monotone property* if for all $x,\ y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ gx_1 \preceq gx_2 \implies f(x_1,\ y) \preceq f(x_2,\ y), \\ y_1,\ y_2 &\in X,\ gy_1 \preceq gy_2 \implies f(x,\ y_1) \succeq f(x,\ y_2). \end{aligned}$$

If g is the identity mapping on X, then f satisfies the mixed monotone property.

Definition 3.7 ([5]). Mappings $f : X^2 \to X$ and $g : X \to X$ are said to be *compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(gf(x_n,\ y_n),\ f(gx_n,\ gy_n)) &= 0, \\ \lim_{n\to\infty} d(gf(y_n,\ x_n),\ f(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} f(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x \in X, \\ \lim_{n\to\infty} f(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y \in X. \end{aligned}$$

Definition 3.8 ([6]). Let X be a non-empty set. Two mappings $g : X \to X$ and $f : X^2 \to X$ are said to be *weakly compatible* if they commute at their coupled coincidence points, that is, if $f(x,\ y) = gx$ and $f(y,\ x) = gy$ for some $(x,\ y) \in X^2$, then $gf(x,\ y) = f(gx,\ gy)$ and $gf(y,\ x) = f(gy,\ gx)$.

Lemma 3.1 ([7, 15, 16]). *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space, $f : X^2 \to X$, $g : X \to X$ and $\Phi,\ \Theta : X^2 \to X^2$ be mappings. Then*

(1) $(X,\ d)$ is complete if and only if $(X^2,\ \delta)$ is complete.

(2) If $(X,\ d,\ \preceq)$ is regular, then $(X^2,\ \delta,\ \sqsubseteq)$ is also regular.

(3) If f is d-continuous, then Φ is δ-continuous.

(4) f has the mixed monotone property with respect to $\preceq$ if and only if Φ is $\sqsubseteq$ $-$non-decreasing.

(5) f has the mixed $g-$monotone property with respect to $\preceq$ if and only if then Φ is $(\Theta,\ \sqsubseteq)$-non-decreasing.

(6) If there exist two elements x_0, $y_0 \in X$ with $gx_0 \preceq f(x_0, y_0)$ and $gy_0 \succeq f(y_0, x_0)$, then there exists a point $V_0 = (x_0, y_0) \in X^2$ such that $\Theta(V_0) \sqsubseteq \Phi(V_0)$.

(7) If $f(X^2) \subseteq g(X)$, then $\Phi(X^2) \subseteq \Theta(X^2)$.

(8) If f and g are commuting in $(X, d, \preceq)$, then Φ and Θ are also commuting in $(X^2, \delta, \sqsubseteq)$.

(9) If f and g are compatible in $(X, d, \preceq)$, then Φ and Θ are also compatible in $(X^2, \delta, \sqsubseteq)$.

(10) If f and g are weak compatible in $(X, d, \preceq)$, then Φ and Θ are also weak compatible in $(X^2, \delta, \sqsubseteq)$.

(11) A point $(x, y) \in X^2$ is a coupled coincidence point of f and g if and only if it is a coincidence point of Φ and Θ.

(12) A point $(x, y) \in X^2$ is a coupled fixed point of f if and only if it is a fixed point of Φ.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $f : X^2 \to X$ and $g : X \to X$ are two mappings such that f has the mixed $g-$monotone property with respect to $\preceq$ on X and there exist an altering distance function ψ and a right upper semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(f(x,\ y),\ f(u,\ v))) \leq \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}), \tag{3.1}$$

for all x, y, u, $v \in X$, with $gx \preceq gu$ and $gy \succeq gv$. Suppose that $f(X^2) \subseteq g(X)$, g is continuous and monotone non-decreasing and the pair $\{f, g\}$ is compatible. Also suppose that either

(a) f is continuous or

(b) $(X, d, \preceq)$ is regular.

Assume that there exist two elements x_0, $y_0 \in X$ with

$$gx_0 \preceq f(x_0,\ y_0) \text{ and } gy_0 \succeq f(y_0,\ x_0).$$

Then f and g have a coupled coincidence point. Moreover, if for every (x, y), $(x^*, y^*) \in X^2$, there exists a point $(u, v) \in X^2$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(x^*, y^*), f(y^*, x^*))$, and also the pair (f, g) is weakly compatible. Then f and g have a unique common coupled fixed point.

Proof. Let x, y, u, $v \in X$ be such that $gx \preceq gu$ and $gy \succeq gv$. Then by using contractive condition (3.1), we have

$$\psi(d(f(x,\ y),\ f(u,\ v))) \leq \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).$$

Furthermore since $gy \succeq gv$ and $gx \preceq gu$, the contractive condition (3.1) also ensure that

$$\psi(d(f(y,\ x),\ f(v,\ u))) \leq \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}).$$

Combining them, we get

$$\begin{aligned} & \max\{\psi(d(f(x,\ y),\ f(u,\ v))),\ \psi(d(f(y,\ x),\ f(v,\ u)))\} \\ \leq\ & \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \end{aligned}$$

It follows, from the monotonicity of ψ, that

$$\begin{aligned} & \psi(\max\{d(f(x,\ y),\ f(u,\ v)),\ d(f(y,\ x),\ f(v,\ u))\}) \\ \leq\ & \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}). \end{aligned} \tag{3.2}$$

Thus by using (3.2), for each $V = (x,\ y)$ and $W = (u,\ v) \in X^2$, we have

$$\begin{aligned} & \psi(\delta(\Phi(V),\ \Phi(W))) \\ =\ & \psi(\max\{d(f(x,\ y),\ f(u,\ v)),\ d(f(y,\ x),\ f(v,\ u))\}) \\ \leq\ & \varphi(\max\{d(gx,\ gu),\ d(gy,\ gv)\}) \\ \leq\ & \varphi(\delta(\Theta(V),\ \Theta(W))). \end{aligned}$$

It is only require to use Theorem 2.1 to the mappings $\alpha = \Phi$ and $\beta = \Theta$ in the partially ordered metric space $(X^2,\ \delta,\ \sqsubseteq)$ with the help of Lemma 3.1. $\square$

Corollary 3.2. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $f : X^2 \to X$ has mixed monotone property with respect to $\preceq$ on X and there exist an altering distance function ψ and a right upper semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\psi(d(f(x,\ y),\ f(u,\ v))) \leq \varphi(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all $x,\ y,\ u,\ v \in X$, with $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) f is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Assume that there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq f(x_0,\ y_0) \text{ and } y_0 \succeq f(y_0,\ x_0).$$

Then f has a coupled fixed point.

In a similar manner, we may state the results analogous to Corollary 2.3 for Theorem 3.1 and Corollary 3.2.

4. Application to Ordinary Differential Equations

In this section, first we obtain the solution for the following first-order periodic problem:

$$u'(t) = f(t,\ u(t)),\ t \in [0,\ T] \text{ and } u(0) = u(T), \tag{4.1}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $X = C(I, \mathbb{R})$ $(I = [0, T])$ of all continuous functions from I to $\mathbb{R}$. It is obvious that X is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{t \in I} |x(t) - y(t)|\,,\ \text{for all } x,\ y \in X.$$

Also X can be equipped with a partial order given by, for all x, $y \in X$,

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),\ \text{for all } t \in I.$$

Definition 4.1. A lower solution for (4.1) is a function $x \in C^1(I, \mathbb{R})$ such that

$$x'(t) \leq f(t,\ x(t)) \text{ for } t \in I \text{ and } x(0) = x(T) = 0.$$

Theorem 4.1. *Consider problem* (4.1) *with* $f : I \times \mathbb{R} \to \mathbb{R}$ *continuous and for* x, $y \in X$ *with* $x \succeq y$,

$$0 \leq f(t,\ x) + \lambda x - f(t,\ y) - \lambda y \leq \frac{\lambda}{2}(x - y).$$

Then the existence of a lower solution of (4.1) *provides the existence of a solution of* (4.1).

Proof. Problem (4.1) is equivalent to the following integral equation

$$u(t) = \int_0^T G(t,\ s)[f(s,\ u(s)) + \lambda u(s)]ds,$$

where $G(t, s)$ is the Green function given by

$$G(t,\ s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1},\ 0 \leq s < t \leq T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1},\ 0 \leq t < s \leq T. \end{cases}$$

Define the mapping $\alpha : X \to X$ by

$$\alpha(x)(t) = \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s)]ds.$$

Suppose that $x_1 \succeq x_2$, then by using our assumption, we can obtain $f(t, x_1)+\lambda x_1 \geq f(t, x_2) + \lambda x_2$. As $G(t, s) > 0$, for $t \in I$, it follows that

$$\begin{aligned}\alpha(x_1)(t) &= \int_0^T G(t,\ s)[f(s,\ x_1(s)) + \lambda x_1(s)]ds \\ &\geq \int_0^T G(t,\ s)[f(s,\ x_2(s)) + \lambda x_2(s)]ds = \alpha(x_2)(t).\end{aligned}$$

Thus α is $\preceq$ −non-decreasing. Now, for each $x \succeq y$, we have

$$\begin{aligned}d(\alpha(x),\ \alpha(y)) &= \sup_{t\in I} |\alpha(x)(t) - \alpha(y)(t)| \\ &= \sup_{t\in I} \left| \int_0^T G(t,\ s)[f(s,\ x(s)) + \lambda x(s) - f(s,\ y(s)) - \lambda y(s)]ds \right| \\ &\leq \sup_{t\in I} \left| \int_0^T G(t,\ s) \cdot \frac{\lambda}{2}(x(s) - y(s))ds \right| \\ &\leq \frac{\lambda}{2} d(x,\ y) \sup_{t\in I} \left| \int_0^T G(t,\ s)ds \right| \\ &\leq \frac{\lambda}{2} d(x,\ y) \sup_{t\in I} \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right| \\ &\leq \frac{1}{2} d(x,\ y).\end{aligned}$$

Thus the contractive condition of Corollary 2.4 is satisfied with $k = 1/2 < 1$. Finally, let $x \in X$ be a lower solution of (4.1), then we have

$$x'(s) + \lambda x(s) \leq f(s,\ x(s)) + \lambda x(s),\ \text{for all } t \in I.$$

Multiplying by $G(t, s)$, we get

$$\int_0^T x'(s)G(t,\ s)ds + \lambda \int_0^T x(s)G(t,\ s)ds \leq \alpha(x)(t),\ \text{for all } t \in I.$$

Then, for all $t \in I$, we have

$$\int_0^t x'(s)\frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T x'(s)\frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds + \lambda \int_0^T x(s)G(t,\ s)ds \leq \alpha(x)(t).$$

Using integration by parts and since $x(0) = x(T) = 0$, for all $t \in I$, we get $x(t) \leq \alpha(x)(t)$. This implies that $x \preceq \alpha(x)$. Thus all the hypothesis of Corollary 2.4 are satisfied. Consequently, α has a fixed point $x \in X$ which is the solution to (4.1) in $X = C(I, \mathbb{R})$. $\square$

Next we obtain the solution of the following two-point boundary value problem.

$$\begin{cases} -x''(t) = f(t,\ x(t),\ x(t)),\ t \in [0,\ 1], \\ \quad x(0) = x(1) = 0, \end{cases} \tag{4.2}$$

where $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The space $X = C(I,\ \mathbb{R})$ ($I = [0,\ 1]$) denote the set of all continuous functions from I to $\mathbb{R}$. Obviously X is a complete metric space with respect to the sup metric and equipped with a partial order given by, for all $x,\ y \in X$,

$$x \preceq y \Longleftrightarrow x(t) \le y(t),\ \text{for all } t \in I.$$

Theorem 4.2. *Assume that the following hold:*

(a) $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(b) *Suppose that there exists* $0 \le \gamma \le 8$ *such that for all* $t \in I$, $x \succeq u$ *and* $y \preceq v$,

$$0 \le f(t,\ x,\ y) - f(t,\ u,\ v) \le \gamma\sqrt{\frac{g(x-u) + g(y-v)}{2}},$$

where $g(t) : [0,\ +\infty) \to [0,\ +\infty)$ *is a right upper semi-continuous and non-decreasing function with* $g(0) = 0$, $g(t) < t^2$, *for all* $t > 0$.

(c) *There exists* $(\alpha,\ \beta) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ *such that*

$$\begin{cases} -\alpha''(t) \le f(t,\ \alpha(t),\ \beta(t)),\ t \in [0,\ 1], \\ -\beta''(t) \ge f(t,\ \beta(t),\ \alpha(t)),\ t \in [0,\ 1], \\ \quad \alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0. \end{cases}$$

Then problem (4.2) *has a solution in* $C^2(I,\ \mathbb{R})$.

Proof. It is well known that the solution (in $C^2(I,\ \mathbb{R})$) of problem (4.2) is equivalent to the solution (in $C(I,\ \mathbb{R})$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ x(s))ds \text{ for } t \in [0,\ 1],$$

where $G(t,\ s)$ is the Green function of differential operator $-\frac{d^2}{dt^2}$ with Dirichlet boundary condition $x(0) = x(1) = 0$, that is,

$$G(t,\ s) = \begin{cases} t(1-s),\ 0 \le t \le s \le 1, \\ s(1-t),\ 0 \le s \le t \le 1. \end{cases}$$

Define the mapping $F : X^2 \to X$ by

$$F(x,\ y)(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ y(s))ds,\ t \in [0,\ 1] \text{ and } x,\ y \in X.$$

It follows from (b) that, F has the mixed monotone property with respect to the partial order $\preceq$ in X. Let $x,\ y,\ u,\ v \in X$ such that $x \succeq u$ and $y \preceq v$. From (b), we have

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &= \sup_{t \in I} |F(x,\ y)(t) - F(u,\ v)(t)| \\ &= \sup_{t \in I} \int_0^1 G(t,\ s)[f(s,\ x(s),\ y(s)) - f(s,\ u(s),\ v(s))]ds \\ &\leq \gamma \sup_{t \in I} \int_0^1 G(t,\ s) \cdot \sqrt{\frac{g(x(s) - u(s)) + g(y(s) - v(s))}{2}} ds \\ &\leq \gamma \sqrt{\frac{g(d(x,\ u)) + g(d(y,\ v))}{2}} \sup_{t \in I} \int_0^1 G(t,\ s)ds. \end{aligned}$$

Now, since g is non-decreasing, we have

$$\begin{aligned} g(d(x,\ u)) &\leq g(\max\{d(x,\ u),\ d(y,\ v)\}), \\ g(d(y,\ v)) &\leq g(\max\{d(x,\ u),\ d(y,\ v)\}), \end{aligned}$$

which implies

$$\frac{g(d(x,\ u)) + g(d(y,\ v))}{2} \leq g(\max\{d(x,\ u),\ d(y,\ v)\}).$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \gamma \sqrt{g(\max\{d(x,\ u),\ d(y,\ v)\})} \sup_{t \in I} \int_0^1 G(t,\ s)ds. \tag{4.3}$$

It is easy to obtain that

$$\int_0^1 G(t,\ s)ds = -\frac{t^2}{2} + \frac{t}{2} \text{ and } \sup_{t \in [0,\ 1]} \int_0^1 G(t,\ s)ds = \frac{1}{8}.$$

These facts, (4.3) and the hypothesis $0 < \gamma \leq 8$ give us

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &\leq \frac{\gamma}{8} \sqrt{g(\max\{d(x,\ u),\ d(y,\ v)\})} \\ &\leq \sqrt{g(\max\{d(x,\ u),\ d(y,\ v)\})}. \end{aligned}$$

Hence

$$d(F(x,\ y),\ F(u,\ v))^2 \leq g(\max\{d(x,\ u),\ d(y,\ v)\}).$$

Put $\psi(t) = t^2$ and $\varphi(t) = g(t)$. Obviously, ψ is an altering distance function, $\psi(t)$ and $\varphi(t)$ satisfy the condition of $\psi(t) > \varphi(t)$ for $t > 0$. From the last inequality, we have

$$\psi(d(F(x,\ y),\ F(u,\ v))) \leq \varphi(\max\{d(x,\ u),\ d(y,\ v)\}).$$

Now, let $(\alpha,\, \beta) \in C^2(I,\, \mathbb{R}) \times C^2(I,\, \mathbb{R})$ satisfying (c), then

$$-\alpha''(s) \leq f(s,\ \alpha(s),\ \beta(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t,\, s)$, we get

$$\int_0^1 -\alpha''(s)G(t,\ s)ds \leq F(\alpha,\ \beta)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0,\, 1]$, we have

$$-(1-t)\int_0^t s\alpha''(s)ds - t\int_t^1 (1-s)\alpha''(s)ds \leq F(\alpha,\ \beta)(t).$$

Using integration by parts and since $\alpha(0) = \alpha(1) = 0$, for all $t \in [0,\, 1]$, we get

$$-(1-t)(t\alpha'(t) - \alpha(t)) - t(-(1-t)\alpha'(t) - \alpha(t)) \leq F(\alpha,\ \beta)(t).$$

Thus, we have

$$\alpha(t) \preceq F(\alpha,\ \beta)(t),\ \text{for } t \in [0,\ 1].$$

This implies that $\alpha \preceq F(\alpha,\, \beta)$. Similarly, one can prove that $\beta \succeq F(\beta,\, \alpha)$. Thus all the hypothesis of Corollary 3.2 are satisfied. Consequently, F has a coupled fixed point $(x,\, y) \in X^2$ which is the solution to (4.2) in $X = C^2(I,\, \mathbb{R})$. □

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