Bull. Korean Math. Soc. **60** (2023), No. 4, pp. 1071–1083
https://doi.org/10.4134/BKMS.b220516
pISSN: 1015-8634 / eISSN: 2234-3016

ON SEMI-REGULAR INJECTIVE MODULES AND STRONG DEDEKIND RINGS

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Abstract. The main motivation of this paper is to introduce and study the notions of strong Dedekind rings and semi-regular injective modules. Specifically, a ring R is called strong Dedekind if every semi-regular ideal is Q_0-invertible, and an R-module E is called a semi-regular injective module provided $\mathrm{Ext}^1_R(T,E)=0$ for every $\mathcal{Q}$-torsion module T. In this paper, we first characterize rings over which all semi-regular injective modules are injective, and then study the semi-regular injective envelopes of R-modules. Moreover, we introduce and study the semi-regular global dimensions sr-gl.dim(R) of commutative rings R. Finally, we obtain that a ring R is a DQ-ring if and only if sr-gl.dim$(R)=0$, and a ring R is a strong Dedekind ring if and only if sr-gl.dim$(R)\leq 1$, if and only if any semi-regular ideal is projective. Besides, we show that the semi-regular dimensions of strong Dedekind rings are at most one.

1. Introduction

In this paper, we always assume R is a commutative ring with identity and $\mathrm{T}(R)$ is the total ring of fractions of R. An ideal I of R is said to be dense if $(0:_R I):=\{r\in R\,|\,Ir=0\}$ is 0, or be semi-regular if it contains a finitely generated dense sub-ideal, or be regular if it contains a regular element. Let I be an ideal of R. Denote by $I^{-1}=\{z\in \mathrm{T}(R)\,|\,Iz\subseteq R\}$. If an ideal I of R satisfies $II^{-1}=R$, then I is said to be an invertible ideal. It is well-known that Dedekind domains are domains over which every nonzero ideal is invertible. Dedekind domains have many classical characterizations from various points of views, such as a domain is a Dedekind domain if and only if every ideal is projective, if and only if every quotient module of injective module is injective, if and only its global dimension is at most one (see [15] for example). So it is an important thing to generalize Dedekind domains to commutative rings with zero divisors. Recently, Elliott [5] defined Dedekind rings to be rings over

Received July 26, 2022; Revised October 12, 2022; Accepted October 28, 2022.
2020 *Mathematics Subject Classification.* 13F05, 13C11.
Key words and phrases. Strong Dedekind ring, DQ-ring, semi-regular injective module, semi-regular global dimension.

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which every regular ideal is invertible. He also gave some characterizations of Dedekind rings (see [5, Theorem 3.7.29]).

In 1932, Prüfer [13] introduced integral domains over which all finitely generated non-zero ideals are invertible, which are named as Prüfer domains by Krull [10]. Since Prüfer domains are of great importance to the study of integral domains, many scholars generalized the notion of integral domains to these of commutative rings with zero-divisors. In 1967, Butts and Smith [3] introduced the notion of Prüfer rings over which every finitely generated regular ideal is invertible. Since the notion of Prüfer rings is very simple, it is very hard to delve deeper (note that all total rings of quotients are Prüfer rings). For better understanding Prüfer rings, Anderson et al. [1] introduced the notion of strong Prüfer rings, over which every finitely generated semi-regular ideal is locally principal. In order to characterize strong Prüfer rings by "invertible" ideals, Lucas [11] developed the ring $Q_0(R)$ of finite fractions:

$$Q_0(R)=\left\{\frac{b_nx^n+\cdots+b_0}{a_nx^n+\cdots+a_0}\in \mathrm{T}(R[x])\mid b_ia_j=a_ib_j \text{ for any } i,j\right\}.$$

An ideal I of R is said to be Q_0-invertible if there is an R-submodule J of $Q_0(R)$ such that $IJ=R$. Then Lucas [11] proved that a ring R is a strong Prüfer ring if and only if every finitely generated semi-regular ideal of R is Q_0-invertible.

The main motivation of this paper is to introduce and study commutative rings over which every semi-regular ideal is Q_0-invertible (which are called strong Dedekind rings in Definition 4.7). It is well-known that an integral domain R is a Dedekind domain if and only every quotient of injective R-module is injective, if and only if the global dimension of R is at most one. In order to give a homological characterization of strong Dedekind rings, we introduce the notion of semi-regular injective modules using $\mathcal{Q}$-torsion theories. We obtain the Baer's Criterion for semi-regular injective modules (see Theorem 2.2) and characterize commutative rings with all semi-regular injective modules injective when R is a WQ-ring (see Theorem 2.4). We also introduce and study the semi-regular injective envelopes of R-modules (see Definition 2.6 and Proposition 2.7). Then, we introduce and study the semi-regular injective dimensions of R-modules and semi-regular global dimensions sr-gl.dim(R) of commutative rings R. Finally, we show that a ring R is a DQ-ring if and only if sr-gl.dim$(R)=0$, and a ring R is a strong Dedekind ring if and only if sr-gl.dim$(R)\leq 1$, if and only if every semi-regular ideal is projective.

2. Semi-regular injective modules

Denote by $\mathcal{Q}$ the set of all finitely generated semi-regular ideals of R. We recall from [16] about some basic notions on $\mathcal{Q}$-torsion theory. Let M be an R-module. We denote by

$$\mathrm{Tor}_{\mathcal{Q}}(M)=\{m\in M\mid \text{there exists } I\in\mathcal{Q} \text{ such that } Im=0\}.$$

Then M is said to be $\mathcal{Q}$-torsion (resp., $\mathcal{Q}$-torsion free) provided that $\mathrm{Tor}_{\mathcal{Q}}(M) = 0$ (resp., $\mathrm{Tor}_{\mathcal{Q}}(M) = M$). Certainly, an ideal I such that R/I is $\mathcal{Q}$-torsion is semi-regular.

Definition 2.1. An R-module E is said to be a semi-regular injective module provided that $\mathrm{Ext}^1_R(T, E) = 0$ for every $\mathcal{Q}$-torsion module T. The class of all semi-regular injective modules is denoted by $\mathcal{I}_{sr}$.

Obviously, an R-module E is semi-regular injective if and only if for every short exact sequence $0 \to M \xrightarrow{i} N \to T \to 0$ with T $\mathcal{Q}$-torsion and every R-homomorphism $f : M \to E$, there exists an R-homomorphism $g : N \to E$ such that $g \circ i = f$, if and only if every short exact sequence $0 \to E \to N \to T \to 0$ with T $\mathcal{Q}$-torsion splits. Next, we establish Baer's Criterion for semi-regular injective modules.

Theorem 2.2 (Baer's criterion for semi-regular injective modules)**.** *An R-module E is semi-regular injective if and only if for every semi-regular ideal I, $\mathrm{Ext}^1_R(R/I, E) = 0$.*

Proof. Suppose E is a semi-regular injective R-module. Then, trivially, $\mathrm{Ext}^1_R(R/I, E) = 0$ for every semi-regular ideal I.

On the other hand, suppose E is an R-module satisfying $\mathrm{Ext}^1_R(R/I, E) = 0$ for every semi-regular ideal I. Let B be an R-module, A a submodule of B such that B/A is $\mathcal{Q}$-torsion. Let $f : A \to E$ be an R-homomorphism. Set

$$\Gamma = \{(C, d) \mid C \text{ is a submodule of } B \text{ containing } A \text{ and } d|_A = f\}.$$

Since $(A, f) \in \Gamma$, Γ is nonempty. Set $(C_1, d_1) \leq (C_2, d_2)$ if and only if $C_1 \subseteq C_2$ and $d_2|_{C_1} = d_1$. Then Γ is a partial order. For every chain (C_j, d_j), let $C_0 = \bigcup_j C_j$ and $d_0(c) = d_j(c)$ if $c \in C_j$. Then (C_0, d_0) is the upper bound of the chain (C_j, d_j). By Zorn's Lemma, there is a maximal element (C, d) in Γ.

We claim that $C = B$. On the contrary, let $x \in B - C$. Denote $I = \{r \in R \mid rx \in C\}$. Since B/A is $\mathcal{Q}$-torsion, so is the quotient module B/C. Thus the submodule $(Rx + C)/C \cong R/I$ is also $\mathcal{Q}$-torsion. It follows that I is a semi-regular ideal of R. Let $h : I \to E$ be an R-homomorphism satisfying $h(r) = d(rx)$. By assumption, there is an R-homomorphism $g : R \to E$ such that $g(r) = h(r) = d(rx)$ for every $r \in I$. Let $C_1 = C + Rx$ and $d_1(c + rx) = d(c) + g(r)$ where $c \in C$ and $r \in R$. If $c + rx = 0$, then $r \in I$ and thus $d(c) + g(r) = d(c) + h(r) = d(c) + d(rx) = d(c + rx) = 0$. Hence d_1 is a well-defined homomorphism such that $d_1|_A = f$. So $(C_1, d_1) \in \Gamma$. However, $(C_1, d_1) > (C, d)$ which contradicts the maximality of (C, d). □

Lemma 2.3. *Consider the following commutative diagram with rows exact*:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i_1"] \arrow[d, "k_1"] & A_2 \arrow[r, "i_2"] \arrow[d, "k_2"] & A_3 \arrow[r] \arrow[d, "k_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "j_1"] & B_2 \arrow[r, "j_1"] & B_3 \arrow[r] & 0
\end{tikzcd}$$

where A_1 is the pull-back of j_1 and k_2. If k_2 is a monomorphism, then k_1 and k_3 are monomorphisms.

Proof. Suppose k_2 is a monomorphism. Then $\mathrm{Ker}k_1 \cong \mathrm{Ker}k_2 = 0$ by [8, Theorem 6.2]. So k_1 is a monomorphism. To show k_3 is also a monomorphism, by Snake Lemma, we just need to prove the induced homomorphism $\alpha : \mathrm{Cok}(k_1) \to \mathrm{Cok}(k_2)$ is a monomorphism. Set $B = \mathrm{Cok}\binom{i_1}{k_1}$. Then there is a short exact sequence

$$0 \to A_1 \xrightarrow{\binom{i_1}{k_1}} A_2 \oplus B_1 \xrightarrow{(l_1,-l_2)} B \to 0.$$

So, by [8, Exercise 6.7], the following diagram is a push-out:

$$\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "k_1"] & A_2 \arrow[d, "l_1"] \\
B_1 \arrow[r, "l_2"] & B
\end{tikzcd}$$

So by the dual of [8, Theorem 6.2], we have $\mathrm{Cok}(k_1) \cong \mathrm{Cok}(l_1)$ and $\mathrm{Cok}(i_1) \cong \mathrm{Cok}(l_2)$. Note that we have the following commutative diagram with rows exact:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "\binom{i_1}{k_1}"] \arrow[d, equal] & A_2 \oplus B_1 \arrow[r, "{(l_1,-l_2)}"] \arrow[d, equal] & B \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "\binom{i_1}{k_1}"] & A_2 \oplus B_1 \arrow[r, "{(k_2,-j_1)}"] & B_2 &
\end{tikzcd}$$

By Snake Lemma, l is a monomorphism. Note that $l \circ l_1 = k_2$ and $l \circ l_2 = j_1$. So we have the following commutative diagram with rows exact:

$$\begin{tikzcd}
A_2 \arrow[r, "l_1"] \arrow[d, equal] & B \arrow[r, "i_2"] \arrow[d, hook, "l"] & \mathrm{Cok}(l_1) \arrow[r] \arrow[d, "t"] & 0 \\
A_2 \arrow[r, "k_2"] & B_2 \arrow[r, "j_1"] & \mathrm{Cok}(k_2) \arrow[r] & 0
\end{tikzcd}$$

So, by the Five Lemma, we have t is also a monomorphism. Hence k_3 is a monomorphism. $\square$

Trivially, every injective module is semi-regular injective. However, the converse is not true (see Example 4.2). Following [15], a finitely generated ideal J of R is said to be a GV-ideal if and only if $\mathrm{Hom}_R(R/J, R) = \mathrm{Ext}^1_R(R/J, R) = 0$. The set of all GV-ideals of R is denoted by $\mathrm{GV}(R)$. A ring R is said to be a DW ring provided that $\mathrm{GV}(R) = \{R\}$. Recall from [18] that a ring R is said to be a WQ-ring if every finitely generated semi-regular ideal is a GV-ideal, which is equivalent to every $\mathcal{Q}$-torsion module is GV-torsion. We now characterize commutative rings R with all semi-regular injective modules injective, when R is a WQ-ring.

Theorem 2.4. *Let R be a* WQ*-ring. Then every semi-regular injective R-module is injective if and only if for every ideal I of R, there exist a semi-regular ideal J and a projective ideal K of R such that $J = I \oplus K$.*

Proof. Suppose R is a WQ-ring satisfying that every semi-regular injective R-module is injective. Let I be an ideal of R. Consider the cotorsion pair $(\mathcal{T}, \mathcal{I}_{sr})$ where $\mathcal{T} = ^{\perp_1} \mathcal{I}_{sr}$. Then we have $\mathcal{T} = R\text{-}Mod$. In particular, $R/I \in \mathcal{T}$. Following [7, Corollary 6.13(b)], there is an R-module N such that R/I is a direct summand of N and an exact sequence $0 \to F \xrightarrow{i} N \to T \to 0$ where F is free and T is $\mathcal{Q}$-torsion. Let $j : R/I \to N$ be the splitting monomorphism. Consider the pull-back of i and j:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "s"] \arrow[d, "l"] & R/I \arrow[r] \arrow[d, hook, "j"] & T_1 \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & F \arrow[r, "i"] & N \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

Then k is a monomorphism by Lemma 2.3. Since T is $\mathcal{Q}$-torsion, T_1 is also $\mathcal{Q}$-torsion. Hence $T_1 \cong R/J$ for some semi-regular ideal J which contains I. Now we claim l is a splitting monomorphism. Indeed consider the following pull-back and push-out:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "s"] \arrow[d, "l"] & R/I \arrow[r] \arrow[d, "j'"] & T_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r, "i'"] \arrow[d, "l'"] & X \arrow[r] \arrow[d, "k'"] & T_1 \arrow[r] & 0 \\
 & Y \arrow[r, equal] & Y & &
\end{tikzcd}$$

By the proof of Lemma 2.3, there exists a monomorphism $l : X \to N$ such that $l \circ j' = j$. Let $\pi : N \to R/I$ be the retraction of j. Then $\pi \circ l \circ j' = \pi \circ j = \mathrm{Id}_{R/I}$. Hence j' is also a splitting monomorphism. Set $\pi_1 : X \to R/I$ to be the retraction of j'. Since R is a WQ ring, T_1 is $\mathcal{Q}$-torsion, then T_1 is GV-torsion. Since F is free, the bottom short exact sequence splits. Set $\pi_2 : X \to F$ to be the retraction of i'. Consider the short exact sequence $0 \to P \xrightarrow{\binom{s}{l}} R/I \oplus F \xrightarrow{(j',-i)} X \to 0$. Then $(\pi_1, -\pi_2)$ is a retraction of $(j', -i)$. Hence l is a splitting monomorphism. Since F is free, we have $P \cong J/I$ is projective. Consider the splitting short exact sequence $0 \to I \to J \to J/I \to 0$. Then there is a projective ideal K of R such that $J = I \oplus K$.

On the other hand, suppose M is a semi-regular injective module. Let I be an ideal of R. Then there exists a semi-regular ideal J such that $J = I \oplus K$ for some projective ideal K of R. Then we have an exact sequence $0 = \mathrm{Ext}^1_R(K, M) \to \mathrm{Ext}^1_R(R/I, M) \to \mathrm{Ext}^1_R(R/J, M) = 0$ by Theorem 2.2. Hence $\mathrm{Ext}^1_R(R/I, M) = 0$. So M is injective. $\square$

Remark 2.5. If R is a finite direct product of integral domains, then every semi-regular injective R-module is injective. So the WQ property is not necessary for semi-regular injective R-modules to be injective. However, we do not know whether the condition "R is a WQ-ring" could be omitted in Theorem 2.4.

It is well-known that every R-module has an injective envelope. In the rest of this section, we introduce the semi-regular injective envelopes of R-modules and show that every R-module also has a semi-regular injective envelope.

Definition 2.6. Let M be an R-module. Suppose E is a semi-regular injective module containing M such that E/M is $\mathcal{Q}$-torsion. If E is an essential extension of M, then E is said to be a semi-regular injective envelope of M.

Let $\mathcal{I}$ be a class of R-modules and M an R-module. Following [7, Definition 5.1], an R-homomorphism $f : M \to I$ with $I \in \mathcal{I}$ is said to be an $\mathcal{I}$-preenvelope provided the natural homomorphism $\mathrm{Hom}_R(I, I') \to \mathrm{Hom}_R(M, I')$ is an epimorphism for every $I' \in \mathcal{I}$. If, moreover, every endomorphism h of I such that $f = f \circ h$ is an isomorphism, then f is said to be an $\mathcal{I}$-envelope.

Proposition 2.7. *Let R be a ring. The following statements hold.*

(1) *Any R-module has a semi-regular injective envelope.*
(2) *Suppose E is a semi-regular injective envelope of M. Then there is no other semi-regular injective module strictly between M and E.*
(3) *Suppose E is a semi-regular injective envelope of M. Then the embedding map $M \hookrightarrow E$ is an $\mathcal{I}_{sr}$-envelope.*
(4) *The semi-regular injective envelope of an R-module is unique up to isomorphism.*

Proof. (1) Let M be an R-module and $E(M)$ the injective envelope of M. Consider the following set

$$\Gamma := \{N \mid M \le N \le E(M) \text{ and } N/M \text{ is } sr\text{-torsion}\}.$$

Clearly Γ is a partially ordered set by conclusion. Let Γ' be a chain in Γ. Then $(\bigcup_{N_i \in \Gamma'} N_i)/M$ is an upper bound of the chain. So there is a maximal element E in Γ by Zorn Lemma. We claim that E is semi-regular injective. Indeed, consider an exact sequence $0 \to E \xrightarrow{f} N \to T \to 0$ with T $\mathcal{Q}$-torsion. Since $i : E \hookrightarrow E(M)$ is the injective envelope of E, there exists an R-homomorphism $g : N \to E(M)$ such that the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[rr, "f"] \arrow[dr, "i"'] & & N \arrow[r] \arrow[dl, "g"] & T \arrow[r] & 0 \\
& & E(M) & & &
\end{tikzcd}$$

So $E \subseteq g(N) \subseteq E(M)$. And hence $T = N/E \to g(N)/M \to 0$ is exact. So $g(N) \in \Gamma$. Since E is maximal in Γ, we have $g(N) = E$. Consequently, the exact sequence $0 \to E \xrightarrow{f} N \to T \to 0$ splits. So E is semi-regular injective.

(2) Suppose E' is an semi-regular injective module strictly between M and E. Consider the exact sequence $0 \to E' \to E \to E/E' \to 0$. Since E/E' is the quotient of $\mathcal{Q}$-torsion module E/M, so E/E' is also $\mathcal{Q}$-torsion. Hence E' is a direct summand of E, which is a contradiction to E is an essential extension of M.

(3) First, we will show the embedding map $M \hookrightarrow E$ is an $\mathcal{I}_{sr}$-preenvelope. Let $f : M \to E'$ be an R-homomorphism with E' semi-regular injective. Consider the exact sequence $0 \to M \xrightarrow{i} E \to T \to 0$. Then T is $\mathcal{Q}$-torsion. So there is an R-homomorphism $g : E \to E'$ such that $g \circ i = f$. Hence $M \hookrightarrow E$ is an $\mathcal{I}_{sr}$-preenvelope. Next, we will show $M \hookrightarrow E$ is an $\mathcal{I}_{sr}$-envelope. Let $h : E \to E$ be an R-homomorphism such that $h \circ i = i$. Since i is an essential extension, we have h is a monomorphism. So $\mathrm{Im}(h) \cong E$ is a semi-regular injective module. Hence $\mathrm{Im}(h) = E$ by (2). So h is an isomorphism. Hence $M \hookrightarrow E$ is an $\mathcal{I}_{sr}$-envelope.

(4) Follows from (3). $\square$

3. Semi-regular injective dimensions and semi-regular global dimensions

In this section, we introduce semi-regular injective dimensions of R-modules and semi-regular global dimensions of commutative rings.

Definition 3.1. Let R be a ring and M an R-module. We write $Inj_{sr}.\dim(M) \leq n$ ($Inj_{sr}.\dim$ abbreviates *semi-regular injective dimension*) if there is an exact sequence of R-modules

$$(\diamondsuit) \qquad 0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

with each E_i injective ($i = 0, \ldots, n-1$) and E_n semi-regular injective. The exact sequence ($\diamondsuit$) is said to be a semi-regular injective resolution of length n of M. The semi-regular injective dimension $Inj_{sr}.\dim(M)$ is defined to be the length of the shortest semi-regular injective resolution of M. If no such finite resolution ($\diamondsuit$) exists, then we say $Inj_{sr}.\dim(M) = \infty$.

Proposition 3.2. *Let R be a ring. The following statements are equivalent for a non-semi-regular injective R-module M:*

1. $Inj_{sr}.\dim(M) \leq n$;
2. $\mathrm{Ext}_R^{n+1}(T, M) = 0$ *for every $\mathcal{Q}$-torsion module T;*
3. $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ *for every semi-regular ideal I;*
4. *If $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ with each E_i injective ($i = 0, \ldots, n-1$), then E_n is semi-regular injective.*

Proof. (1) $\Rightarrow$ (2) We prove (2) by induction on n. For the case $n = 0$, (2) holds naturally. If $n > 0$, then there is an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ with each E_i injective ($i = 0, \ldots, n-1$) and E_n semi-regular injective. Let $L_0 = \ker(M \to E_0)$. We have two exact sequences $0 \to M \to E_0 \to L_0 \to 0$ and $0 \to L_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$. Note that

$Inj_{sr}.\dim(L_0) \leq n$. By induction, $\mathrm{Ext}_R^n(T, L_0) = 0$ for every $\mathcal{Q}$-torsion module T. Since E_0 is an injective module, we have $\mathrm{Ext}_R^{n+1}(T, M) \cong \mathrm{Ext}_R^n(T, L_0) = 0$.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (4) Let $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$ be an exact sequence with each E_i injective ($i = 0, \ldots, n-1$) and E_n semi-regular injective. Set $L_n = E_n$ and $L_i = \mathrm{Im}(E_i \to E_{i+1})$, where $i = 0, \ldots, n-1$. Then both $0 \to L_{i-1} \to E_i \to L_i \to 0$ and $0 \to L_1 \to E_0 \to M \to 0$ are exact sequences. By dimension shift repeatedly, we can obtain that $\mathrm{Ext}_R^1(R/I, E_n) \cong \mathrm{Ext}_R^{n+1}(R/I, M) = 0$ for all semi-regular ideals I. Thus E_n is semi-regular injective by Proposition 2.2.

(4) $\Rightarrow$ (1) It follows from that every R-module has an injective envelope. $\square$

Definition 3.3. The *semi-regular global dimension* of a ring R is defined by

$$sr\text{-gl.dim}(R) = \sup\{Inj_{sr}.\dim(M) \mid M \text{ is an } R\text{-module}\}.$$

The following result can easily be deduced by Proposition 3.2.

Proposition 3.4. *The following statements are equivalent for a ring R:*

(1) *sr-$gl.dim(R) \leq n$;*
(2) *$Inj_{sr}.\dim(M) \leq n$ for all R-modules M;*
(3) *$\mathrm{Ext}_R^{n+1}(T, M) = 0$ for all R-modules M and all $\mathcal{Q}$-torsion R-module T;*
(4) *$\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ for all R-modules M and all semi-regular ideals I of R.*

4. Rings with semi-regular global dimensions at most one

Recall from [17, Porposition 2.2], a ring R is called a DQ ring provided that the only finitely generated semi-regular ideal of R is R itself. It is well-known that a ring R is a semi-simple ring if and only if every R-module is injective. Now we characterize commutative rings over which all R-modules are semi-regular injective.

Proposition 4.1. *The following statements are equivalent for a ring R:*

(1) *sr-$gl.dim(R) = 0$;*
(2) *All R-modules are semi-regular injective;*
(3) *For every semi-regular ideal I and every ideal J, we have $I \cap J = IJ$;*
(4) *For every semi-regular ideal I, R/I is flat module;*
(5) *For every semi-regular ideal I and $a \in I$, there is $c \in I$ such that $(1-c)a = 0$;*
(6) *R is a* DQ *ring;*
(7) $\mathrm{fPD}(R) = 0$;
(8) *The only semi-regular ideal of R is R itself.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (8) Trivial.

(6) $\Leftrightarrow$ (7) See [17, Proposition 2.2].

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) See [6, Theorem 1.2.15].

(2) $\Rightarrow$ (3) Let I be a semi-regular ideal of R and J a ideal of R. Then $\mathrm{Hom}_{\mathbb{Z}}(R/J,\mathbb{Q}/\mathbb{Z})$ is a semi-regular injective module. So

$$\mathrm{Ext}^1_R(R/I,\mathrm{Hom}_{\mathbb{Z}}(R/J,\mathbb{Q}/\mathbb{Z}))\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(R/I,R/J),\mathbb{Q}/\mathbb{Z})=0.$$

Thus $\mathrm{Tor}^R_1(R/I,R/J)=0$, that is, $I\cap J=IJ$ (see [15, Exercise 3.20]).

(5) $\Rightarrow$ (6) Suppose $I=\langle a_1,\dots,a_n\rangle$ is a finitely generated semi-regular ideal. Then, for every $i=1,\dots,n$, there exists $c_i\in I$ such that $(1-c_i)a_i=0$. Set $c=\prod_{i=1}^n(1-c_i)$. Then $ca_i=0(i=1,\dots,n)$. Thus $c\in(0:_R I)=0$. Note that $1-c\in I$. Thus $1\in I$ and $I=R$.

(6) $\Rightarrow$ (8) Let I be a semi-regular ideal of R. Then there is a finitely generated dense sub-ideal J of I. By (6), $J=R$ and thus $I=R$. $\square$

The following example shows that semi-regular injective modules are not always injective.

Example 4.2. Let $(R,\mathfrak{m})$ be a non-field local ring such that $\mathfrak{m}$ is not semiregular (such as the residue class ring $R=\mathbb{Z}_4$, or the idealization $R=k(+)k$ of a field k with itself). Then R is a DQ-ring but is not semi-simple (see [16, Corollary 3.11]). Then every R-module is semi-regular injective by Proposition 4.1. However, there certainly exists a non-injective R-module since R is not semi-simple.

Recall from [15] that an R-module M is said to be divisible if and only if $rM=M$ for every regular element R, which is equivalent to $\mathrm{Ext}^1_R(R/\langle r\rangle,M)=0$ for every regular element R. Trivially, semi-regular injective modules are divisible.

Proposition 4.3. *A ring R is a total ring of quotients (i.e., every regular element is a unit) if and only if all R-modules are divisible.*

Proof. Suppose R is a total rings of quotient. Then trivially all R-modules are divisible. On the other hand, let r be a regular element of R. Since all R-modules are divisible, we have $R/\langle r\rangle$ is a projective ideal. So $\langle r\rangle$ is a direct summand of R. Since r is regular, we have r is a unit. $\square$

The following example shows that divisible modules are also not always semi-regular injective.

Example 4.4 ([19, Example 3.10])**.** Let $D=k[x]$ be the polynomial ring over a field k, and $\mathfrak{m}=\langle x\rangle$ a maximal ideal of D and $\mathcal{P}=\mathrm{Max}(R)-\{\mathfrak{m}\}$. Set $R=D(+)B$ the idealization of D with B, where $B=\bigoplus_{\mathfrak{p}\in\mathcal{P}}D/\mathfrak{p}$. Then R is a total ring of quotients by [11, Theorem 11(a)]. However, $\mathrm{fPD}(R)=1$. So R is not a DQ-ring. Hence there exists a divisible R-module which is not semi-regular injective.

Recall from [11] that an ideal I of R is said to be Q_0-invertible if there is an R-submodule J of $Q_0(R)$ such that $IJ=R$, which is equivalent to $I\mathrm{Hom}_R(I,R)=R$.

Theorem 4.5. *Let I be a semi-regular ideal of R. Then I is Q_0-invertible if and only if I is projective. Consequently, every semi-regular projective ideal is finitely generated.*

Proof. Let I be a semi-regular ideal of R. Suppose I is Q_0-invertible. Then there exist $a_i \in I$ and $f_i \in \mathrm{Hom}_R(I, R)$ with $i = 1, \ldots, n$ such that $\sum_{i=1}^n a_i f_i = 1$. So $\sum_{i=1}^n a_i f_i(r) = r$ for every $r \in R$. Hence I is a finitely generated projective ideal by the projective basis lemma (see [15, Theorem 2.3.6]). On the other hand, suppose I is a projective semi-regular ideal of R. Then, by the projective basis lemma, there exist elements $\{a_i \in I\}$ and $\{f_i\} \subseteq \mathrm{Hom}_R(I, R)$ such that

(1) if $x \in I$, then almost all $f_i(x) = 0$;
(2) if $x \in I$, then $x = \sum f_i(x)$.

Since I is semi-regular, there is a finitely generated dense subideal $I_0 = \langle x_0, \ldots, x_n \rangle$ of I. Set $g_i = f_i \circ \delta$, where $\delta : I_0 \hookrightarrow I$ is the natural embedding map. Set $g_i(x_j) = y_{ij}$. Then there are finite elements $i = 1, \ldots, m$ such that $y_{ij} \neq 0$. Thus $g_i = \frac{\sum_{j=0}^n y_{ij} X^j}{\sum_{j=0}^n x_j X^j} \in Q_0(R)$ for each $i = 1, \ldots, m$. So

$$x_j = \sum_{i=1}^m a_i g_i(x_j) = \sum_{i=1}^m a_i y_{ij} = \sum_{i=1}^m a_i g_i x_j.$$

It follows that $x_j(1 - \sum_{i=1}^m a_i g_i) = 0$ for all j. Since I_0 is dense, we have $\sum_{i=1}^m a_i g_i = 1$. Note that $I g_i \subseteq R$ for each $i = 1, \ldots, m$. Indeed, $x \in I$. Then, in the complete ring of quotients $Q(R)$ (see [9] for example), we have $x g_i = f_i(x) \in R$ for each $i = 1, \ldots, m$. Consequently, I is Q_0-invertible. □

Remark 4.6. It follows from Theorem 4.5 that semi-regular projective ideals are always finitely generated. However, dense projective ideals are not always finitely generated in general. Indeed, let $R = \prod_{i=1}^{\infty} \mathbb{F}_2$ be the countably infinite direct product of copies of $\mathbb{F}_2$, and $e_i = (1, \ldots, 1, 0, \ldots)$ where the sequence of 1's has length i. Set I to be the ideal generated by all e_is. Then I is a dense projective ideal which is obviously not finitely generated (see [4, 05WH]).

Now, we are ready to introduce and study the notion of strong Dedekind rings.

Definition 4.7. A ring R is said to be a strong Dedekind ring provided that every semi-regular ideal is Q_0-invertible.

We say a ring R semi-regular Noetherian provided that every semi-regular ideal is finitely generated. We have the following result whose proof is similar with the classical one, so we omit it.

Lemma 4.8. *Let R be a ring. The following statements are equivalent*:

(1) *R is a semi-regular Noetherian ring*;
(2) *Every set of semi-regular ideals has a maximal element*;

(3) *Every ascending chain of semi-regular ideals stabilizes.*

Recall from [11] that a ring R is called a strong Prüfer ring if every finitely generated semiregular ideal is Q_0-invertible. And recall from [12] that a ring R is called a Q_0-Krull ring if every semiregular ideal is t-invertible (which is equivalent to every semiregular ideal is w-invertible by [14, Proposition 4.17]). Obviously, strong Prüfer rings and Q_0-Krull rings are both generalizations of strong Dedekind rings. Moreover, we have the following result.

Theorem 4.9. *Let R be a ring. Then the following statements are equivalent*:

(1) sr-$gl.dim(R) \leq 1$;
(2) *R is a strong Dedekind ring*;
(3) *R is both an sr-Noetherian ring and a strong* Prüfer *ring*;
(4) *R is both a* DW *ring and a Q_0-Krull ring*;
(5) *Any quotient of a semi-regular injective R-module is semi-regular injective*;
(6) *Any quotient of a injective R-module is semi-regular injective*;
(7) *Any semi-regular ideal of R is projective.*

Proof. (1) $\Rightarrow$ (6) Let $0 \to L \to E \to M \to 0$ be an exact sequence, where E is an injective module. Since $Inj_{sr}.\dim(L) \leq 1$ by (1), we have M is semi-regular injective by Proposition 3.2.

(6) $\Rightarrow$ (7) Let I be a semi-regular ideal of R and M an R-module. Consider the following short exact sequences: $0 \to I \to R \to R/I \to 0$ and $0 \to M \to E(M) \to E(M)/M \to 0$ where $E(M)$ is the injective envelope of M. Then we can deduce $\mathrm{Ext}_R^1(I, M) \cong \mathrm{Ext}_R^1(R/I, E(M)/M) = 0$ as $E(M)/M$ is semi-regular injective by (6). So I is projective.

(7) $\Leftrightarrow$ (2) It follows from Theorem 4.5.

(2) $\Rightarrow$ (3) Suppose R is a strong Dedekind ring. Then R is strong Prüfer. Also R is an sr-Noetherian ring by Theorem 4.5.

(2) $\Rightarrow$ (4) Suppose R is a strong Dedekind ring. Then R is strong Prüfer and Q_0-Krull. Now we claim that R is a DW-ring. Indeed, let $J \in \mathrm{GV}(R)$. Then J is a finitely generated semi-regular ideal of R. Since R is strong Prüfer, for every $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a regular element $\frac{a}{b} \in J_{\mathfrak{p}}$ such that $J_{\mathfrak{p}} = \langle \frac{a}{b} \rangle$. Therefore $J_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$. Thus J is flat. By [15, Theorem 6.7.24] and [15, Exercise 6.10(1)], we have $J = R$.

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) Trivial.

(7) $\Rightarrow$ (1) Let I be a semi-regular ideal of R and M an R-module. Then $\mathrm{Ext}_R^2(R/I, M) \cong \mathrm{Ext}_R^1(I, M) = 0$ as I is projective by (7). So sr-gl.dim$(R) \leq 1$ by Proposition 3.4.

(7) $\Rightarrow$ (5) Let E be a semi-regular injective module and M a submodule of E. Then we have an exact sequence $0 = \mathrm{Ext}_R^1(R/I, E) \to \mathrm{Ext}_R^1(R/I, E/M) \to \mathrm{Ext}_R^2(R/I, M)$. Since I is projective by (7), we have

$$\mathrm{Ext}_R^2(R/I, M) \cong \mathrm{Ext}_R^1(I, M) = 0.$$

So $\mathrm{Ext}^1_R(R/I, E/M)$, and hence E/M is semi-regular injective by Theorem 2.2. $\square$

Recall from [5, Definition 2.2.19] that a ring R is said to be a Dedekind ring provided that every regular ideal is invertible, which is equivalent to every regular ideal is projective by Theorem 4.5. Obviously every strong Dedekind ring is a Dedekind ring. However, the converse does not always hold.

Example 4.10 ([11, Example 12]). Let $D = L[X^2, X^3, Y]$ where L is a field and $\mathcal{P} = \mathrm{Max}(R) - \{\langle X^2, X^3, Y\rangle\}$. Set $R = D(+)B$ the idealization of D with B, where $B = \bigoplus_{\mathfrak{p}\in\mathcal{P}} D/\mathfrak{p}$. Then the only semi-regular ideals of R are those of the form $J(+)B$ where $\sqrt{J} = \langle X^2, X^3, Y\rangle$. Then $R = \mathrm{T}(R)$. So R is a Dedekind ring. Since D is a Noetherian ring, each ideal J is a finitely generated ideal of D. Hence each $J(+)B$ is also a finitely generated ideal of R. So R is an *sr*-Noetherian ring. But R is not a strong Prüfer ring. Thus R is also not strong Dedekind by Theorem 4.9.

The semi-regular height of a prime ideal $\mathfrak{p}$ of a ring R is defined by the supremum of the number of prime ideals in every chain of semi-regular prime ideals contained in $\mathfrak{p}$. And the semi-regular dimension $\dim_{sr}(R)$ of a ring R is the supremum of the semi-regular heights of all prime ideals of R.

Corollary 4.11. *Let R be a strong Dedekind ring. Then every semi-regular ideal of R has a unique representation in the form of a product of* (*semi-regular*) *maximal ideals. Consequently,* $\dim_{sr}(R) \leq 1$.

Proof. Set Λ to be set of semi-regular ideals that cannot be written as a product of (semi-regular) maximal ideals. Suppose Λ is non-empty. Then, by Lemma 4.8, Λ has a maximal element I since R is *sr*-Noether. Let $\mathfrak{p}$ be a maximal ideal of R containing I and set $J = I\mathfrak{p}^{-1}$. Then J is a semi-regular ideal containing I. If $J = I$, then we have $\mathfrak{p} = R$ since I is $\mathcal{Q}_0$-invertible. So $I \subsetneq J$. By maximality of I, $J = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ for some semi-regular maximal ideal $\mathfrak{p}_i$ $(i = 1, \dots, k)$. So $I = \mathfrak{p}\mathfrak{p}_1 \cdots \mathfrak{p}_k$ which is a contradiction. Next we prove uniqueness. Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_k = \mathfrak{q}_1 \cdots \mathfrak{q}_l$ where $\mathfrak{p}_i$ and $\mathfrak{q}_j$ are semi-regular maximal ideals $(i = 1, \dots, k; j = 1, \dots, l)$. Since $\mathfrak{p}_1 \supseteq \mathfrak{q}_1 \cdots \mathfrak{q}_l$, we have $\mathfrak{p}_1 \supseteq \mathfrak{q}_j$ for some j. Since $\mathfrak{q}_j$ is maximal, we have $\mathfrak{p}_1 = \mathfrak{q}_j$. Since $\mathfrak{p}_1$ is Q_0-invertible, we can proceed by induction. So the uniqueness holds. Next we claim that each semi-regular prime ideal is maximal. Indeed, let $\mathfrak{p}$ be a semi-regular prime ideal of R. Then $\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ for some maximal ideal $\mathfrak{p}_i$ $(i = 1, \dots, k)$. So $\mathfrak{p} = \mathfrak{p}_i$ for some i by [2, Theorem 1.1.7]. Hence $\dim_{sr}(R) \leq 1$. $\square$

References

[1] D. D. Anderson, D. F. Anderson, and R. Markanda, *The rings $R(X)$ and $R\langle X\rangle$*, J. Algebra **95** (1985), no. 1, 96–115. `https://doi.org/10.1016/0021-8693(85)90096-1`

[2] S. Balcerzyk and T. Józefiak, *Commutative Noetherian and Krull rings*, translated from the Polish by Maciej Juniewicz and Sergiusz Kowalski, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1989.

[3] H. S. Butts and W. Smith, *Prüfer rings*, Math. Z. **95** (1967), 196–211. https://doi.org/10.1007/BF01111523

[4] A. J. de Jong et al., *The Stacks project*, https://stacks.math.columbia.edu

[5] J. Elliott, *Rings, Modules, and Closure Operations*, Switzerland: Springer Nature Switzerland AG, 2019.

[6] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989. https://doi.org/10.1007/BFb0084570

[7] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules. Volume* 2, second revised and extended edition, De Gruyter Expositions in Mathematics, 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2012. https://doi.org/10.1515/9783110218114

[8] P. J. Hilton and U. Stammbach, *A course in homological algebra*, second edition, Graduate Texts in Mathematics, 4, Springer-Verlag, New York, 1997. https://doi.org/10.1007/978-1-4419-8566-8

[9] J. A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, Inc., New York, 1988.

[10] W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, Math. Z. **42** (1937), no. 1, 745–766. https://doi.org/10.1007/BF01160110

[11] T. G. Lucas, *Strong Prüfer rings and the ring of finite fractions*, J. Pure Appl. Algebra **84** (1993), no. 1, 59–71. https://doi.org/10.1016/0022-4049(93)90162-M

[12] T. G. Lucas, *Krull rings, Prüfer v-multiplication rings and the ring of finite fractions*, Rocky Mountain J. Math. **35** (2005), no. 4, 1251–1325. https://doi.org/10.1216/rmjm/1181069687

[13] H. Prüfer, *Untersuchungen über Teilbarkeitseigenschaften in Körpern*, J. Reine Angew. Math. **168** (1932), 1–36. https://doi.org/10.1515/crll.1932.168.1

[14] F. Wang and H. Kim, *Two generalizations of projective modules and their applications*, J. Pure Appl. Algebra **219** (2015), no. 6, 2099–2123. https://doi.org/10.1016/j.jpaa.2014.07.025

[15] F. Wang and H. Kim, *Foundations of commutative rings and their modules*, Algebra and Applications, 22, Springer, Singapore, 2016. https://doi.org/10.1007/978-981-10-3337-7

[16] F. G. Wang, D. C. Zhou, and D. Chen, *Module-theoretic characterizations of the ring of finite fractions of a commutative ring*, J. Commut. Algebra **14** (2022), no. 1, 141–154. https://doi.org/10.1216/jca.2022.14.141

[17] F. G. Wang, D. C. Zhou, H. Kim, T. Xiong, and X. W. Sun, *Every Prüfer ring does not have small finitistic dimension at most one*, Comm. Algebra **48** (2020), no. 12, 5311–5320. https://doi.org/10.1080/00927872.2020.1787422

[18] X. Zhang, *A homological characterization of* Q_0-Prüfer *v-multiplication rings*, Int. Electron. J. Algebra **32** (2022), 228–240.

[19] X. L. Zhang and F. G. Wang, *The small finitistic dimensions over commutative rings*, J. Commut. Algebra, to appear.

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