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PRODUCTS OF COMPOSITION, DIFFERENTIATION AND MULTIPLICATION FROM THE CAUCHY SPACES TO THE ZYGMUND SPACE

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Abstract. In this paper, we study products of composition, multiplication and differentiation acting on the fractional Cauchy spaces and mapping into the Zygmund space. Characterizations are provided for boundedness and compactness of these operators.

1. Introduction

For $\alpha > 0$, the space F_α is defined as the collection of functions f analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$ of the form

$$f(z) = \int_T \frac{1}{(1 - \overline{x}z)^\alpha} \, d\mu(x), \tag{1}$$

where μ varies over complex-valued Borel measures on $T = \{x \in \mathbf{C} : |x| = 1\}$. The space is normed by

$$\|f\|_{F_\alpha} = \inf \|\mu\|,$$

where the infimum extends over all measures μ participating in the integral representation of f. The families F_α have been studied extensively [3].

The Zygmund space Z is the set of functions analytic in U for which

$$\|f\|_Z = |f(0)| + |f'(0)| + \sup_{z \in U} (1 - |z|^2)|f''(z)| < \infty.$$

Let $H(U)$ denote the space of functions analytic in U. Let Φ be an analytic self-map of U. The composition operator C_Φ is defined by $C_\Phi(f) = f \circ \Phi$ for $f \in H(U)$. A function $\Psi \in H(U)$ induces the multiplication operator $M_\Psi(f) = \Psi f$ for $f \in H(U)$. Finally the differentiation operator D is defined by $D(f) = f'$. The operators to be studied here are products of C_Φ, M_Ψ and D, acting on functions $f \in F_\alpha$. Characterizations are provided on the inducing symbols Φ and Ψ so that the associated operators are bounded or compact.

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2. The operators DC_Φ and DM_Ψ

Various test functions will be used throughout the paper. Fix $\alpha > 0$. It is known [3] that $\|1/(1-\overline{w}z)^\alpha\|_{F_\alpha} \le 1$ for all $w \in U$. There is a constant C independent of $w \in U$ such that

$$\|\frac{1-|w|^2}{(1-\overline{w}z)^{\alpha+1}}\|_{F_\alpha} \le C \text{ and } \|\frac{(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+2}}\|_{F_\alpha} \le C$$

for all $w \in U$ [1,2]. Here we follow the convention that C will denote a positive constant, which may vary from one appearance to the next.

Theorem 2.1. *Fix $\alpha > 0$ and let Φ be an analytic self-map of U. Define $DC_\Phi(f) = (f \circ \Phi)'$ for $f \in H(U)$. Then*

$$\begin{aligned} &DC_\Phi : F_\alpha \to Z \text{ is bounded} \\ \Leftrightarrow\ &\Phi' \in Z,\ \Phi\Phi' \in Z,\ \Phi^2\Phi' \in Z \text{ and } \|\Phi\|_\infty < 1 \\ \Leftrightarrow\ &DC_\Phi : F_\alpha \to Z \text{ is compact.} \end{aligned}$$

Proof. First assume that there is a constant C independent of $f \in F_\alpha$ such that $\|(DC_\Phi)(f)\|_Z \le C\|f\|_{F_\alpha}$ for all $f \in F_\alpha$. Since polynomials belong to F_α it is immediate that Φ', $\Phi\Phi'$ and $\Phi^2\Phi' \in Z$. In particular

$$\begin{aligned} &(1-|z|^2)|(\Phi^2(z)\Phi'(z))''| \\ &= (1-|z|^2)|\Phi^2(z)\Phi'''(z) + 6\Phi(z)\Phi'(z)\Phi''(z) + 2(\Phi'(z))^3| \\ &\le \|\Phi^2\Phi'\|_Z \end{aligned}$$

for all $z \in U$ and a triangle inequality argument yields

$$(1-|z|^2)|2(\Phi'(z))^3| \le C + \|\Phi'\|_Z + 6(1-|z|^2)|\Phi'(z)\Phi''(z)| \tag{2}$$

for all $z \in U$. Since

$$(1-|z|^2)|3\Phi'(z)\Phi''(z) + \Phi(z)\Phi'''(z)| \le \|\Phi\Phi'\|_Z$$

for all $z \in U$, a second application of the triangle inequality yields

$$(1-|z|^2)|3\Phi'(z)\Phi''(z)| \le \|\Phi\Phi'\|_Z + \|\Phi'\|_Z \tag{3}$$

for all $z \in U$. Substitution into (2) yields

$$\sup_{z\in U}(1-|z|^2)|\Phi'(z)|^3 < \infty. \tag{4}$$

Let $\beta > 0$. An analytic function g belongs to the Bloch space B^β if $\sup_{z\in U}(1-|z|^2)^\beta|g'(z)| < \infty$. The relation (4) implies that $\Phi \in B^{1/3}$. Furthermore (4) implies that

$$\sup_{|\Phi(z)|\le 1/2} \frac{(1-|z|^2)^{1/3}|\Phi'(z)|}{(1-|\Phi(z)|^2)^{1/3}} < \infty. \tag{5}$$

For $w \in U$, define the function f_w by

$$f_w(z) = \frac{\beta}{(1-\overline{\Phi(w)}z)^{\alpha}} + \frac{\gamma(1-|\Phi(w)|^2)}{(1-\overline{\Phi(w)}z)^{\alpha+1}} + \frac{\lambda(1-|\Phi(w)|^2)^2}{(1-\overline{\Phi(w)}z)^{\alpha+2}}, \tag{6}$$

where $z \in U$ and $\beta = (\alpha+1)(\alpha+2)$, $\gamma = -2\alpha(\alpha+2)$ and $\lambda = \alpha(\alpha+1)$. By the preliminary remarks there is a constant C depending only on α such that $\|f_w\|_{F_\alpha} \le C$ for all $w \in U$. Therefore $\|(DC_\Phi)(f_w)\|_Z \le C$ for all w. For ease of notation let $g_w = (f_w' \circ \Phi)\Phi'$. Then

$$\sup_{z\in U}(1-|z|^2)|g_w''(z)| \le C \tag{7}$$

for all w. By a calculation $g_w'' = (f_w' \circ \Phi)\Phi''' + 3(f_w'' \circ \Phi)\Phi'\Phi'' + (f_w''' \circ \Phi)(\Phi')^3$. The relation (7) and the calculations $f_w'(\Phi(w)) = f_w''(\Phi(w)) = 0$ and $f_w'''(\Phi(w)) = 2\alpha(\alpha+1)(\alpha+2)\overline{\Phi(w)}^3/(1-|\Phi(w)|^2)^{\alpha+3}$ yield

$$\frac{(1-|w|^2)|\Phi'(w)|^3|\Phi(w)|^3}{(1-|\Phi(w)|^2)^{\alpha+3}} \le C$$

for all $w \in U$. It follows that

$$\sup_{1/2<|\Phi(w)|} \frac{(1-|w|^2)^{1/3}|\Phi'(w)|}{(1-|\Phi(w)|^2)^{(\alpha+3)/3}} < \infty. \tag{8}$$

The relations (5) and (8) imply

$$\sup_{z\in U} \frac{(1-|z|^2)^{1/3}|\Phi'(z)|}{(1-|\Phi(z)|^2)^{1/3}} < \infty. \tag{9}$$

Thus $C_\Phi : B^{1/3} \to B^{1/3}$ is bounded [6]. Furthermore (8) implies

$$\lim_{|\Phi(z)|\to 1} \frac{(1-|z|^2)^{1/3}|\Phi'(z)|}{(1-|\Phi(z)|^2)^{1/3}} = 0$$

and as in [6] it follows that $C_\Phi : B^{1/3} \to B^{1/3}$ is compact. A result of J. H. Shapiro [5] now yields $\|\Phi\|_\infty < 1$ and the proof of the first implication of the theorem is complete.

Next assume $\Phi' \in Z$, $\Phi\Phi' \in Z$, $\Phi^2\Phi' \in Z$ and $\|\Phi\|_\infty < 1$. In order to prove that $DC_\Phi : F_\alpha \to Z$ is compact, let (f_n) be a bounded sequence in F_α with $f_n \to 0$ uniformly on compact subsets of U as $n \to \infty$. It is enough to prove that $\|(DC_\Phi)(f_n)\|_Z \to 0$ as $n \to \infty$. For ease of notation, let $g_n = DC_\Phi(f_n)$. It is clear that $|g_n(0)| \to 0$ and $|g_n'(0)| \to 0$ as $n \to \infty$. An argument will be given to prove that $\sup_{z\in U}(1-|z|^2)|g_n''(z)| \to 0$ as $n \to \infty$.

First, since $f_n' \to 0$ uniformly on compact subsets as $n \to \infty$,

$$\sup_{z\in U}(1-|z|^2)|f_n'(\Phi(z))\Phi'''(z)| \le \|\Phi'\|_Z \max_{|w|\le\|\Phi\|_\infty} |f_n'(w)| \to 0 \text{ as } n \to \infty. \tag{10}$$

Since $f_n'' \to 0$ uniformly on compact subsets, the relation (3) yields

$$\sup_{z\in U}(1-|z|^2)|f_n''(\Phi(z))\Phi'(z)\Phi''(z)| \le C \max_{|w|\le\|\Phi\|_\infty} |f_n''(w)| \to 0 \text{ as } n \to \infty. \tag{11}$$

Finally, by relation (4),

$$\sup_{z\in U}(1-|z|^2)|f_n'''(\Phi(z))||\Phi'(z)|^3 \leq C \max_{|w|\leq\|\Phi\|_\infty} |f_n'''(w)| \to 0 \text{ as } n\to\infty. \tag{12}$$

The relations (10), (11) and (12) now yield $\|(DC_\Phi)(f_n)\|_Z \to 0$ as $n\to\infty$, and thus $DC_\Phi : F_\alpha \to Z$ is compact.

The final implication in the proof is immediate. □

The test functions for Theorem 2.3 require the following lemma.

Lemma 2.2. *Fix $\alpha > 0$ and let $w \in U$. There is a constant C depending only on α such that*

$$\left\|\frac{(1-|w|^2)^3}{(1-\overline{w}z)^{\alpha+3}}\right\|_{F_\alpha} \leq C$$

for all $w \in U$.

Proof. Let $w\in U$ and define $g_w(z) = (1-|w|^2)/(1-\overline{w}z)^{(\alpha+3)/3}$. By previous remarks, $g_w \in F_{\alpha/3}$ and there is a constant C with $\|g_w\|_{F_{\alpha/3}} \leq C$ for all w. A known result [4] states that if $f\in F_\alpha$ and $g \in F_\beta$, then $fg \in F_{\alpha+\beta}$ and $\|fg\|_{F_{\alpha+\beta}} \leq \|f\|_{F_\alpha}\|g\|_{F_\beta}$. Thus $g_w^3 \in F_\alpha$ and $\|g_w^3\|_{F_\alpha} \leq C$. □

Theorem 2.3. *Fix $\alpha > 0$ and let $\Psi \in H(U)$. Define $(DM_\Psi)(f) = (\Psi f)'$ for $f\in H(U)$. Then*

$$DM_\Psi : F_\alpha \to Z \textit{ is bounded}$$
$$\Leftrightarrow \Psi \equiv 0 \Leftrightarrow DM_\Psi : F_\alpha \to Z \textit{ is compact.}$$

Proof. First assume $\|(DM_\Psi)(f)\|_Z \leq C\|f\|_{F_\alpha}$ for a constant C independent of f. It will be shown that $\Psi\equiv 0$ and this is enough to complete the proof.

For $w \in U$, define the test function f_w by

$$f_w(z) = \frac{-1}{(1-\overline{w}z)^\alpha} + \frac{3(1-|w|^2)}{(1-\overline{w}z)^{\alpha+1}} - \frac{3(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+2}} + \frac{(1-|w|^2)^3}{(1-\overline{w}z)^{\alpha+3}} \quad (z\in U).$$

Since $\|f_w\|_{F_\alpha} \leq C$ for all $w\in U$ it follows that $\|(DM_\Psi)(f_w)\|_Z = \|(\Psi f_w)'\|_Z \leq C$ for all w. Calculations yield

$$\sup_{z\in U}(1-|z|^2)|f_w(z)\Psi'''(z) + 3f_w'(z)\Psi''(z) + 3f_w''(z)\Psi'(z) + f_w'''(z)\Psi(z)| \leq C$$

for all w. Since $f_w(w) = 0 = f_w'(w) = f_w''(w)$ and $f_w'''(w) = 6\overline{w}^3/(1-|w|^2)^{\alpha+3}$ it follows that

$$\frac{|w|^3|\Psi(w)|}{(1-|w|^2)^{\alpha+2}} \leq C$$

for all $w\in U$. For w with $1/2 < |w|$, $|\Psi(w)| \leq C(1-|w|^2)^{\alpha+2}$. An argument using the Maximum Modulus Theorem now proves that $\Phi \equiv 0$. The proof is complete. □

3. The operators $C_\Phi D$ and $M_\Psi D$

A known result states that $f \in F_\alpha \Leftrightarrow f' \in F_{\alpha+1}$ [3]. In particular, if $f' \in F_{\alpha+1}$, then $\|f\|_{F_\alpha} \le |f(0)| + C\|f'\|_{F_{\alpha+1}}$ for a constant C depending only on α. Based on this fact, information about the operator $C_\Phi D : F_\alpha \to Z$ can be obtained from the following theorem.

Theorem 3.1. *Fix $\alpha > 0$ and let Φ be an analytic self-map of U. Then*

$$\begin{aligned} &C_\Phi : F_\alpha \to Z \text{ is bounded} \\ \Leftrightarrow\ &\Phi \in Z,\ \Phi^2 \in Z \text{ and } \|\Phi\|_\infty < 1 \\ \Leftrightarrow\ &C_\Phi : F_\alpha \to Z \text{ is compact.} \end{aligned}$$

Proof. First assume there exists $C > 0$ such that $\|C_\Phi f\|_Z \le C\|f\|_{F_\alpha}$. For $w \in U$ define

$$f_w(z) = \frac{-1}{\alpha}\frac{1}{(1-\overline{\Phi(w)}z)^\alpha} + \frac{1}{\alpha+1}\frac{1-|\Phi(w)|^2}{(1-\overline{\Phi(w)}z)^{\alpha+1}}\ (z \in U).$$

Since $\|f_w\|_{F_\alpha} \le C$ for all w it follows that $\|f_w \circ \Phi\|_Z \le C$ and therefore

$$\sup_{z\in U}\,(1-|z|^2)|f_w'(\Phi(z))\Phi''(z) + (\Phi'(z))^2 f_w''(\Phi(z))| \le C$$

for all $w \in U$. Since $f_w'(\Phi(w)) = 0$ and $f_w''(\Phi(w)) = \overline{\Phi(w)}^2/(1-|\Phi(w)|^2)^{\alpha+2}$ it follows that

$$\frac{(1-|w|^2)|\Phi'(w)|^2|\Phi(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} \le C.$$

In particular,

$$\sup_{1/2<|\Phi(w)|}\frac{(1-|w|^2)|\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} < \infty. \tag{13}$$

The hypothesis implies that $\Phi \in Z$ and $\Phi^2/2 \in Z$. Therefore

$$\sup_{z\in U}(1-|z|^2)|\Phi''(z)| \le \|\Phi\|_Z \tag{14}$$

and

$$\sup_{z\in U}(1-|z|^2)|\Phi(z)\Phi''(z) + (\Phi'(z))^2| \le \|\Phi^2/2\|_Z.$$

A triangle inequality argument yields

$$(1-|z|^2)|\Phi'(z)|^2 \le \|\Phi\|_Z + \|\Phi^2/2\|_Z \text{ for all } z \in U \tag{15}$$

and thus

$$\sup_{|\Phi(w)|\le 1/2}\frac{(1-|w|^2)|\Phi'(w)|^2}{(1-|\Phi(w)|^2)^{\alpha+2}} < \infty. \tag{16}$$

The equations (13) and (16) yield

$$\sup_{z\in U}\frac{(1-|z|^2)|\Phi'(z)|^2}{(1-|\Phi(z)|^2)^{\alpha+2}} < \infty.$$

By an argument as in Theorem 2.1 and Xiao's result [6], $C_\Phi : B^{1/2} \to B^{1/2}$ is compact. The details are omitted. Shapiro's result [5] yields $\|\Phi\|_\infty < 1$.

To complete the theorem, assume that $\Phi \in Z$, $\Phi^2 \in Z$ and $\|\Phi\|_\infty < 1$. Suppose that (f_n) is a bounded sequence in F_α such that $f_n \to 0$ uniformly on compact subsets. Relations (14) and (15) apply and yield

$$\begin{aligned}(1-|z|^2)|(f_n \circ \Phi)''(z)| &= (1-|z|^2)|f_n'(\Phi(z))\Phi''(z) + (\Phi'(z))^2 f_n''(\Phi(z))| \\ &\leq C(\max_{|w|\leq\|\Phi\|_\infty} |f_n'(w)| + \max_{|w|\leq\|\Phi\|_\infty} |f_n''(w)|).\end{aligned}$$

Since $f_n \to 0$ uniformly on compact subsets, it now follows as in Theorem 2.1 that $\|f_n \circ \Phi\|_Z \to 0$ as $n \to \infty$. Thus $C_\Phi : F_\alpha \to Z$ is compact.

The remaining implication is immediate and the proof is complete. □

Theorem 3.2. *Fix $\alpha > 0$ and let Φ be an analytic self-map of U. Then*

$$\begin{aligned}&C_\Phi D : F_\alpha \to Z \text{ is bounded} \\ &\Leftrightarrow \Phi \in Z,\ \Phi^2 \in Z \text{ and } \|\Phi\|_\infty < 1 \\ &\Leftrightarrow C_\Phi D : F_\alpha \to Z \text{ is compact.}\end{aligned}$$

Proof. Assume that $C_\Phi D : F_\alpha \to Z$ is bounded and let $g \in F_{\alpha+1}$. Define f by

$$f(z) = \int_0^z g(w)\, dw.$$

It follows that $f \in F_\alpha$ and $\|f\|_{F_\alpha} \leq C\|g\|_{F_{\alpha+1}}$ for a constant C depending only on α [3]. Therefore $\|(C_\Phi D)(f)\|_Z \leq C\|f\|_{F_\alpha}$ and

$$\|g \circ \Phi\|_Z = \|f' \circ \Phi\|_Z \leq C\|f\|_{F_\alpha} \leq C\|g\|_{F_{\alpha+1}}$$

and thus $C_\Phi : F_{\alpha+1} \to Z$ is bounded. By Theorem 3.1, $\Phi \in Z, \Phi^2 \in Z$ and $\|\Phi\|_\infty < 1$.

Now assume that $\Phi \in Z$, $\Phi^2 \in Z$ and $\|\Phi\|_\infty < 1$. Let (f_n) be a bounded sequence in F_α with $f_n \to 0$ uniformly on compact subsets as $n \to \infty$. Then $\|f_n'\|_{F_{\alpha+1}} \leq C$ [3] and $f_n' \to 0$ uniformly on compact subsets. By Theorem 3.1, $C_\Phi : F_{\alpha+1} \to Z$ is compact and thus $\|f_n' \circ \Phi\|_Z \to 0$. This shows that $C_\Phi D : F_\alpha \to Z$ is compact. The remaining implication is immediate. □

Theorem 3.3. *Fix $\alpha > 0$ and let $\Psi \in H(U)$. Then*

$$M_\Psi : F_\alpha \to Z \text{ is bounded } \Leftrightarrow \Phi \equiv 0 \Leftrightarrow M_\Psi : F_\alpha \to Z \ \text{ is compact.}$$

Proof. First assume that there is a constant C independent of $f \in F_\alpha$ such that $\|M_\Psi(f)\|_Z \leq C\|f\|_{F_\alpha}$. For $w \in U$, define g_w by

$$g_w(z) = \frac{1}{(1-\overline{w}z)^\alpha} - 2\frac{1-|w|^2}{(1-\overline{w}z)^{\alpha+1}} + \frac{(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+2}} \quad (z \in U).$$

There is a constant C independent of w such that $\|g_w\|_{F_\alpha} \leq C$ and it follows that

$$\sup_{z\in U}(1-|z|^2)|g_w(z)\Psi''(z) + 2g_w'(z)\Psi'(z) + g_w''(z)\Psi(z)| \leq C.$$

Since $g_w(w) = g_w'(w) = 0$ and $g_w''(w) = 2\overline{w}^2/(1-|w|^2)^{\alpha+2}$, it follows as in the proof of Theorem 2.3 that

$$\frac{|\Psi(w)|}{(1-|w|^2)^{\alpha+1}} \leq C \text{ for } w \text{ with } |w| > 1/2.$$

As in the previous argument, $\Psi \equiv 0$.

The remaining implications are immediate and the proof is complete. □

Theorem 3.4. *Fix $\alpha > 0$ and $\Psi \in H(U)$. Then*

$$M_\Psi D : F_\alpha \to Z \textit{ is bounded } \Leftrightarrow \Psi \equiv 0 \Leftrightarrow M_\Psi D : F_\alpha \to Z \textit{ is compact.}$$

Proof. Assume that $M_\Psi D : F_\alpha \to Z$ is bounded. By an argument as in the proof of Theorem 3.2, it follows that $M_\Psi : F_{\alpha+1} \to Z$ is bounded. Therefore $\Psi \equiv 0$ and the proof is complete. □

4. The weighted composition operator $\Psi C_\Phi : F_\alpha \to Z$

Fix $\Psi \in H(U)$ and let Φ be an analytic self-map of U. The weighted composition operator ΨC_Φ is defined by $\Psi C_\Phi(f) = \Psi(f \circ \Phi)$ for $f \in H(U)$.

Fix $z \in U$ and $\alpha > 0$. For $f \in F_\alpha$, there is a constant depending only on α and $n = 0, 1, 2, \ldots$ such that $|f^{(n)}(z)| \leq C\|f\|_{F_\alpha}/(1-|z|^2)^{\alpha+n}$.

Theorem 4.1. *Fix $\alpha > 0$. Let Φ be an analytic self-map of U and let $\Psi \in H(U)$. Then*

$$\Psi C_\Phi : F_\alpha \to Z \textit{ is bounded } \Leftrightarrow$$

$$\sup_{z \in U} \frac{(1-|z|^2)|\Psi''(z)|}{(1-|\Phi(z)|^2)^\alpha} < \infty \textit{ and} \tag{17}$$

$$\sup_{z \in U} \frac{(1-|z|^2)|2\Phi'(z)\Psi'(z) + \Psi(z)\Phi''(z)|}{(1-|\Phi(z)|^2)^{\alpha+1}} < \infty \textit{ and} \tag{18}$$

$$\sup_{z \in U} \frac{(1-|z|^2)|\Psi(z)||\Phi'(z)|^2}{(1-|\Phi(z)|^2)^{\alpha+2}} < \infty. \tag{19}$$

Proof. First assume conditions (17), (18) and (19) and let $f \in F_\alpha$. For convenience denote the operator ΨC_Φ by W. By a calculation

$$(Wf)'' = (f \circ \Phi)\Psi'' + (f' \circ \Phi)\ (2\Phi'\Psi' + \Psi\Phi'') + (f'' \circ \Phi)\Psi(\Phi')^2. \tag{20}$$

The preliminary remarks and (17)-(19) imply

$$|(Wf)(0)| + |(Wf)'(0)| + \sup_{z \in U}(1-|z|^2)|(Wf)''(z)| \leq C\|f\|_{F_\alpha} \tag{21}$$

and thus $\|Wf\|_Z \leq C\|f\|_{F_\alpha}$.

To prove necessity of the conditions, assume that $\|Wf\|_Z \leq C\|f\|_{F_\alpha}$ for a constant C independent of $f \in F_\alpha$. The test functions f_w defined at (6) will be used with various choices for β, γ and λ. Recall $\|f_w\|_{F_\alpha} \le C$ for a constant C independent of w. Thus there is a constant C such that

$$\sup_{z\in U}(1-|z|^2)|(Wf_w)''(z)| < C \text{ for all } w \in U. \tag{22}$$

First let $\beta = (\alpha+1)(\alpha+2)$, $\gamma = -2\alpha(\alpha+2)$ and $\lambda = \alpha(\alpha+1)$. Calculations yield $f_w(\Phi(w)) = 2/(1-|\Phi(w)|^2)^\alpha$, $f_w'(\Phi(w)) = 0$ and $f_w''(\Phi(w)) = 0$. The relations (20) and (22) yield

$$(1-|w|^2)\frac{|\Psi''(w)|}{(1-|\Phi(w)|^2)^\alpha} \le C$$

for $w \in U$. Condition (17) is established.

In order to obtain (18), let $\beta = -2(\alpha+2)$, $\gamma = 4\alpha+6$ and $\lambda = -2(\alpha+1)$. Then $f_w(\Phi(w)) = 0 = f_w''(\Phi(w))$ and $f_w'(\Phi(w)) = 2\overline{\Phi(w)}/(1-|\Phi(w)|^2)^{\alpha+1}$. By relation (22),

$$(1-|w|^2)\frac{|\Phi(w)|}{(1-|\Phi(w)|^2)^{\alpha+1}}|2\Phi'(w)\Psi'(w)+\Psi(w)\Phi''(w)| \le C$$

for $w \in U$. Thus

$$\sup_{|\Phi(w)|>1/2}\frac{(1-|w|^2)|2\Phi'(w)\Psi'(w)+\Psi(w)\Phi''(w)|}{(1-|\Phi(w)|^2)^{\alpha+1}} < \infty. \tag{23}$$

A triangle inequality argument using $\Psi \in Z$, $\Psi\Phi \in Z$ yields

$$(1-|w|^2)|2\Phi'(w)\Psi'(w)+\Psi(w)\Phi''(w)| \le \|\Psi\Phi\|_Z + \|\Psi\|_Z \tag{24}$$

and thus

$$\sup_{|\Phi(w)|\le 1/2}\frac{(1-|w|^2)|2\Phi'(w)\Psi'(w)+\Psi(w)\Phi''(w)|}{(1-|\Phi(w)|^2)^{\alpha+1}} < \infty. \tag{25}$$

The relations (23) and (25) yield (18).

The final argument uses $\beta = 1$, $\gamma = -2$ and $\lambda = 1$. Calculations yield $f_w(\Phi(w)) = 0 = f_w'(\Phi(w))$ and $f_w''(\Phi(w)) = 2\overline{\Phi(w)}^2/(1-|\Phi(w)|^2)^{\alpha+2}$. Substitution into (22) yields

$$\sup_{|\Phi(w)|>1/2}\frac{(1-|z|^2)|\Psi(z)||\Phi'(z)|^2}{(1-|\Phi(z)|^2)^{\alpha+2}} < \infty.$$

To complete the argument note that $\Psi\Phi^2 \in Z$, $\Psi\Phi \in Z$ and $\Psi \in Z$. A triangle inequality argument yields

$$(1-|w|^2)|\Psi(w)||\Phi'(w)|^2 \le C\ (\|\Psi\Phi^2\|_Z + \|\Psi\Phi\|_Z + \|\Psi\|_Z).$$

The relation (19) now follows. The proof is complete. □

A sketch is given for the proof of the last theorem.

Theorem 4.2. *Assume that $\Psi C_\Phi : F_\alpha \to Z$ is bounded. Then $\Psi C_\Phi : F_\alpha \to Z$ is compact if and only if*

$$\lim_{|\Phi(z)|\to 1} \frac{(1-|z|^2)|\Psi''(z)|}{(1-|\Phi(z)|^2)^\alpha} = 0 \tag{26}$$

and

$$\lim_{|\Phi(z)|\to 1} \frac{(1-|z|^2)|2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z)|}{(1-|\Phi(z)|^2)^{\alpha+1}} = 0 \tag{27}$$

and

$$\lim_{|\Phi(z)|\to 1} \frac{(1-|z|^2)|\Psi(z)||\Phi'(z)|^2}{(1-|\Phi(z)|^2)^{\alpha+2}} = 0. \tag{28}$$

Proof. First assume that $W = \Psi C_\Phi : F_\alpha \to Z$ is bounded and the relations (26)-(28) hold. Assume that $\|f_n\|_{F_\alpha} \leq 1$ for $n = 1, 2, \dots$ and $f_n \to 0$ uniformly on compact subsets of U as $n \to \infty$. It follows immediately that $|(Wf_n)(0)| \to 0$ and $|(Wf_n)'(0)| \to 0$ as $n \to \infty$. To establish that $W : F_\alpha \to Z$ is compact, it remains to show that

$$\sup_{z\in U}(1-|z|^2)|(Wf_n)''(z)| \to 0$$

as $n \to \infty$. By a triangle inequality argument, it is enough to prove that

$$\sup_{z\in U}(1-|z|^2)|f_n(\Phi(z))\Psi''(z)| \to 0, \tag{29}$$

$$\sup_{z\in U}(1-|z|^2)|f_n'(\Phi(z))\ (2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z))| \to 0, \tag{30}$$

and

$$\sup_{z\in U}(1-|z|^2)|f_n''(\Phi(z))\Psi(z)\ (\Phi'(z))^2| \to 0 \tag{31}$$

as $n \to \infty$.

An argument will be given to establish (30). The arguments for (29) and (31) are similar, and will be omitted. By the initial remarks, there is a constant C depending only on α such that

$$\begin{aligned} &(1-|z|^2)|f_n'(\Phi(z))||2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z)| \\ &\leq \frac{C(1-|z|^2)}{(1-|\Phi(z)|^2)^{\alpha+1}}|2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z)|. \end{aligned}$$

Given $\epsilon > 0$, condition (27) now provides r, $0 < r < 1$ such that

$$\sup_{|\Phi(z)|>r}(1-|z|^2)|f_n'(\Phi(z))||2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z)| < \epsilon \tag{32}$$

for all n. Since $f_n' \to 0$ uniformly on compact subsets and since relation (18) holds, there exists $N > 0$ such that

$$\sup_{|\Phi(z)|\leq r}(1-|z|^2)|f_n'(\Phi(z))||2\Phi'(z)\Psi'(z)+\Psi(z)\Phi''(z)| < \epsilon \tag{33}$$

for all $n > N$. The relations (32) and (33) yield (30).

For the opposite implication, assume $\Psi C_\Phi : F_\alpha \to Z$ is compact. To establish (27), it is enough to consider any sequence z_n in U with $|\Phi(z_n)| \to 1$ as $n \to \infty$. Let

$$f_n(z) = \frac{C_1(1-|\Phi(z_n)|^2)}{(1-\overline{\Phi(z_n)}z)^{\alpha+1}} + \frac{C_2(1-|\Phi(z_n)|^2)^2}{(1-\overline{\Phi(z_n)}z)^{\alpha+2}} + \frac{C_3(1-|\Phi(z_n)|^2)^3}{(1-\overline{\Phi(z_n)}z)^{\alpha+3}},$$

where $C_1 = -2(\alpha+3)$, $C_2 = 4\alpha+10$ and $C_3 = -2(\alpha+2)$. Then $\|f_n\|_{F_\alpha} \le C$ and $f_n \to 0$ uniformly on compact subsets as $n \to \infty$. Therefore $\|W(f_n)\|_Z \to 0$ as $n \to \infty$. Since $|\Phi(z_n)| \to 1$, calculations yield

$$\frac{(1-|z_n|^2)|2\Phi'(z_n)\Psi'(z_n) + \Psi(z)\Phi''(z)|}{(1-|\Phi(z_n)|^2)^{\alpha+1}} \to 0$$

as $n \to \infty$ and thus (27) is established.

The relation (26) is derived in a similar way with $C_1 = (\alpha+2)(\alpha+3)$, $C_2 = -2(\alpha+1)(\alpha+3)$ and $C_3 = (\alpha+1)(\alpha+2)$. Relation (28) is derived using $C_1 = 1 = C_3$ and $C_2 = -2$. The sketch is complete. □

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