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A MODEL OF RETIREMENT AND CONSUMPTION-PORTFOLIO CHOICE

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Abstract. In this study we propose a model of optimal retirement, consumption and portfolio choice of an individual agent, which encompasses a large class of the models in the literature, and provide a methodology to solve the model. Different from the traditional approach, we consider the problems before and after retirement simultaneously and identify the difference in the dual value functions as the utility value of lifetime labor. The utility value has an option nature, namely, it is the maximized value of choosing the retirement time optimally and we discover it by solving a variational inequality. Then, we discover the dual value functions by using the utility value. We discover the value function and optimal policies by establishing a duality between the value function and the dual value function. The model and approach offer a significant advantage for computation of optimal policies for a large class of problems.

1. Introduction

In this study, we propose and investigate a model of optimal retirement, consumption and portfolio choice of an individual agent. The optimal retirement decision is an important determinant of labor supply and human capital. The option of voluntary retirement makes the beta of human capital negative, since one can work longer after poor realizations of investment outcomes and has an important effect on risk taking attitudes ([7]). Researchers have developed economic models of the retirement decision in conjunction with the consumption and portfolio choice. In particular, they have shown that the retirement option induces people to increase savings and to make aggressive investments before retirement and to increase risky positions in a stock market boom even after making large profits ([4, 5, 8, 17]). The models, thus, have provided a profound insight into the interaction between the retirement decision and the life cycle choice of consumption and investment.

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The models in the literature, however, are highly specialized and often make contradicting predictions. For example, the models of [7] and [8] predict jumps of consumption at retirement, whereas those of [4] and [17] do not. The difference in the models originates from the special utility functions they employ, and hence, the predictions are not robust to changes in the utility specifications.

Our objective is (i) to propose a unified model, which encompasses a large class of the models in the literature and (ii) to provide a methodology to solve the model. For the first objective, we propose a model with minimum assumptions among those employing time separable expected utility functions with a constant subjective discount rate and stationary felicity functions before and after retirement in a single-good economy. The minimum assumptions consist of two; Assumption 1 is the condition for the problem to be well-defined, which is much less restrictive than those in the literature, and Assumption 2 is the necessary and sufficient condition for the retirement option to have a positive value. We next solve the problem by the dual martingale approach. However, we do not take the traditional approach of taking sequential steps; solving the problem after retirement in the first step and then solving the problem before retirement in the second step, taking the solution in the second step as an essential ingredient.[1] Instead we consider the problems before and after retirement simultaneously and identify the difference in the dual value functions as the utility value of lifetime labor. The utility value has an option nature, namely, it is the maximized value of choosing the retirement time optimally, and can be obtained as a solution to a variational inequality. Then, we find the dual value functions by using the utility value. We discover the value function and optimal policies by establishing a duality between the value function and the dual value function.

As an illustrative example we provide a model which subsumes major models in the literature as special cases. We obtain optimal policies in closed form when the felicity function has constant relative risk aversion. The model and approach offer a significant advantage for computation of optimal policies for a large class of problems, for which only specialized approaches were available.

[17] have considered the utility value of lifetime labor and derived its value by solving a variational inequality. Their method is, however, still the two-step method. [11] make an observation that the dual value functions before and after retirement differ by the utility value of lifetime labor. [12] study a consumption/portfolio choice and retirement problem with flexible labor supply and use a method similar to ours. However, in these studies the felicity functions before and after retirement differ by disutility of labor, while we consider generally different felicity functions before and after retirement in this paper. In the absence of voluntary retirement option [2] and [3] study flexible labor supply with an exogenously fixed retirement date. Recently, [10] investigate an

[1] Most of the papers in the literature take this two-step approach. See [17] for a systematic treatment.

optimal investment and consumption problem with heterogeneous consumption of basic and luxury goods, together with choice of a retirement time.

The paper is organized as follows. Section 2 explains the model. Section 3 sets up the dual optimization problem and provides a solution. Section 4 establishes the duality between the value function and the dual value function and derives the optimal policies. Section 5 discusses examples and Section 6 concludes. All the auxiliary lemmas are contained in the appendix.

2. Model

We consider an agent who lives in a single good economy. The agent is currently working and has an option to retire voluntarily. The agent cannot come back to work after retirement, and hence, retirement is an irreversible decision. The agent's preference over consumption is represented by the following expected utility function:

$$U \equiv \mathbb{E}\left[\int_0^\tau e^{-\rho t} u_B(c_t)dt + \int_\tau^\infty e^{-\rho t} u_A(c_t)dt\right]. \tag{1}$$

Here $\rho > 0$ is the subjective discount rate, τ is the time of, c_t is the rate of consumption, and $u_B : \mathcal{D} \to \mathbb{R}$ and $u_A : \mathcal{D} \to \mathbb{R}$ ($\mathcal{D} = [0, \infty)$ or $(0, \infty)$) are the agent's felicity functions before and after retirement, respectively.[2]

There are two assets in the financial market, a risk-free asset and a risky asset. The return on the risk-free asset is a constant $r > 0$. The cum-dividend price S_t of the risky asset follows the dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \tag{2}$$

where $\mu > 0$ and σ are constants, describing the mean and standard deviation of the returns on the risky asset, and B_t is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. We assume that filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration generated by B_t; it describes the information available to the agent at each instance. The two asset model is without loss of generality if the market consists of n-risky assets and a single risk-free asset with constant covariance matrix of risky asset returns. In this case the two-fund separation theorem is valid and the risky asset corresponds to the market portfolio ([9]). There are no trading frictions, i.e., no transaction costs nor taxes nor short-selling restrictions.

We assume that the agent receives constant wage income equal to $\epsilon > 0$ until retirement. The constant wage rate assumption is standard in literature.[3] Let

[2] We consider an infinite horizon model for simplicity of exposition, and extension to a finite horizon is technically complex, but can be done following the methods in [17].

[3] For example, [8] states, "we maintain the assumption that agents receive a constant wage. This is done not only for simplicity, but more importantly because it makes the results more surprising." (p. 91)

π_t denote the agent's investment in the risky asset at time t. Then, her wealth X_t at time t satisfies the dynamics:

$$dX_t = [rX_t + (\mu - r)\pi_t + \epsilon \mathbf{1}_{\{t<\tau\}} - c_t]dt + \sigma\pi_t dB_t \text{ with } X_0 = x, \tag{3}$$

where x is her initial wealth and $\mathbf{1}_A$ denotes the indicator function of set A. The natural limit for the wealth is the following:

$$X_t \geq -\frac{\epsilon}{r}\mathbf{1}_{\{t<\tau\}}, \tag{4}$$

that is, the agent cannot borrow more than the maximum possible present value of labor income. Constraint (4) does not impose any further borrowing constraints than implied by the present value of income. Hence the financial market is dynamically complete.

We give technical admissibility conditions for (c, π, τ). Throughout the paper, (c, π, τ) belongs to the admissible class $\mathcal{A}(x)$ if they are $\mathcal{F}_t$-progressively measurable processes satisfying the following conditions:

(a) τ belongs to $\mathcal{S}$, the set of all $\mathcal{F}$-stopping times taking values in $[0, \infty]$,
(b) (c, π, τ) satisfies the agent's wealth dynamics in (3) together with constraint (4),
(c) c and π satisfy

$$c_t \geq 0, \text{ a.s.}, \int_0^t c_s ds < \infty, \text{ a.s. and } \int_0^t \pi_s^2 ds < \infty \text{ a.s.}, \quad \forall\, t \geq 0. \tag{5}$$

We define the class Σ of all set of felicity functions u satisfying the following conditions:

(i) The felicity function $u : \mathcal{D} \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable, and $\lim_{c\to+\infty} u'(c) = 0$.
The strictly decreasing and continuous function $u' : (0,\infty) \overset{\text{onto}}{\longrightarrow} (0, u'(0))$ has a strictly decreasing, continuous inverse $I_u : (0, u'(0)) \overset{\text{onto}}{\longrightarrow} (0,\infty)$. We extend I_u by setting $I_u(y) = 0$ for $y \geq u'(0)$. Then, we have

$$u'(I_u(y)) = \begin{cases} y, & 0 < y < u'(0), \\ u'(0), & y \geq u'(0), \end{cases} \tag{6}$$

and $I_u(u'(c)) = c$ for $0 < c < \infty$. Note that $\lim_{y\to\infty} I_u(y) = 0$.
(ii) For any $y > 0$,

$$\int_0^y \eta^{-n_2} I_u(\eta) d\eta < \infty, \tag{7}$$

where $n_1 > 0$ and $n_2 < 0$ are two roots of the quadratic equation:

$$\frac{\theta^2}{2}n^2 + \left(\rho - r - \frac{\theta^2}{2}\right)n - \rho = 0, \tag{8}$$

where

$$\theta \equiv \frac{\mu - r}{\sigma}.$$

Note that the constant θ is equal to the Sharpe ratio of the risky asset.

To guarantee that the problem is well-defined, we make the following assumption:

Assumption 1.
(i) $u_B, u_A \in \Sigma$.
(ii) For all $y > 0$, the following inequality holds:

$$u_B(I_{u_B}(y)) < u_A(I_{u_A}(y)).$$

Assumption 1(ii) states that the post-retirement felicity (period utility) is greater than the pre-retirement felicity if the felicities are measured at consumption levels which provide the same marginal utility y before and after retirement. [4] have proposed a model with disutility of labor before retirement. In the model the assumption means that disutility is positive. [8] and [7] have studied a model where consumption is augmented by a multiplicative factor $k > 1$ for calculation of the post-retirement felicity, i.e., $u_A(c) = u_B(kc)$. Here the marginal utility of consumption is magnified by k after retirement, and hence the same marginal utility is obtained if the marginal utility is reduced by $\frac{1}{k}$ and the level of consumption is also reduced by the same factor, i.e., for the given pre-retirement marginal utility y, the same marginal utility y is obtained after retirement when post-retirement consumption is equal to $\frac{1}{k} I_{u_B}(\frac{y}{k})$. Hence $u_A(I_{u_A}(y)) = u_B(I_{u_B}(\frac{y}{k}))$. Since u_B exhibits diminishing marginal utility, i.e., strictly concave, $I_{u_B}(\frac{y}{k}) > I_{u_B}(y)$, and the assumption is satisfied. Under Assumption 1(ii) we can show that the normalized marginal benefit of work defined below in (19) is an increasing function of marginal utility (Lemma 3.1), which is necessary to establish the existence of a unique optimal retirement boundary. We provide a class of felicity functions which satisfy the assumption and subsume the previous two cases in Section 5.

We now state the agent's problem:

Problem 1. Given $x > -\frac{\epsilon}{r}$, we consider the following optimization problem:

$$V(x) = \sup_{(c,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t} u_B(c_t)dt + \int_\tau^\infty e^{-\rho t} u_A(c_t)dt\right].$$

3. Optimization

We derive a solution to the agent's optimization problem by the standard dual martingale approach developed by [6] and [13]. They show that in a dynamically complete market the stochastic discount factor of the economy takes the form

$$\xi_t = e^{-rt-\frac{1}{2}\theta^2 t-\theta B_t}, \tag{9}$$

and the wealth dynamics (3) can be transformed into a budget constraint in static form, as we state in the following proposition.

Proposition 3.1 (Theorem 9.4 in [14])**.** *Let $x > -\frac{\epsilon}{r}$. Suppose $(c,\pi,\tau) \in \mathcal{A}(x)$. Then $c = (c_t)_{t=0}^{\infty}$ satisfies the following static budget constraint*

$$\mathbb{E}\left[\int_0^{\infty} \xi_t \left(c_t - \epsilon \mathbf{1}_{\{t<\tau\}}\right) dt\right] = x. \tag{10}$$

Conversely, suppose that $c = (c_t)_{t=0}^{\infty}$ satisfies the first condition in (5) *and $\tau \in \mathcal{S}$ such that* (10) *is satisfied. Then there exists a portfolio process π_t such that $(c,\pi,\tau) \in \mathcal{A}(x)$. The corresponding wealth process $X^{x,c,\pi}$ is*

$$dX_t^{x,c,\pi} = [rX_t^{x,c,\pi} + (\mu - r)\pi_t - c_t + \epsilon \mathbf{1}_{\{t<\tau\}}]dt + \sigma\pi_t dB_t, \quad t \geq 0$$

and

$$X_t^{x,c,\pi} = \mathbb{E}_t\left[\int_t^{\infty} \frac{\xi_s}{\xi_t}(c_s - \epsilon 1_{\{s<\tau\}})ds\right], \tag{11}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t]$ is the conditional expectation at time t on the σ-algebra $\mathcal{F}_t$.

3.1. Dual formulation

For $y > 0$, the conjugate function $\tilde{u}$ of $u \in \Sigma$ is defined as

$$\tilde{u}(y) = \sup_{c\geq 0}(u(c) - yc) = u(I_u(y)) - yI_u(y).$$

We formulate the following Lagrangian:

(12)

$$\begin{aligned}
\mathfrak{L} &= \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}u_B(c_t)dt + \int_{\tau}^{\infty} e^{-\rho t}u_A(c_t)dt\right] + y\left(x - \mathbb{E}\left[\int_0^{\infty} \xi_t(c_t - \epsilon\mathbf{1}_{\{t<\tau\}})dt\right]\right) \\
&= \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}(u_B(c_t) - \mathcal{Y}_t^y c_t + \mathcal{Y}_t^y\epsilon)dt + e^{-\rho\tau}\mathbb{E}_{\tau}\left[\int_{\tau}^{\infty} e^{-\rho(t-\tau)}\left(u_A(c_t) - \mathcal{Y}_t^y c_t\right)dt\right]\right] + yx \\
&\leq \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}(u_B(c_t) - \mathcal{Y}_t^y c_t + \mathcal{Y}_t^y\epsilon)dt + e^{-\rho\tau}\mathbb{E}_{\tau}\left[\int_{\tau}^{\infty} e^{-\rho(t-\tau)}\tilde{u}_A(\mathcal{Y}_t^y)dt\right]\right] + yx \\
&\leq \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt + \int_{\tau}^{\infty} e^{-\rho t}\tilde{u}_A(\mathcal{Y}_t^y)dt\right] + yx \\
&= \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt + \int_0^{\infty} e^{-\rho t}\tilde{u}_A(\mathcal{Y}_t^y)dt\right] + yx \\
&= J_A(y) + \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt\right] + yx,
\end{aligned}$$

where $y > 0$ is a Lagrangian multiplier of the static budget constraint (10), $\mathcal{Y}_t^y = ye^{\rho t}\xi_t$,

$$J_A(y) = \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\tilde{u}_A(\mathcal{Y}_t^y)dt\right], \tag{13}$$

and $\tilde{u}_B$, $\tilde{u}_A$ are the conjugate functions of u_B, u_A. The dual variable $\mathcal{Y}_t$ is the *marginal value* of wealth, as will be shown in Theorem 4.1. Note that

$$d\mathcal{Y}_t^y = (\rho - r)\mathcal{Y}_t^y dt - \theta\mathcal{Y}_t^y dB_t \ \text{ with } \ \mathcal{Y}_0^y = y. \tag{14}$$

Thus, for $y > 0$ the candidate of optimal consumption $(\hat{c}(\mathcal{Y}_t^y))_{t=0}^{\infty}$ is given by

$$\hat{c}(\mathcal{Y}_t^y) = \begin{cases} I_{u_B}(\mathcal{Y}_t^y) & \text{for } 0 \le t < \tau, \\ I_{u_A}(\mathcal{Y}_t^y) & \text{for } t \ge \tau. \end{cases} \tag{15}$$

Considering the Lagrangian (12), we formulate the dual problem which chooses the optimal time of retirement by observing the changes of the marginal value of wealth $\mathcal{Y}_t$.

Problem 2. Consider the following optimal stopping problem:

$$J(y) = J_A(y) + \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y \epsilon\right) dt\right], \tag{16}$$

where $\mathcal{S}$ denotes the set of $\mathcal{F}$-stopping times. We call $J(y)$ the *dual value function.*

Problem 2 states that the dual value function consists of J_A and the optimized value of the optimal stopping problem. If $\tau = 0$, i.e., the agent retires immediately, then the dual value function is equal to J_A, and hence, J_A can be regarded as the *dual value function after retirement.* The quantity

$$\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y \epsilon \tag{17}$$

inside the integral in (16) consists of two components: the first is the difference in the conjugate felicity functions, $\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y)$, which can be interpreted as the difference in the utility values before and after retirement, and the second is $\mathcal{Y}_t^y \epsilon$, labor income adjusted by the marginal utility of wealth and can be interpreted as the utility value of labor income. Accordingly, quantity (17) is the *marginal benefit of work* relative to retirement. The agent chooses the retirement time τ that maximizes the present value of the marginal benefit. In this sense we will call the optimized value of the optimal stopping problem as the *utility value of lifetime labor.*

Let us denote the utility value of lifetime labor by $\mathcal{P}(y)$, i.e.,

$$\mathcal{P}(y) = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y \epsilon\right) dt\right]. \tag{18}$$

Let us denote the current normalized marginal benefit of work by $\Psi(y)$, i.e.,

$$\Psi(y) = \frac{1}{y}\left(\tilde{u}_B(y) - \tilde{u}_A(y)\right) + \epsilon. \tag{19}$$

We will now investigate properties of the normalized marginal benefit. The following lemma provides its asymptotic behavior, whose proof is given in Appendix B.

Lemma 3.1. *$\Psi(y)$ is a strictly increasing function of $y > 0$ and*

$$\lim_{y \to \infty} \Psi(y) = \epsilon.$$

If $\lim_{y\to 0+}\Psi(y)\ge 0$, Lemma 3.1 implies that $\Psi(y)>0$ for all $y>0$. Thus, we deduce that for any $\tau\in\mathcal{S}$

$$\mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y)-\tilde{u}_A(\mathcal{Y}_t^y)+\mathcal{Y}_t^y\epsilon\right)dt\right]$$
$$\le \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y)-\tilde{u}_A(\mathcal{Y}_t^y)+\mathcal{Y}_t^y\epsilon\right)dt\right].$$

That is, the agent does not choose the option to retire in the case when $\lim_{y\to 0+}\Psi(y)\ge 0$, that is, the marginal benefit of work is always positive. Consequently, the necessary condition for the agent to retire at a finite time τ is the following, which we make a standing assumption:

Assumption 2.

$$\lim_{y\to 0+}\Psi(y)<0.$$

Under Assumption 2, Lemma 3.1 implies that there exists a unique $\bar{z}>0$ such that

$$\Psi(\bar{z})=0.$$

By the standard theory of the optimal stopping problem ([16]), $\mathcal{P}(y)$ satisfies the following variational inequality:

$$\begin{cases}\mathbb{L}\mathcal{P}+h(y)\le 0 & \text{if } \mathcal{P}(y)=0,\\ \mathbb{L}\mathcal{P}+h(y)=0 & \text{if } \mathcal{P}(y)>0,\end{cases} \tag{20}$$

where $h(y)\equiv y\Psi(y)=(\tilde{u}_B(y)-\tilde{u}_A(y)+\epsilon y)$ and the differential operator $\mathbb{L}$ is given by

$$\mathbb{L}\equiv\frac{\theta^2}{2}y^2\frac{d^2}{dy^2}+(\rho-r)y\frac{d}{dy}-\rho.$$

We will now provide a verification theorem. The proof is similar to that of Theorem 3.2 and Lemma 3.4 in [15], and we omit the proof.

Theorem 3.1 (Verification theorem)**.** *Suppose that variational inequality* (20) *has a solution $\mathcal{Q}(y)$ which is continuously differentiable in $y>0$ and twice continuously differentiable in $(0,\infty)\backslash\{b\}$ for some point $b>0$, and there exist positive constants ζ_1 and ζ_2 such that*

$$|\mathcal{Q}'(y)|\le\zeta_1\left(y^{-\zeta_2}+y^{\zeta_2}\right)\quad \textit{for}\ \ y>0.$$

Then,

(a) $\mathcal{P}(y)\le\mathcal{Q}(y)$.
(b) *If $\lim_{t\to\infty}e^{-\rho t}\mathbb{E}\left[\mathcal{Q}(\mathcal{Y}_t)\right]=0$, then $\mathcal{Q}(y)=\mathcal{P}(y)$ and the optimal solution to the problem in* (18) *is given by*

$$\tau_R(y)=\inf\{t\ge 0\mid \mathcal{Y}_t^y\in\{y>0\mid \mathcal{P}(y)=0\}\}.$$

From Theorem 3.1, we know that the transversality condition is necessary to guarantee that the solution to the HJB equation is the option value of voluntary retirement:

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[\mathcal{P}(\mathcal{Y}_t^y)\right] = 0. \tag{21}$$

We now derive the utility value of lifetime labor, $\mathcal{P}(y)$. We first explain the derivation in a heuristic and intuitive manner, and later provide a formal statement and its proof. Suppose that there exists a boundary z_R such that if the agent's marginal utility $\mathcal{Y}_t^y$ hits the boundary, the agent retires. The option value becomes zero after the agent retires. Hence, by the *smooth pasting condition* we have

$$\mathcal{P}(z_R) = \mathcal{P}'(z_R) = 0, \quad \mathcal{P}(y) = 0 \quad \text{for } 0 < y \le z_R. \tag{22}$$

When the agent is still working, $\mathcal{P}(y)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\theta^2}{2} y^2 \mathcal{P}''(y) + (\rho - r) y \mathcal{P}(y) - \rho \mathcal{P} + h(y) = 0. \tag{23}$$

That is, for $y > z_R$, a general solution to the HJB equation (23) can be represented as the sum of a general solution to the homogeneous equation and a particular solution:

$$\mathcal{P}(y) = D_1 y^{n_1} + D_2 y^{n_2} + \Xi_h(y),$$

where the particular solution $\Xi_h(y)$ is given by

$$\Xi_h(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[y^{n_2} \int_0^y \nu^{-n_2-1} h(\nu) d\nu + y^{n_1} \int_y^\infty \nu^{-n_1-1} h(\nu) d\nu \right]. \tag{24}$$

More generally, for any measurable function $f : \mathbb{R}^+ \to \mathbb{R}$, we define two operators Ξ and Γ by

$$\Xi_f(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[y^{n_2} \int_0^y \nu^{-n_2-1} f(\nu) d\nu + y^{n_1} \int_y^\infty \nu^{-n_1-1} f(\nu) d\nu \right] \tag{25}$$

and

$$\Gamma_f(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[y^{n_2-1} \int_0^y \nu^{-n_2} f(\nu) d\nu + y^{n_1-1} \int_y^\infty \nu^{-n_1} f(\nu) d\nu \right]. \tag{26}$$

To satisfy the transversality condition (21) for $\mathcal{P}(y)$, the coefficient D_1 of y^{n_1} should be zero, i.e.,

$$D_1 = 0.$$

Thus, we can write $\mathcal{P}(y)$ as

$$\mathcal{P}(y) = D y^{n_2} + \Xi_h(y).$$

By using the smooth pasting condition ($\mathcal{P}(z_R) = \mathcal{P}'(z_R) = 0$), we have

$$\int_{z_R}^\infty \nu^{-n_1-1} h(\nu) d\nu = 0 \quad \text{and} \quad D = -\frac{2}{\theta^2(n_1 - n_2)} \int_0^{z_R} \nu^{-n_2-1} h(\nu) d\nu.$$

We now state a formal proposition.

Proposition 3.2. *The utility value of lifetime labor, $\mathcal{P}(y)$, is given by*

$$\mathcal{P}(y) = \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^y) - \tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y \epsilon\right) dt\right] = \begin{cases} Dy^{n_2} + \Xi_h(y) & \text{for } y \geq z_R, \\ 0 & \text{for } 0 < y \leq z_R, \end{cases} \tag{27}$$

where $z_R \in (0, \bar{z})$ is a unique solution of the following equation

$$\int_{z_R}^\infty \nu^{-n_1-1} h(\nu) d\nu = 0$$

and

$$D = -\frac{2}{\theta^2(n_1 - n_2)} \int_0^{z_R} \nu^{-n_2-1} h(\nu) d\nu.$$

Moreover, the optimal stopping time $\tau_R(y)$ is given by

$$\tau_R(y) = \inf\{t \geq 0 \mid \mathcal{Y}_t^y \leq z_R\}.$$

Proof. Let us denote $\mathcal{G}(y)$ by

$$\mathcal{G}(y) = \int_y^\infty \nu^{-n_1-1} h(\nu) d\nu = \int_y^\infty \nu^{-n_1} \Psi(\nu) d\nu.$$

By Lemma 3.1, we deduce that $\mathcal{G}(y)$ is strictly increasing on $(0, \bar{z})$ and strictly decreasing on $(\bar{z}, \infty)$. Moreover,

$$\mathcal{G}(z) > 0 \quad \text{for all} \quad z \geq \bar{z}.$$

By Assumption 2, there exist $\delta > 0$ and $y_\delta > 0$ such that

$$\Psi(y) < -\delta \quad \text{for all} \quad y \in (0, y_\delta).$$

For a sufficiently small $y < y_\delta$, we have

$$\begin{aligned} \mathcal{G}(y) = \int_y^\infty \nu^{-n_1} \Psi(\nu) d\nu = & \int_y^{y_\delta} \nu^{-n_1} \Psi(\nu) d\nu + \int_{y_\delta}^\infty \nu^{-n_1} \Psi(\nu) d\nu \\ < & -\delta \int_y^{y_\delta} \nu^{-n_1} d\nu + \int_{y_\delta}^\infty \nu^{-n_1} \Psi(\nu) d\nu \\ = & -\delta \frac{1}{1-n_1}({y_\delta}^{1-n_1} - y^{1-n_1}) + \int_{y_\delta}^\infty \nu^{-n_1} \Psi(\nu) d\nu. \end{aligned}$$

Since $\int_{y_\delta}^\infty \nu^{-n_1} |\Psi(\nu)| d\nu < \infty$,

$$\lim_{y\to 0+} \mathcal{G}(y) < \delta \frac{1}{n_1 - 1}({y_\delta}^{1-n_1} - \lim_{y\to 0+} y^{1-n_1}) + \int_{y_\delta}^\infty \nu^{-n_1} \Psi(\nu) d\nu = -\infty.$$

Thus, there exists a unique $z_R \in (0, \bar{z})$ such that $\mathcal{G}(z_R) = 0$.

Claim 1. $\mathcal{P}(y)$, defined by

$$\mathcal{P}(y) = \begin{cases} Dy^{n_2} + \Xi_h(y) & \text{for } \ y > z_R \\ 0 & \text{for } \ 0 < y \leq z_R, \end{cases}$$

is continuously differentiable in $(0, \infty)$ and twice continuously differentiable in $(0, \infty)\backslash\{z_R\}$ and satisfies the variational inequality (20).

Proof of Claim 1. By construction of $\mathcal{P}(y)$, it is clear that $\mathcal{P}(y)$ is continuously differentiable in $(0, \infty)$ and twice continuously differentiable in $(0, \infty)\backslash\{z_R\}$.

Note that for $y \in (0, z_R]$

$$\mathbb{L}\mathcal{P}(y) + h(y) = h(y) = y\Psi(y) < 0,$$

where we have used that fact that $\Psi(y)$ is strictly increasing $y > 0$ and $\Psi(\bar{z}) = 0$.

Since $h(y) = \tilde{u}_B(y) - \tilde{u}_A(y) + \epsilon y$, we have

$$\Xi_h(y) = \Xi_{\tilde{u}_B}(y) - \Xi_{\tilde{u}_A}(y) + \frac{\epsilon}{r}y.$$

It follows from Lemma A.3 that

$$\Xi_h'(y) = \Gamma_{I_{u_A}}(y) - \Gamma_{I_{u_B}}(y) + \frac{\epsilon}{r}$$

and

$$\lim_{y\uparrow\infty} \Xi_h'(y) = \lim_{y\uparrow\infty} \Gamma_{I_{u_A}}(y) - \lim_{y\uparrow\infty} \Gamma_{I_{u_B}}(y) + \frac{\epsilon}{r} = \frac{\epsilon}{r}.$$

Since $\mathcal{P}(y) = Dy^{n_2} + \Xi_h(y)$ for $y > z_R$, we have

$$\lim_{y\uparrow\infty} \mathcal{P}'(y) = \lim_{y\uparrow\infty} (n_2 D y^{n_2 - 1}) + \lim_{y\uparrow\infty} \Xi_h'(y) = \frac{\epsilon}{r} > 0.$$

Note that for $y > z_R$,

$$\begin{aligned}
\mathcal{P}(y) &= Dy^{n_2} + \Xi_h(y) \\
&= -\frac{2}{\theta^2(n_1 - n_2)} y^{n_2} \int_0^{z_R} \nu^{-n_2 - 1} h(\nu) d\nu \\
&\quad + \frac{2}{\theta^2(n_1 - n_2)} \left[y^{n_2} \int_0^y \nu^{-n_2 - 1} h(\nu) d\nu + y^{n_1} \int_y^\infty \nu^{-n_1 - 1} h(\nu) d\nu \right] \\
&= \frac{2}{\theta^2(n_1 - n_2)} \left[y^{n_2} \int_{z_R}^y \nu^{-n_2 - 1} h(\nu) d\nu + y^{n_1} \int_y^\infty \nu^{-n_1 - 1} h(\nu) d\nu \right].
\end{aligned}$$

Thus,

$$\mathcal{P}'(y) = \frac{2}{\theta^2(n_1 - n_2)} \left[n_2 y^{n_2 - 1} \int_{z_R}^y \nu^{-n_2 - 1} h(\nu) d\nu + n_1 y^{n_1 - 1} \int_y^\infty \nu^{-n_1 - 1} h(\nu) d\nu \right]$$

for $y > z_R$.

Since $h(y) \geq 0$ for $y \geq \bar{z}$ and $h(y) < 0$ for $y < \bar{z}$, we deduce that $\mathcal{P}'(y)$ is strictly increasing in $y \in (z_R, \bar{z})$ and strictly decreasing in $y > \bar{z}$.

It follows from $\mathcal{P}'(z_R) = 0$ and $\lim_{y\uparrow\infty} \mathcal{P}'(y) = \frac{\epsilon}{r}$ that

$$\mathcal{P}'(y) > 0 \ \text{ for } \ y > z_R.$$

That is, $\mathcal{P}(y)$ is strictly increasing in $y \in (z_R, \infty)$. Since $\mathcal{P}(z_R) = 0$, we have

$$\mathcal{P}(y) > 0 \ \text{ for } \ y > z_R.$$

Therefore, $\mathcal{P}(y)$ satisfies the variational inequality (20). This completes the proof of Claim 1. □

Claim 2. There exists a positive constant ζ such that

$$|\mathcal{P}'(y)| \le \zeta \left(y^{n_1-1} + y^{n_2-1}\right)$$

and $\lim_{y\to\infty} e^{-\rho t}\mathbb{E}\left[\mathcal{P}(\mathcal{Y}_t^y)\right] = 0$.

Proof of Claim 2. Since

$$\mathcal{P}(y) = \left(Dy^{n_2} + \Xi_h(y)\right)\mathbf{1}_{\{y>z_R\}} = \left(Dy^{n_2} + \Xi_{\tilde{u}_B}(y) - \Xi_{\tilde{u}_A}(y) + \frac{\epsilon}{r}y\right)\mathbf{1}_{\{y>z_R\}},$$

we have

$$\begin{aligned}|\mathcal{P}'(y)| &\le \left|n_2 D y^{n_2-1} + \Xi'_{\tilde{u}_B}(y) - \Xi'_{\tilde{u}_A}(y) + \frac{\epsilon}{r}\right| \\ &\le n_2|D|y^{n_2-1} + |\Xi'_{\tilde{u}_B}(y)| + |\Xi'_{\tilde{u}_A}(y)| + \frac{\epsilon}{r}.\end{aligned}$$

Since $u_A, u_B \in \Sigma$, it follows from Lemma A.1 and Proposition 4.1 [15][4] that

$$|\mathcal{P}'(y)| \le \zeta \left(y^{n_1-1} + y^{n_2-1}\right)$$

for some constant $\zeta > 0$.

Moreover,

$$\begin{aligned}|\mathcal{P}(y)| &= \left|\left(Dy^{n_2} + \Xi_h(y)\right)\mathbf{1}_{\{y>z_R\}}\right| \\ &\le |D|y^{n_2}\mathbf{1}_{\{y>z_R\}} + |\Xi_{\tilde{u}_A}(y)| + |\Xi_{\tilde{u}_B}(y)| + \frac{\epsilon}{r}y. \\ &\le |D|z_R^{n_2} + |\Xi_{\tilde{u}_A}(y)| + |\Xi_{\tilde{u}_B}(y)| + \frac{\epsilon}{r}y.\end{aligned}$$

By Proposition 4.1 in [15], we have

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[|\Xi_{\tilde{u}_A}(\mathcal{Y}_t^y)|\right] = \lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[|\Xi_{\tilde{u}_B}(\mathcal{Y}_t^y)|\right] = 0.$$

Since $\lim_{t\to\infty} e^{-\rho t}|D|z_R^{n_2} = 0$ and $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[\mathcal{Y}_t^y\right] = \lim_{t\to\infty} e^{-rt}y = 0$, it follows that

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[|\mathcal{P}(\mathcal{Y}_t^y)|\right] = 0.$$

This completes the proof of Claim 2. □

By Claim 1, Claim 2 and Theorem 3.1, we derive the desired results. □

[4]Tbe proposition is frequently used in proofs and we provide a review of the proposition in Appendix D for the reader's convenience.

Note that

$$\Xi_h(y) = \Xi_{\tilde{u}_B}(y) - \Xi_{\tilde{u}_A}(y) + \frac{\epsilon}{r}y \;\text{ and }\; J_A(y) = \Xi_{\tilde{u}_A}(y).$$

From (16), we derive the following corollary.

Corollary 3.1. *The dual value function $J(y)$ is given by*

$$J(y) = \begin{cases} Dy^{n_2} + \Xi_{\tilde{u}_B}(y) + \frac{\epsilon}{r}y & \text{for } y \geq z_R, \\ \Xi_{\tilde{u}_A}(y) & \text{for } 0 < y \leq z_R. \end{cases} \tag{28}$$

4. Optimal policies and human wealth

In this section we derive the optimal policies of the agent and discuss their properties.

We derive the agent's optimal policies by establishing the duality between the value function and the dual value function.

Theorem 4.1. *Given $x > -\frac{\epsilon}{r}$.*

(a) *$V(x)$ and $J(y)$ satisfy the duality relationship*:

$$V(x) = \inf_{y>0}\left(J(y) + yx\right), \quad J(y) = \sup_{x>-\frac{\epsilon}{r}}\left(V(x) - yx\right). \tag{29}$$

There exists a unique $y^* > 0$ such that

$$x = -J'(y^*).$$

(b) *(c^*, π^*, τ^*) given by*

$$c_t^* = \hat{c}(\mathcal{Y}_t^{y^*}) = \begin{cases} I_B(\mathcal{Y}_t^{y^*}) & \text{for } 0 \leq t < \tau^*, \\ I_A(\mathcal{Y}_t^{y^*}) & \text{for } t \geq \tau^*, \end{cases}$$

$$\tau^* = \tau_R(y^*) = \inf\{t \geq 0 \,|\, \mathcal{Y}_t^{y^*} \leq z_R\},$$

and

$$\pi_t^* = \frac{\theta}{\sigma}\mathcal{Y}_t^* J''(\mathcal{Y}_t^{y^*})$$

are admissible and optimal, i.e.,

$$V(x) = \sup_{(c,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}u_B(c_t)dt + \int_\tau^\infty e^{-\rho t}u_A(c_t)dt\right] \tag{30}$$

$$= \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t}u_B(c_t^*)dt + \int_{\tau^*}^\infty e^{-\rho t}u_A(c_t^*)dt\right]. \tag{31}$$

Here, $X_t = -J'(\mathcal{Y}_t^{y^*})$ and $\mathcal{Y}_t^{y^*} = y^* e^{\rho t}\xi_t$.

Proof. Since τ^* is the first hitting time of geometric Brownian motion, it is clear that $\tau^* \in \mathcal{S}$. Moreover, it follows from (i) in Assumption 1 and Proposition 4.1 in [15] that

$$\mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-rt}I_A(\mathcal{Y}_t^{y^*})dt] < \infty \;\text{ and }\; \mathbb{E}^{\mathbb{Q}}[\int_0^\infty e^{-rt}I_B(\mathcal{Y}_t^{y^*})dt] < \infty. \tag{32}$$

This implies that

$$\int_0^t c_s^* ds < \infty \quad \text{a.s..} \tag{33}$$

Since $z_R < \bar{z}$ and $h(y) < 0$ for $y \in (0, \bar{z})$, we have

$$D = -\frac{2}{\theta^2(n_1 - n_2)} \int_0^{z_R} \nu^{-n_2-1} h(\nu) d\nu > 0.$$

By (28), it follows from Lemma A.3 and $D > 0$ that

$$J''(y) = n_2(n_2 - 1) D y^{n_2 - 2} + \Xi''_{\tilde{u}_B}(y) > 0 \ \text{ for } \ y > z_R.$$

Since $J(y) = J_A(y) = \Xi_{\tilde{u}_A}(y)$ for $0 < y \leq z_R$, it is clear that

$$J''(y) = \Xi''_{\tilde{u}_A}(y) > 0 \ \text{ for } \ 0 < y \leq z_R.$$

Hence, $J(y)$ is strictly convex in $y > 0$.

Note that

$$\lim_{y \uparrow \infty} \mathcal{P}'(y) = \frac{\epsilon}{r}, \ \lim_{y \downarrow 0} \mathcal{P}'(y) = 0$$

and

$$\lim_{y \uparrow \infty} J'_A(y) = 0, \ \lim_{y \downarrow 0} J'_A(y) = -\infty.$$

Thus, it follows that

$$\lim_{y \uparrow \infty} J'(y) = \frac{\epsilon}{r},$$
$$\lim_{y \downarrow 0} J'(y) = -\infty.$$

Therefore, there exists a unique $y^* > 0$ such that for given $x > -\frac{\epsilon}{r}$

$$x = -J'(y^*).$$

It follows from Lemma A.4 that

$$x = -J'(y^*) = \mathcal{X}(y^*) = \mathbb{E}\left[\int_0^\infty \xi_t \left(\hat{c}(\mathcal{Y}_t^{y^*}) - \epsilon \mathbf{1}_{\{t < \tau^*\}}\right) dt\right].$$

By Proposition 3.1, there exists a portfolio π_t^* such that $(c^*, \pi^*, \tau^*) \in \mathcal{A}(x)$ and the corresponding wealth process X^{x,c^*,π^*} is

$$dX_t^{x,c^*,\pi^*} = [rX_t^{x,c^*,\pi^*} + (\mu - r)\pi_t^* - c_t^* + \epsilon \mathbf{1}_{\{t < \tau^*\}}]dt + \sigma \pi_t^* dB_t, \ \ t \geq 0 \tag{34}$$

and

$$X_t^{x,c^*,\pi^*} = \mathbb{E}_t\left[\int_t^\infty \frac{\xi_s}{\xi_t}(c_s^* - \epsilon 1_{\{s < \tau^*\}}) ds\right].$$

Hence, we have

$$\begin{aligned}
y^*x &= y^*\mathbb{E}\left[\int_0^\infty \xi_t\left(\hat{c}(\mathcal{Y}_t^{y^*}) - \epsilon\mathbf{1}_{\{t<\tau^*\}}\right)dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathcal{Y}_t^{y^*}\left(\hat{c}(\mathcal{Y}_t^{y^*}) - \epsilon\mathbf{1}_{\{t<\tau^*\}}\right)dt\right] \\
&= \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t}u_A(\hat{c}(\mathcal{Y}_t^{y^*}))dt + \int_{\tau^*}^\infty e^{-\rho t}u_B(\hat{c}(\mathcal{Y}_t^{y^*}))dt\right] \\
&\quad - \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t}\left(\tilde{u}_B(\mathcal{Y}_t^{y^*}) + \mathcal{Y}_t^{y^*}\epsilon\right)dt + \int_0^{\tau^*} e^{-\rho t}\tilde{u}_A(\mathcal{Y}_t^{y^*})dt\right] \\
&= \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t}u_A(\hat{c}(\mathcal{Y}_t^{y^*}))dt + \int_{\tau^*}^\infty e^{-\rho t}u_B(\hat{c}(\mathcal{Y}_t^{y^*}))dt\right] - J(y^*).
\end{aligned}$$

Hence,

$$\begin{aligned}
J(y^*) + y^*x &= \mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t}u_A(\hat{c}(\mathcal{Y}_t^{y^*}))dt + \int_{\tau^*}^\infty e^{-\rho t}u_B(\hat{c}(\mathcal{Y}_t^{y^*}))dt\right] \\
&\le \sup_{(c,\pi,\tau\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}u_A(c_t)dt + \int_\tau^\infty e^{-\rho t}u_B(c_t)dt\right].
\end{aligned}$$

For any $y > 0$ and $(c, \pi, \tau) \in \mathcal{A}(x)$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\tau e^{-\rho t}u_A(c_t)dt + \int_\tau^\infty e^{-\rho t}u_B(c_t)dt\right] \\
&\le \mathbb{E}\left[\int_0^\tau e^{-\rho t}u_A(c_t)dt + e^{-\rho\tau}V_A(X_\tau)\right] + y\left(x - \mathbb{E}\left[\int_0^\tau \xi_t(c_t - \epsilon)dt + \xi_\tau X_\tau\right]\right) \\
&\le \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left(\tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt + e^{-\rho\tau}J_A(\mathcal{Y}_\tau^y)\right] + yx.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sup_{(c,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}u_A(c_t)dt + \int_\tau^\infty e^{-\rho t}u_B(c_t)dt\right] \\
&\le \sup_{\tau\in\mathcal{S}}\inf_{y>0}\left(\mathbb{E}\left[\int_0^\tau e^{-\rho t}\left(\tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt + e^{-\rho\tau}J_A(\mathcal{Y}_\tau^y)\right] + yx\right) \\
&\le \inf_{y>0}\sup_{\tau\in\mathcal{S}}\left(\mathbb{E}\left[\int_0^\tau e^{-\rho t}\left(\tilde{u}_A(\mathcal{Y}_t^y) + \mathcal{Y}_t^y\epsilon\right)dt + e^{-\rho\tau}J_A(\mathcal{Y}_\tau^y)\right] + yx\right) \\
&= \inf_{y>0}\left(J(y) + yx\right).
\end{aligned}$$

Overall, we have

$$J(y^*) + y^*x \le \inf_{y>0}\left(J(y) + yx\right) \le J(y^*) + y^*x$$

and

$$\mathbb{E}\left[\int_0^{\tau^*} e^{-\rho t} u_A(\hat{c}(\mathcal{Y}_t^{y^*}))dt + \int_{\tau^*}^{\infty} e^{-\rho t} u_B(\hat{c}(\mathcal{Y}_t^{y^*}))dt\right]$$
$$= \sup_{(c,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^{\tau} e^{-\rho t} u_A(c_t)dt + \int_{\tau}^{\infty} e^{-\rho t} u_B(c_t)dt\right].$$

That is, $c_t^* = \hat{c}(\mathcal{Y}_t^{y^*})$ and $\tau^* = \tau^*(y^*)$ are optimal.

The *strong Markov property* implies that

$$\xi_t \mathcal{X}(\mathcal{Y}_t^{y^*}) = \mathbb{E}\left[\int_t^{\infty} \xi_s \left(\hat{c}(\mathcal{Y}_s^{y^*}) - \epsilon \mathbf{1}_{\{s<\tau_R(\mathcal{Y}_t^{y^*})\}}\right) dt\right] = \xi_t X_t^{x,c^*,\pi^*}.$$

That is,

$$\mathcal{X}(\mathcal{Y}_t^{y^*}) = X_t^{x,c^*,\pi^*} \quad \text{for } t \geq 0. \tag{35}$$

By Proposition 3.1,

$$\begin{aligned} d\mathcal{X}(\mathcal{Y}_t^{y^*}) = &\left(r\mathcal{X}(\mathcal{Y}_t^{y^*}) + (\mu - r)\Pi(\mathcal{Y}_t^{y^*}) - \hat{c}(\mathcal{Y}_t^{y^*}) + \epsilon \mathbf{1}_{\{t<\tau_R(y^*)\}}\right) dt \\ &+ \sigma \Pi(\mathcal{Y}_t^{y^*}) dB_t \end{aligned} \tag{36}$$

It follows from (34), (35), and (36) that

$$\pi_t^* = \Pi(\mathcal{Y}_t^{y^*}) = \frac{\theta}{\sigma} \mathcal{Y}_t^{y^*} J''(\mathcal{Y}_t^{y^*}).$$

□

Lemma A.4 and Theorem 4.1 imply

$$X_t = -J'(\mathcal{Y}_t^{y^*}) = \mathbb{E}\left[\int_t^{\infty} e^{-\rho(s-t)} \frac{\xi_s}{\xi_t} \left(\hat{c}(\mathcal{Y}_s^{y^*}) - \epsilon \mathbf{1}_{\{s<\tau_R(y^*)\}}\right) ds\right].$$

Thus, we establish the following relationship:

$$\mathbb{E}\left[\int_t^{\infty} e^{-\rho(s-t)} \frac{\xi_s}{\xi_t} c_s^* ds\right] = X_t + \mathbb{E}\left[\int_t^{\tau^*} e^{-\rho(s-t)} \frac{\xi_s}{\xi_t} \epsilon ds\right]. \tag{37}$$

The left-hand side is the present value of lifetime consumption. The right-hand side consists of two components, financial wealth X_t and the present value of labor income, which can be regarded as *human wealth*. Thus, the right-hand side is equal to the agent's *total wealth*, composed of financial wealth and human wealth. Equation (37) says that life time consumption is financed by financial wealth and human wealth.

Theorem 4.1 also implies that the agent retires when the marginal utility of wealth reaches the threshold level z_R, or equivalently, the agent's wealth level reaches threshold x_R, satisfying

$$x_R = -J'(z_R) = -J_A'(z_R), \tag{38}$$

where we have used $\mathcal{P}'(z_R) = 0$, i.e., human wealth is equal to 0 at retirement.

We now provide a comparative static result for the optimal retirement threshold.

Proposition 4.1. *If ϵ increases, then the optimal retirement threshold x_R increases.*

Proof. Let $\Psi_i(y)$ be the current normalized marginal benefit of work for given $\epsilon_i > 0$ $(i = 1, 2)$.

Suppose that $\epsilon_1 > \epsilon_2$. Then,

$$\Psi_1(y) > \Psi_2(y) \quad \text{for any} \;\; y > 0.$$

Note that there exists a unique $\bar{z}_i > 0$ such that $\Psi_i(\bar{z}_i) = 0$ for $i = 1, 2$.

Since $0 = \Psi_1(\bar{z}_1) = \Psi_2(\bar{z}_2) > \Psi_2(\bar{z}_1)$, it follows from Lemma 3.1 that

$$\bar{z}_2 > \bar{z}_1.$$

For $i = 1, 2$, there exists a unique $z_{R,i} \in (0, \bar{z}_i)$ such that

$$\mathcal{G}_i(z_{R,i}) = 0,$$

where

$$\mathcal{G}_i(y) \equiv \int_y^\infty \nu^{-n_1} \Psi_i(\nu) d\nu.$$

It follows from $\Psi_1(y) > \Psi_2(y)$ for all $y > 0$ that

$$0 = \mathcal{G}_1(z_{R,1}) = \mathcal{G}_2(z_{R,2}) > \mathcal{G}_2(z_{R,1}).$$

Since $\mathcal{G}_2(y)$ is strictly increasing in $y \in (0, \bar{z}_2)$, it follows from $\bar{z}_2 > \bar{z}_1 > z_{R,1}$ that

$$z_{R,2} > z_{R,1}.$$

That is, the free boundary z_R decreases as ϵ increases.

Let $x_{R,i}$ be the optimal retirement threshold corresponding to $z_{R,i}$.

Since $J_A(y)$ is strictly convex in $y > 0$, it follows from $x_{R,i} = -J_A'(z_{R,i})$ that

$$x_{R,1} > x_{R,2}.$$

Therefore, if $\epsilon > 0$ increases, the optimal retirement threshold x_R increases. □

The following proposition shows that the agent adjusts the portfolio discretely at the retirement time, consistent with results by [4] and [5].

Proposition 4.2. *If $\mu - r > 0$ $(\mu - r < 0)$, the optimal portfolio π^* jumps downward (upward) just after retirement. In other words,*

$$\pi_{\tau^*-} - \pi_{\tau^*+} = \lim_{y \to z_R+} \Pi(y) - \lim_{y \to z_R-} \Pi(y) = -\frac{2}{\mu - r} \Psi(z_R).$$

Proof. Since $J(y) = J_A(y) + \mathcal{P}(y)$, it is easy to obtain that

$$\lim_{y \to z_R+} \Pi(y) - \lim_{y \to z_R-} \Pi(y) = \frac{\theta}{\sigma} z_R \mathcal{P}''(z_R). \tag{39}$$

Note that for $y > z_R$, $\mathcal{P}(y)$ satisfies

$$\frac{\theta^2}{2} y^2 \mathcal{P}''(y) + (\rho - r) y \mathcal{P}(y) - \rho \mathcal{P} + h(y) = 0.$$

Since $\mathcal{P}(z_R) = \mathcal{P}'(z_R) = 0$, we have

$$\lim_{y\to z_R+} \mathcal{P}''(y) = -\lim_{y\to z_R+} \frac{2}{\theta^2}\frac{1}{y^2}\left((\rho - r)y\mathcal{P}(y) - \rho\mathcal{P} + h(y)\right) = -\frac{2}{\theta^2}\frac{1}{z_R}h(z_R) = -\frac{2}{\theta^2}\frac{1}{z_R}\Psi(z_R). \tag{40}$$

By (39) and (40), we deduce that

$$\lim_{y\to z_R+} \Pi(y) - \lim_{y\to z_R-} \Pi(y) = -\frac{2}{\mu - r}\Psi(z_R).$$

□

5. Special case

In this section we consider a special. For $u \in \Sigma$, let us consider the following utility functions:

$$u_B(c) = u(c) - l \quad \text{and} \quad u_A(c) = u(kc + b), \;\; l \geq 0, \;\; k \geq 1, \;\; b \geq 0, \quad \text{with} \;\; (k-1)^2 + l^2 \neq 0. \tag{41}$$

The case $l > 0$, $k = 1$, $b = 0$ is the model of [4], the case $l = 0$, $k > 1$, $b = 0$ is that of [8] and [7], and the case $l > 0$, $b > 0$ is that of [1]. Thus, the example encompasses important models in the literature[5].

By Lemma A.5, all results in Sections 3-4 can be applied when u_A, u_B are given by (41).

Recall that $z_R \in (0, \bar{z})$ is a unique solution satisfying

$$\mathcal{G}(y) = 0,$$

where

$$\mathcal{G}(y) = \int_y^\infty \nu^{-n_1-1}h(\nu)d\nu = \int_y^\infty \nu^{-n_1}\Psi(\nu)d\nu.$$

Moreover, $\mathcal{G}(y)$ is strictly increasing and decreasing in $y \in (0, \bar{z})$ and $y \in (\bar{z}, \infty)$, respectively. Since $\lim_{y\to 0+} \mathcal{G}(y) = -\infty$ and $\mathcal{G}(y) > 0$ for $y \geq \bar{z}$, we can easily derive the following lemma:

Lemma 5.1. *$z_R < ku'(b)$ if and only if $\mathcal{G}(ku'(b)) > 0$.*

In order to obtain a more concrete solution, we assume that the agent has constant relative risk aversion (CRRA), i.e.,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1, \; \gamma > 0, \tag{42}$$

where γ denotes the coefficient of relative risk aversion.

Then,

$$u_B(c) = \frac{c^{1-\gamma}}{1-\gamma} - l \quad \text{and} \quad u_A(c) = \frac{(kc+b)^{1-\gamma}}{1-\gamma},$$

where $l \geq 0$, $k \geq 1$, $b \geq 0$, with $(k-1)^2 + l^2 \neq 0$.

[5][8], [7] and [1] only consider the constant relative risk aversion (CRRA) felicity function.

Note that, under the CRRA utility (42), the integrabiity condition (7) is equivalent to the following assumption:

Assumption 3. The Merton constant M defined by

$$M \equiv r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{\gamma^2}\frac{\theta^2}{2} > 0 \tag{43}$$

is positive.

We can easily obtain that

$$\begin{aligned}
&I_{u_B}(y) = y^{-\frac{1}{\gamma}}, \quad I_{u_A}(y) = \frac{1}{k}\left(\left(\frac{y}{k}\right)^{-\frac{1}{\gamma}} - b\right)\mathbf{1}_{\{y \le kb^{-\gamma}\}},\\
&\tilde{u}_B(y) = \frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}} - l,\\
&\tilde{u}_A(y) = \left(\frac{\gamma}{1-\gamma}\left(\frac{y}{k}\right)^{-\frac{1-\gamma}{\gamma}} + b\frac{y}{k}\right)\mathbf{1}_{\{y \le kb^{-\gamma}\}} + \frac{b^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y > kb^{-\gamma}\}}.
\end{aligned}$$

By Corollary 3.1, the dual value function $J(y)$ is given by

$$J(y) = \begin{cases} Dy^{n_2} + \Xi_{\tilde{u}_B}(y) + \dfrac{\epsilon}{r}y & \text{for } \; y \ge z_R, \\ \Xi_{\tilde{u}_A}(y) & \text{for } \; 0 < y \le z_R, \end{cases}$$

where D is given by equations (62) and (63) in Appendix C.

6. Concluding remarks

We have studied a retirement and consumption/portfolio choice of an individual economic agent. We have identified minimum assumptions for the problem well-defined and the retirement option valuable. We have solved the problem by formulating a Lagrangian which allows us to consider the difference between the dual value functions before retirement and after retirement as the option value of lifetime labor. We have shown that the option value is the present value of the marginal benefit of work compared to retirement.

In this paper we have assumed a constant investment opportunity facing the agent. Consideration of the model to allow general market environment as in [18] would be a useful topic for future research. We have not considered borrowing constraints, which can play an important role for individuals' choice. The extension incorporating borrowing constraints would also be an interesting topic for future research.

Appendix A. Auxiliary lemmas

We state and prove auxiliary lemmas in this section.

Lemma A.1. *For a felicity function $u \in \Sigma$, the following integrability conditions hold*:

$$\int_0^y \nu^{-n_2-1}|u(I_u(\nu))|d\nu + \int_y^\infty \nu^{-n_1-1}|u(I_u(\nu))|d\nu < \infty$$

and

$$\int_0^y \nu^{-n_2-1}|\tilde{u}(\nu)|d\nu + \int_y^\infty \nu^{-n_1-1}|\tilde{u}(\nu)|d\nu < \infty.$$

Proof. Under condition (i) of Section 2, condition (ii) of the section is equivalent to the following: for any $u \in \Sigma$

$$\int_0^y \nu^{-n_2} I_u(\nu)d\nu + \int_y^\infty \nu^{-n_1} I_u(\nu)d\nu < \infty.$$

Note that for any $y > 0$

$$u(I_u(y)) - yI_u(y) = \tilde{u}(y) \text{ and } \tilde{u}'(y) = -I_u(y),$$

and thus

$$u(I_u(\nu)) = u(I_u(y)) - yI_u(y) + \nu I_u(\nu) + \int_\nu^y I_u(\eta)d\eta.$$

It follows that

$$\begin{aligned}
(44) \qquad & \int_0^y \nu^{-n_2-1}|u(I_u(\nu))|d\nu + \int_y^\infty \nu^{-n_1-1}|u(I_u(\nu))|d\nu \\
\leq\ & -\frac{1}{n_2}y^{-n_2}|u(I_u(y)) - yI_u(y)| + \int_0^y \nu^{-n_2} I_u(\nu)d\nu \\
& + \int_0^y\int_\nu^y \nu^{-n_2-1} I_u(\eta)d\eta d\nu \\
& + \frac{1}{n_1}y^{-n_1}|u(I_u(y)) - yI_u(y)| + \int_y^\infty \nu^{-n_1} I_u(\nu)d\nu \\
& + \int_y^\infty\int_y^\nu \nu^{-n_1-1} I_u(\eta)d\eta d\nu \\
=\ & -\frac{1}{n_2}y^{-n_2}|u(I_u(y)) - yI_u(y)| + \frac{1}{n_1}y^{-n_1}|u(I_u(y)) - yI_u(y)| \\
& + \left(1 + \frac{1}{n_1}\right)\int_y^\infty \nu^{-n_1} I_u(\nu)d\nu + \left(1 - \frac{1}{n_2}\right)\int_0^y \nu^{-n_2} I_u(\nu)d\nu < \infty,
\end{aligned}$$

where we have used Fubini's theorem in last equality.

Since $u(I_u(\nu)) = \tilde{u}(\nu) + \nu I_u(\nu)$, we have

$$(45) \qquad \int_0^y \nu^{-n_2-1}|\tilde{u}(\nu)|d\nu + \int_y^\infty \nu^{-n_1-1}|\tilde{u}(\nu)|d\nu < \infty. \qquad \square$$

The following lemma follows directly from Lemma A.1 and Proposition 4.1 in [15]. Here the operators Ξ and Γ are defined in (25) and (26), respectively.

Lemma A.2. *For any $u \in \Sigma$, Γ_{I_u}, $\Xi_{u \circ I_u}$ and $\Xi_{\tilde{u}}$ are well-defined and the following equalities hold*:

$$\Gamma_{I_u}(y) = \frac{1}{y}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathcal{Y}_t^y I_u(\mathcal{Y}_t^y)dt\right], \quad \Xi_{u\circ I_u}(y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}u(I_u(\mathcal{Y}_t^y))dt\right]$$

and

$$\Xi_{\tilde{u}}(y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\tilde{u}(\mathcal{Y}_t^y)dt\right],$$

where $\mathcal{Y}_t^y \equiv ye^{\rho t}\xi_t$.

Lemma A.3. *For any* $u \in \Xi$, *the following statements are true*:

(a) $\Xi'_{\tilde{u}}(y) = -\Gamma_{I_u}(y)$.

(b) $\Gamma_{I_u}(y)$ *is strictly decreasing in* $y > 0$. *Thus,* $\Xi_{\tilde{u}}(y)$ *is strictly convex in* $y > 0$.

(c) $\lim_{y\to\infty}\Gamma_{I_u}(y) = 0$ *and* $\lim_{y\to 0+}\Gamma_{I_u}(y) = \infty$.

(d) $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[\mathcal{Y}_t^y\Gamma_{I_u}(\mathcal{Y}_t^{y^*})\right] = 0$.

(e) *There exist positive constants* ζ_1 *and* ζ_2 *such that*

$$|y\Gamma'_{I_u}(y)| \le \zeta_1(y^{\zeta_2} + y^{-\zeta_2}).$$

Proof. (a) By the definition of $\Xi_{\tilde{u}}(y)$, we have

$$\Xi'_{\tilde{u}}(y) = \frac{2}{\theta^2(n_1-n_2)}\left[n_2y^{n_2-1}\int_0^y \nu^{-n_2-1}\tilde{u}(\nu)d\nu + n_1y^{n_1-1}\int_y^\infty \nu^{-n_1-1}\tilde{u}(\nu)d\nu\right].$$

It follows from Lemma A.1 and Proposition 4.1 in [15] that

$$\liminf_{y\downarrow 0} y^{-n_2}\tilde{u}(y) = \liminf_{y\uparrow\infty} y^{-n_1}\tilde{u}(y) = 0. \tag{46}$$

By applying the integration by parts for Riemann-Stieltjes integral, it follows from the limiting behavior of $\tilde{u}$ in (46) that

$$\begin{aligned}
\Xi'_{\tilde{u}}(y) &= \frac{2}{\theta^2(n_1-n_2)}\left[n_2y^{n_2-1}\int_0^y \nu^{-n_2-1}\tilde{u}(\nu)d\nu + n_1y^{n_1-1}\int_y^\infty \nu^{-n_1-1}\tilde{u}(\nu)d\nu\right]\\
&= -\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2-1}\int_0^y \nu^{-n_2}I_u(\nu)d\nu + y^{n_1-1}\int_y^\infty \nu^{-n_1}I_u(\nu)d\nu\right]\\
&= -\Gamma_{I_u}(y),
\end{aligned}$$

where we have used the fact $(\tilde{u}(y))' = -I_u(y)$.

(b) It follows from $u \in \Sigma$ and Proposition 4.1 in [15] that

$$\liminf_{y\downarrow 0} y^{-n_2}I_u(y) = \liminf_{y\uparrow\infty} y^{-n_1}I_u(y) = 0. \tag{47}$$

The integration by parts for Riemann-Stieltjes integral implies that

$$\begin{aligned}
&\Gamma'_{I_u}(y)\\
&= \frac{2}{\theta^2(n_1-n_2)}\left[(n_2-1)y^{n_2-2}\int_0^y \nu^{-n_2}I_u(\nu)d\nu + (n_1-1)y^{n_1-2}\int_y^\infty \nu^{-n_1}I_u(\nu)d\nu\right]\\
&= \frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2-2}\int_0^y \nu^{1-n_2}I'_u(\nu)d\nu + y^{n_1-2}\int_y^\infty \nu^{1-n_1}I'_u(\nu)d\nu\right] < 0,
\end{aligned}$$

where we have used the fact I_u is strictly decreasing in $y > 0$.

(c) Note that

$$\Gamma_{I_u}(y) = \frac{1}{y}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathcal{Y}_t^y I_u(\mathcal{Y}_t^y)dt\right] = \mathbb{E}\left[\int_0^\infty \xi_t I_y(\mathcal{Y}_t)dt\right].$$

Since $\mathbb{E}\left[\int_0^\infty \xi_t I_u(\mathcal{Y}_t^y)dt\right] < \infty$, the dominated convergence theorem implies that

$$\lim_{y\downarrow 0}\Gamma_{I_u}(y) = \mathbb{E}\left[\int_0^\infty \lim_{y\downarrow 0}\xi_t I_u(ye^{\rho t}\xi_t)dt\right] = \infty$$

and

$$\lim_{y\uparrow\infty}\Gamma_{I_u}(y) = \mathbb{E}\left[\int_0^\infty \lim_{y\uparrow\infty}\xi_t I_u(ye^{\rho t}\xi_t)dt\right] = 0.$$

(d) Note that

$$y\Gamma_{I_u}(y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathcal{Y}_t^y I_u(\mathcal{Y}_t^y)dt\right] < \infty.$$

Thus, it follows from Proposition 4.1 in [15] that

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[\mathcal{Y}_t^y\Gamma_{I_u}(\mathcal{Y}_t^{y^*})\right] = 0.$$

(e) Since $\Gamma_{I_u}(y) = -\Xi'_{\tilde{u}}(y)$, Proposition 4.1 [15] implies that there exits a constant $C_1 > 0$ satisfying

$$|\Gamma_{I_u}(y)| \le C_1(y^{n_1-1} + y^{n_2-1}). \tag{48}$$

Since $y\Gamma_{I_u}(y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\mathcal{Y}_t^y I_u(\mathcal{Y}_t^y)dt\right]$, there exists a constant $C_2 > 0$ such that

$$|(y\Gamma_{I_u}(y))'| \le C_2(y^{n_1-1} + y^{n_2-1}). \tag{49}$$

Thus, we have

$$|y\Gamma'_{I_u}(y)| \le |(y\Gamma_{I_u}(y))'| + |\Gamma_{I_u}(y)| \le (C_1 + C_2)(y^{n_1-1} + y^{n_2-1}).$$

This completes the proof. □

Lemma A.4. *Let us denote $\mathcal{X}(y)$ and $\Pi(y)$ by*

$$\mathcal{X}(y) = -J'(y) \quad \textit{and} \quad \Pi(y) = \frac{\theta}{\sigma}yJ''(y),$$

respectively.

Then, the dynamics of $\mathcal{X}(\mathcal{Y}_t^y)$ follows

$$d\mathcal{X}(\mathcal{Y}_t^y) = \left(r\mathcal{X}(\mathcal{Y}_t^y) + (\mu - r)\Pi(\mathcal{Y}_t^y) - \hat{c}(\mathcal{Y}_t^y) + \epsilon\mathbf{1}_{\{t<\tau_R(y)\}}\right)dt + \sigma\Pi(\mathcal{Y}_t^y)dB_t$$

and

$$\mathcal{X}(y) = \mathbb{E}\left[\int_0^\infty \xi_t(\hat{c}(\mathcal{Y}_t^y) - \epsilon\mathbf{1}_{\{t<\tau_R(y)\}})dt\right].$$

Proof. Since $J(y)$ is smooth in $y > z_R$, it follows that

$$\frac{\theta^2}{2}y^2\mathcal{X}''(y) + (\rho - r + \theta^2)y\mathcal{X}'(y) - r\mathcal{X}(y) + I_{u_B}(y) - \epsilon = 0 \ \text{ for } \ y > z_R.$$

By Proposition 4.1 in [15], the explicit-form of $J_A(y)$ is given by

$$J_A(y) = \frac{2}{\theta^2(n_1 - n_2)}\left[y^{n_2}\int_0^y \nu^{-n_2-1}\tilde{u}_A(\nu)d\nu + y^{n_1}\int_y^\infty \nu^{-n_1-1}\tilde{u}_A(\nu)d\nu\right].$$

Since $J_A(y) = \Xi_{\tilde{u}_A}(y)$, Lemma A.3 implies that $J_A(y)$ is strictly convex in $y > 0$ and

$$\lim_{y\uparrow\infty} J_A'(y) = 0 \quad \text{and} \quad \lim_{y\downarrow 0+} J_A'(y) = -\infty.$$

Moreover, $J_A(y)$ satisfies the following ordinary differential equation (ODE):

$$\frac{\theta^2}{2}y^2 J_A''(y) + (\rho - r)yJ_A'(y) - \rho J_A + \tilde{u}_A(y) = 0. \tag{50}$$

For $0 < y \le z_R$, it follows from the ODE (50) that

$$\frac{\theta^2}{2}y^2\mathcal{X}''(y) + (\rho - r + \theta^2)y\mathcal{X}'(y) - r\mathcal{X}(y) + I_{u_A}(y) = 0 \ \text{ for } \ 0 < y \le z_R.$$

Since

$$\mathcal{X}(y) = \begin{cases} -Dn_2 y^{n_2-1} - \Xi_{\tilde{u}_B}'(y) - \dfrac{\epsilon}{r} & \text{for } \ y \ge z_R, \\ -\Xi_{\tilde{u}_A}'(y) & \text{for } \ 0 < y \le z_R, \end{cases} \tag{51}$$

it is easy to show that

$$\mathcal{X}(\cdot) \in \mathcal{W}^{2,p}_{loc}(0,\infty) \ \text{ for any } \ p \ge 1,$$

where $\mathcal{W}^{2,p}(0,\infty)$, $p \ge 1$, is the completion of $C^\infty(0,\infty)$ under the norm

$$\|\, V\,\|_{\mathcal{W}^{2,p}(0,\infty)} := \left[\int_0^\infty (\,|\,V\,|^{\,p} + |\,\partial_y V\,|^{\,p} + |\,\partial_{yy} V\,|^{\,p}\,)\ dy\right]^{\frac{1}{p}}$$

and $\mathcal{W}^{2,p}_{loc}(0,\infty)$, $p \ge 1$, is the set of all functions whose restrictions to the domain $\mathcal{K}$ belong to $\mathcal{W}^{2,p}(\mathcal{K})$ for any compact subset $\mathcal{K}$ of $(0,\infty)$.

For a given $T > 0$, the generalized Itô's lemma for $W^{2,p}_{loc}$ to $\mathcal{X}(\mathcal{Y}^y_t)$ yields that

$$d\mathcal{X}(\mathcal{Y}^y_t) = \left(r\mathcal{X}(\mathcal{Y}^y_t) - \hat{c}(\mathcal{Y}^y_t) + \epsilon\mathbf{1}_{\{t<\tau_R(y)\}}\right)dt + \sigma\left(-\frac{\theta}{\sigma}\mathcal{Y}^y_t\mathcal{X}'(\mathcal{Y}^y_t)\right)dB^{\mathbb{Q}}_t. \tag{52}$$

Hence, we have

$$\begin{aligned} d\left(e^{-rt}\mathcal{X}(\mathcal{Y}^y_t)\right) &= e^{-rt}\left(\epsilon\mathbf{1}_{\{t<\tau_R(y)\}} - \hat{c}(\mathcal{Y}^y_t)\right)dt \\ &\quad + e^{-rt}\sigma\left(-\frac{\theta}{\sigma}\mathcal{Y}^y_t\mathcal{X}'(\mathcal{Y}^y_t)\right)dB^{\mathbb{Q}}_t, \end{aligned} \tag{53}$$

where the equivalent martingale measure $\mathbb{Q}$ is define as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}\theta^2 T - \theta B_T}. \tag{54}$$

From (51), we deduce that for any $y > 0$

$$\mathcal{X}(y) \le D|n_2|z_R^{n_2-1} + |\Gamma_{I_{u_A}}(y)| + |\Gamma_{I_{u_B}}(y)| + \frac{\epsilon}{r},$$

and

$$|y\mathcal{X}'(y)| \le D|n_2(n_2-1)|z_R^{n_2-1} + |y\Gamma'_{I_{u_A}}(y)| + |y\Gamma'_{I_{u_B}}(y)|.$$

Thus, it follows from Lemma A.3(d) and (e) that

$$\lim_{T\to\infty} e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathcal{X}(\mathcal{Y}_T^y)\right] = \frac{1}{y}\lim_{T\to\infty} e^{-\rho T}\mathbb{E}\left[\mathcal{Y}_T^y\mathcal{X}(\mathcal{Y}_T^y)\right] = 0$$

and the stopped process for T

$$\mathcal{M}_{t\wedge T} = \int_0^{t\wedge T} e^{-rs}\theta\mathcal{Y}_s^y\mathcal{X}'(\mathcal{Y}_s^y)dB_s^{\mathbb{Q}}$$

is a martingale (see Lemma 3.4 in [15]).

By integrating the both sides of (53) with respect to t, we have

$$\mathcal{X}(y) = \mathbb{E}\left[\int_0^T \xi_t\left(\hat{c}(\mathcal{Y}_t^y) - \epsilon)\mathbf{1}_{\{t<\tau_R(y)\}}\right)dt\right] + e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathcal{X}(\mathcal{Y}_T^y)\right].$$

Since $u_A, u_B \in \Sigma$, it follows from Proposition 4.1 in [15] that

$$\mathbb{E}\left[\int_0^\infty \xi_t I_{\tilde{u}_A}(\mathcal{Y}_t^y)dt\right] < \infty \quad\text{and}\quad \mathbb{E}\left[\int_0^\infty \xi_t I_{\tilde{u}_B}(\mathcal{Y}_t^y)dt\right] < \infty.$$

Thus, letting $T \to \infty$, the *dominated convergence theorem* implies that

$$\mathcal{X}(y) = \mathbb{E}\left[\int_0^\infty \xi_t\left(\hat{c}(\mathcal{Y}_t^y) - \epsilon\mathbf{1}_{\{t<\tau_R(y)\}}\right)dt\right].$$

It follows from (52) that

$$\begin{aligned} d\mathcal{X}(\mathcal{Y}_t^y) = &\left(r\mathcal{X}(\mathcal{Y}_t^y) + (\mu-r)\Pi(\mathcal{Y}_t^y) - \hat{c}(\mathcal{Y}_t^y) + \epsilon\mathbf{1}_{\{t<\tau^*\}}\right)dt \\ &+ \sigma\Pi(\mathcal{Y}_t^y)dB_t \quad \text{for } t \ge 0. \end{aligned}$$

□

Lemma A.5. *u_A and u_B in (41) satisfy Assumptions 1-2.*

Proof. We have

$$\tilde{u}_B(y) = \sup_{c\ge 0}(u(c) - l - yc) = u(I_u(y)) - yI_u(y) - l = \tilde{u}(y) - l,$$

$$\begin{aligned} \tilde{u}_A(y) &= \sup_{c\ge 0}(u(kc+b) - yc) \\ &= \left(u(I_u(\frac{y}{k})) - \frac{y}{k}\left(I_u(\frac{y}{k}) - b\right)\right)\mathbf{1}_{\{y/k\le u'(b)\}} + u(b)\mathbf{1}_{\{y/k>u'(b)\}} \\ &= \left(\tilde{u}(\frac{y}{k}) + b\frac{y}{k}\right)\mathbf{1}_{\{y/k\le u'(b)\}} + u(b)\mathbf{1}_{\{y/k>u'(b)\}} \end{aligned}$$

and

$$I_{u_B}(y) = I_u(y) \quad\text{and}\quad I_{u_A}(y) = \frac{1}{k}\left(I_u(\frac{y}{k}) - b\right)\mathbf{1}_{\{y/k\le u'(b)\}}. \tag{55}$$

Since $u \in \Sigma$, it is easy to check that

$$u_A,\ u_B \in \Sigma. \tag{56}$$

Note that

$$\begin{aligned} & u_A(I_A(y)) - u_B(I_A(y)) \\ &= u\left(I_u(\frac{y}{k})\mathbf{1}_{\{y/k\le u'(b)\}} + b\mathbf{1}_{\{y/k>u'(b)\}}\right) - (u(I_u(y)) - l) \\ &\ge u(I_u(\frac{y}{k})) - u(I_u(y)) + l \\ &> 0, \end{aligned} \tag{57}$$

where we have used the fact $(k-1)^2 + l^2 \neq 0$ in the second inequality.

Since $\tilde{u}_A(y) = \sup_{c\ge 0}(u(kc+b) - yc) \ge \sup_{c\ge 0}(u(kc) - yc) = \tilde{u}(\frac{y}{k})$, we deduce that

$$\begin{aligned} \Psi(y) &= \frac{1}{y}\left(\tilde{u}_B(y) - \tilde{u}_A(y)\right) + \epsilon \\ &\le \frac{1}{y}\left(\tilde{u}(y) - l - \tilde{u}(\frac{y}{k})\right) + \epsilon. \end{aligned} \tag{58}$$

If $k = 1$ and $l > 0$, then

$$\lim_{y\to 0+}\Psi(y) \le \lim_{y\to 0+}\frac{1}{y}\left(\tilde{u}(y) - l - \tilde{u}(y)\right) + \epsilon = -\infty. \tag{59}$$

If $k \neq 1$, then it follows from the mean-value theorem that for any $y > 0$ there exists $y_\delta \in (y, y/k)$ such that

$$\Psi(y) \le \frac{1}{y}(\tilde{u}(y) - l - \tilde{u}(\frac{y}{k})) + \epsilon = -\left(1 - \frac{1}{k}\right) I_u(y_\delta) - \frac{l}{y} + \epsilon.$$

Since $\lim_{y\to 0+} I_u(y) = +\infty$, we deduce that

$$\begin{aligned} \lim_{y\to 0+}\Psi(y) &\le \lim_{y\to 0+}\left[\frac{1}{y}(\tilde{u}(y) - l - \tilde{u}(\frac{y}{k})) + \epsilon\right] \\ &= \lim_{y\to 0+}\left[-\left(1 - \frac{1}{k}\right) I_u(y_\delta) - \frac{l}{y} + \epsilon\right] = -\infty. \end{aligned} \tag{60}$$

From (56), (57), (59), and (60), we conclude that u_A and u_B in (41) satisfy Assumptions 1-2. ☐

Appendix B. Proof of Lemma 3.1

It follows from Assumption 1 that

$$\begin{aligned} \Psi'(y) &= -\frac{1}{y^2}\left(\tilde{u}_B(y) - \tilde{u}_A(y)\right) + \frac{1}{y}(-I_{u_B}(y) + I_{u_A}(y)) \\ &= -\frac{1}{y^2}\left((\tilde{u}_B(y) + yI_{u_B}(y)) - (\tilde{u}_A(y) + yI_{u_A}(y))\right) \\ &= -\left(u_B(I_{u_B}(y)) - u_A(I_{u_A}(y))\right) > 0. \end{aligned}$$

Hence, $\Psi(y)$ is a strictly increasing function of $y>0$.

(i) If $\lim_{y\to\infty} |\,(\tilde{u}_B(y)-\tilde{u}_A(y))\,| = \infty$, it follows form L'Hospital's rule that
$$\lim_{y\to\infty}\frac{\tilde{u}_B(y)-\tilde{u}_A(y)}{y}=\lim_{y\to\infty}\frac{-I_{u_B}(y)+I_{u_A}(y)}{1}=0.$$

(ii) If $\lim_{y\to\infty} |\,(\tilde{u}_B(y)-\tilde{u}_A(y))\,| < \infty$, it is clear that
$$\lim_{y\to\infty}\frac{\tilde{u}_B(y)-\tilde{u}_A(y)}{y}=0.$$

By (i) and (ii), we deduce that
$$\lim_{y\to\infty}\Psi(y)=\epsilon.$$

Appendix C. Derivation of the dual value function for the special case in Section 5

Since
$$\Xi_{\tilde{u}_B}(y)=\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \nu^{-n_2-1}\tilde{u}_B(\nu)d\nu+y^{n_1}\int_y^\infty \nu^{-n_1-1}\tilde{u}_B(\nu)d\nu\right],$$
$$\Xi_{\tilde{u}_A}(y)=\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \nu^{-n_2-1}\tilde{u}_A(\nu)d\nu+y^{n_1}\int_y^\infty \nu^{-n_1-1}\tilde{u}_A(\nu)d\nu\right],$$
we have
$$\begin{aligned}\Xi_{\tilde{u}_B}(y)=&\frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \nu^{-n_2-1}\tilde{u}_B(\nu)d\nu+y^{n_1}\int_y^\infty \nu^{-n_1-1}\tilde{u}_B(\nu)d\nu\right]\\ =&\frac{1}{M}\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}-\frac{l}{\rho},\end{aligned}$$
and
$$\Xi_{\tilde{u}_A}(y)=\begin{cases}\phi_1(y) & \text{for } \ 0<y\le kb^{-\gamma},\\ \phi_2(y) & \text{for } \ y>kb^{-\gamma},\end{cases} \tag{61}$$
where $\phi_1(y)$ and $\phi_2(y)$ are given by
$$\begin{aligned}\phi_1(y)=&\left(\frac{\frac{1}{M}\left(\frac{\gamma n_2}{1-\gamma}+1\right)+\frac{n_2-1}{r}-\frac{n_2}{\rho(1-\gamma)}}{n_1-n_2}\right)b^{1-\gamma+\gamma n_1}\left(\frac{y}{k}\right)^{n_1}\\ &+\frac{1}{K}\frac{\gamma}{1-\gamma}(\frac{y}{k})^{-\frac{1-\gamma}{\gamma}}+\frac{b}{r}\frac{y}{k}\end{aligned}$$
and
$$\phi_2(y)=\left(\frac{\frac{1}{M}\left(\frac{\gamma n_1}{1-\gamma}+1\right)+\frac{n_1-1}{r}-\frac{n_1}{\rho(1-\gamma)}}{n_1-n_2}\right)b^{1-\gamma+\gamma n_2}\left(\frac{y}{k}\right)^{n_2}+\frac{b^{1-\gamma}}{\rho(1-\gamma)},$$
respectively.

Since the current normalized marginal benefit of work $\Psi(y)$ is given by

$$\Psi(y) = \frac{1}{y}\left[\frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}} - l - \left(\frac{\gamma}{1-\gamma}\left(\frac{y}{k}\right)^{-\frac{1-\gamma}{\gamma}} + b\frac{y}{k}\right)\mathbf{1}_{\{y\le kb^{-\gamma}\}} - \frac{b^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y>kb^{-\gamma}\}}\right] + \epsilon,$$

we have

$$\mathcal{G}(kb^{-\gamma}) = -\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_1-\frac{1}{\gamma}}(kb^{-\gamma})^{1-n_1-\frac{1}{\gamma}} + l\frac{(kb^{-\gamma})^{-n_1}}{n_1} + \frac{b^{1-\gamma}}{1-\gamma}\frac{(kb^{-\gamma})^{-n_1}}{n_1} - \epsilon\frac{(kb^{-\gamma})^{1-n_1}}{n_1-1}\right].$$

Lemma 5.1 implies that
(Case 1) $\mathcal{G}(kb^{-\gamma}) > 0$ (or equivalently, $z_R < kb^{-\gamma}$)
In this case, the coefficient D is given by
(62)

$$D = -\frac{2}{\theta^2(n_1-n_2)}\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_2-\frac{1}{\gamma}}\left(1-k^{\frac{1-\gamma}{\gamma}}\right)(z_R)^{1-n_2-\frac{1}{\gamma}} + l\frac{(z_R)^{-n_2}}{n_2} - \frac{(z_R)^{1-n_2}}{n_2-1}\left(\epsilon - \frac{b}{k}\right)\right]$$

and z_R is a unique solution of the following algebraic equation:

$$\begin{aligned}
0 &= \mathcal{G}(z_R)\\
&= \left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_1-\frac{1}{\gamma}}\left(1-k^{\frac{1-\gamma}{\gamma}}\right)(kb^{-\gamma})^{1-n_1-\frac{1}{\gamma}} + l\frac{(kb^{-\gamma})^{-n_1}}{n_1} - \frac{(kb^{-\gamma})^{1-n_1}}{n_1-1}\left(\epsilon-\frac{b}{k}\right)\right]\\
&\quad - \left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_1-\frac{1}{\gamma}}\left(1-k^{\frac{1-\gamma}{\gamma}}\right)(z_R)^{1-n_1-\frac{1}{\gamma}} + l\frac{(z_R)^{-n_1}}{n_1} - \frac{(z_R)^{1-n_1}}{n_1-1}\left(\epsilon-\frac{b}{k}\right)\right]\\
&\quad - \left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_1-\frac{1}{\gamma}}(kb^{-\gamma})^{1-n_1-\frac{1}{\gamma}} + l\frac{(kb^{-\gamma})^{-n_1}}{n_1} + \frac{b^{1-\gamma}}{1-\gamma}\frac{(kb^{-\gamma})^{-n_1}}{n_1} - \epsilon\frac{(kb^{-\gamma})^{1-n_1}}{n_1-1}\right].
\end{aligned}$$

(Case 2) $\mathcal{G}(kb^{-\gamma}) \le 0$ (or equivalently, $z_R \ge kb^{-\gamma}$)
In this case, the coefficient D is given by
(63)

$$\begin{aligned}
D = &-\frac{2}{\theta^2(n_1-n_2)}\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_2-\frac{1}{\gamma}}\left(1-k^{\frac{1-\gamma}{\gamma}}\right)(kb^{-\gamma})^{1-n_2-\frac{1}{\gamma}} + l\frac{(kb^{-\gamma})^{-n_2}}{n_2} - \frac{(kb^{-\gamma})^{1-n_2}}{n_2-1}\left(\epsilon-\frac{b}{k}\right)\right]\\
&-\frac{2}{\theta^2(n_1-n_2)}\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_2-\frac{1}{\gamma}}(z_R)^{1-n_2-\frac{1}{\gamma}} + l\frac{(z_R)^{-n_2}}{n_2} + \frac{b^{1-\gamma}}{1-\gamma}\frac{(z_R)^{-n_2}}{n_2} - \epsilon\frac{(z_R)^{1-n_2}}{n_2-1}\right]\\
&-\frac{2}{\theta^2(n_1-n_2)}\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_2-\frac{1}{\gamma}}(kb^{-\gamma})^{1-n_2-\frac{1}{\gamma}} + l\frac{(kb^{-\gamma})^{-n_2}}{n_2} + \frac{b^{1-\gamma}}{1-\gamma}\frac{(kb^{-\gamma})^{-n_2}}{n_2} - \epsilon\frac{(kb^{-\gamma})^{1-n_2}}{n_2-1}\right]
\end{aligned}$$

and z_R is a unique solution of the following algebraic equation:

$$\begin{aligned}
0 &= \mathcal{G}(z_R)\\
&= -\left[\frac{\gamma}{1-\gamma}\frac{1}{1-n_1-\frac{1}{\gamma}}(z_R)^{1-n_1-\frac{1}{\gamma}} + l\frac{(z_R)^{-n_1}}{n_1} + \frac{b^{1-\gamma}}{1-\gamma}\frac{(z_R)^{-n_1}}{n_1} - \epsilon\frac{(z_R)^{1-n_1}}{n_1-1}\right].
\end{aligned}$$

D. Review of Proposition 4.1 in [15]

Proposition D.1 (Proposition 4.1 in [15])**.** *Let $f(y)$ be an arbitrary measurable function defined on $(0,\infty)$. Then the following conditions are equivalent:*

(i) *for every* $y > 0$

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}|f(\mathcal{Y}_t)|dt\right] < \infty,$$

(ii) *for every* $y > 0$

$$\int_0^y \nu^{-n_2-1}|f(\nu)|d\nu + \int_y^\infty \nu^{-n_1-1}|f(\nu)|d\nu < \infty.$$

Let us denote $\Xi_f(y)$ *by*

$$\Xi_f(y) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} f(\mathcal{Y}_t)dt\right].$$

Under the condition (i) *or* (ii), *the following statements are true*:

(a) $\liminf_{y\downarrow 0} y^{-n_2}|f(y)| = \liminf_{y\uparrow\infty} y^{-n_1}|f(y)| = 0,$

(b) Ξ_f *has a following form*:

$$\Xi_f(y) = \frac{2}{\theta^2(n_1-n_2)}\left[y^{n_2}\int_0^y \nu^{-n_2-1}f(\nu)d\nu + y^{n_1}\int_y^\infty \nu^{-n_1-1}f(\nu)d\nu\right],$$

(c) Ξ_f *is twice differentiable and*

$$\frac{\theta^2}{2}y^2\Xi_f''(y) + (\rho - r)y\Xi_f'(y) - \rho\Xi_f(y) + f(y) = 0,$$

(d) *there exists a positive constant* C *such that*

$$|\Xi_f'(y)| \le C(y^{n_1-1} + y^{n_2-1}) \quad \textit{for all } y > 0,$$

(e) $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[|\Xi_f(\mathcal{Y}_t)|\right] = 0.$

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