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ON THE STRUCTURE OF CERTAIN SUBSET OF FAREY SEQUENCE

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Abstract. Let F_n be the Farey sequence of order n. For $S \subseteq F_n$, let $\mathcal{Q}(S)$ be the set of rational numbers x/y with $x, y \in S$, $x \leq y$ and $y \neq 0$. Recently, Wang found all subsets S of F_n with $|S| = n+1$ for which $\mathcal{Q}(S) \subseteq F_n$. Motivated by this work, we try to determine the structure of $S \subseteq F_n$ such that $|S| = n$ and $\mathcal{Q}(S) \subseteq F_n$. In this paper, we determine all sets $S \subseteq F_n$ satisfying these conditions for $n \in \{p, 2p\}$, where p is prime.

1. Introduction

For a positive integer n, let F_n denote the Farey sequence of order n, that is, the set of irreducible fractions between 0 and 1 whose denominators do not exceed n. For $S \subseteq F_n$, define

$$\mathcal{Q}(S) = \left\{ \frac{x}{y} : x, y \in S,\ x \leq y,\ y \neq 0 \right\}.$$

Recently, Wang [7] found all subsets $S \subseteq F_n$ for which $\mathcal{Q}(S) = F_n$.

Theorem 1.1 ([7, Theorem 3]). *Suppose $S \subseteq F_n$ and $\mathcal{Q}(S) = F_n$. Then S can only be one of the following two sets:*

$$S = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}\right\} \ \text{or}\ S = \left\{0, 1, \frac{1}{n}, \ldots, \frac{n-1}{n}\right\}.$$

Wang [7] also proved the following results.

Theorem 1.2 ([7, Theorem 1]). *If $S \subseteq F_n$ and $\mathcal{Q}(S) \subseteq F_n$, then S has at most $n+1$ elements.*

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Theorem 1.3 ([7, Theorem 4]). *Suppose $S \subseteq F_n$, $|S| = n+1$ and $\mathcal{Q}(S) \subseteq F_n$. Then*

$$S = \left\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}\right\} \text{ or } S = \left\{0, 1, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}$$

except for $n = 4$, where we have an additional set $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\}$.

In 1970, Graham [4] proposed the following conjecture.

Conjecture 1.4. *Let $a_1, a_2, \dots, a_n$ be distinct positive integers. Then*

$$\max_{i \neq j} \frac{a_i}{(a_i, a_j)} \geq n.$$

Vélez [6] pointed out that Graham also made the following stronger conjecture.

Conjecture 1.5. *Let $M_n = [1, 2, \dots, n]$ and $a_1, a_2, \dots, a_n$ be distinct positive integers. If*

$$(a_1, a_2, \dots, a_n) = 1 \text{ and } \max_{i \neq j} \frac{a_i}{(a_i, a_j)} = n,$$

then $\{a_1, a_2, \dots, a_n\}$ can only be $\{1, 2, \dots, n\}$ or $\left\{\frac{M_n}{n}, \frac{M_n}{n-1}, \dots, \frac{M_n}{1}\right\}$ except for $n = 4$, where we have the additional sequence $\{2, 3, 4, 6\}$.

These conjectures have been confirmed by Balasubramanian and Soundararajan [1] based on deep analytical methods. Wang [7] showed that Theorems 1.2 and 1.3 are equivalent to Conjectures 1.4 and 1.5, respectively. In Wang's proofs of Theorems 1.2 and 1.3, Conjectures 1.4 and 1.5 are necessary. But there are potentially other proofs which do not need the conjectures. Wang [7] asked whether one can prove Theorems 1.2 and 1.3 directly and thus providing new proofs for Graham's conjectures. For more results about Graham's conjectures, see [2, 3, 5, 8, 9].

By Theorem 1.2, we know that $n+1$ is critical. Motivated by Theorem 1.3, we study the structure of $S \subseteq F_n$ with $|S| = n$ for which $\mathcal{Q}(S) \subseteq F_n$. Obviously, if $S = \left\{0, 1, \frac{2}{3}, \frac{2}{4} \dots, \frac{2}{n}\right\}$ or $S = \left\{0, 1, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}\right\}$ or S is a subset of the sets in Theorem 1.3, then $\mathcal{Q}(S) \subseteq F_n$. We wonder whether there is other desired set $S \subseteq F_n$. In this paper, we determine the structure of $S \subseteq F_n$ with $|S| = n$ and $\mathcal{Q}(S) \subseteq F_n$ for $n \in \{p, 2p\}$, where p is prime. For other n, we attempted to figure out the structure of the desired set S but without success. In this paper, the following results are proved.

Theorem 1.6. *Let $n \in \{p, 2p\}$, where p is prime. Let S be a subset of F_n with $|S| = n$. Then $\mathcal{Q}(S) \subseteq F_n$ if and only if S satisfies one of the following conditions*:

(i) $S = \left\{0, 1, \frac{2}{3}, \frac{2}{4} \dots, \frac{2}{n}\right\}$,
(ii) $S \subseteq \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right\}$,
(iii) $S \subseteq \left\{0, 1, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right\}$,

(iv) $S=\left\{0,1,\frac{1}{n-1},\frac{2}{n-1},\dots,\frac{n-2}{n-1}\right\}$

except for $n\in\{4,5,6,10\}$. For $n=4$, there are two additional sets $S=\left\{0,\frac{1}{2},\frac{1}{3},\frac{2}{3}\right\}$ and $S=\left\{1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\right\}$. For $n=5$, there are two additional sets $S=\left\{0,1,\frac{1}{5},\frac{2}{5},\frac{1}{2}\right\}$ and $S=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\right\}$. For $n=6$, there is an additional set $S=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{2}{5}\right\}$. For $n=10$, there are two additional sets $S=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{9},\frac{2}{3},\frac{2}{9}\right\}$ and $S=\left\{0,1,\frac{1}{6},\frac{1}{2},\frac{1}{9},\frac{2}{9},\frac{1}{3},\frac{4}{9},\frac{5}{9},\frac{2}{3}\right\}$.

The proof of Theorem 1.6 will be divided into two parts according to $n=p$ and $n=2p$, which we will give in Section 2 and Section 3, respectively. In the following, we denote $\bar{F_n}$ the set of all nonnegative fractions with both denominator and numerator not more than n.

2. Proof of Theorem 1.6 for $n=p$

It is easy to verify that the sufficiency is true. Next, we prove the necessity. If $0\notin S$, then $S\cup\{0\}\subseteq F_p$ and $\mathcal{Q}(S\cup\{0\})\subseteq F_p$. By Theorem 1.3, we have $S\cup\{0\}=\left\{0,1,\frac{1}{2},\dots,\frac{1}{p}\right\}$ or $S\cup\{0\}=\left\{0,1,\frac{1}{p},\dots,\frac{p-1}{p}\right\}$. Thus,

$$S=\left\{1,\frac{1}{2},\dots,\frac{1}{p}\right\}\text{ or }S=\left\{1,\frac{1}{p},\dots,\frac{p-1}{p}\right\}.$$

Similarly, we can get that if $1\notin S$, then

$$S=\left\{0,\frac{1}{2},\dots,\frac{1}{p}\right\}\text{ or }S=\left\{0,\frac{1}{p},\dots,\frac{p-1}{p}\right\}.$$

In the following, we assume that $\{0,1\}\subseteq S$. Clearly, if $p=2$, then $S=\{0,1\}$ which satisfies the condition (i). Now, we suppose that $p\geq 3$. Let

$$S=\left\{0,1,\frac{x_1}{p},\frac{x_2}{p},\dots,\frac{x_r}{p},\frac{y_1}{z_1},\frac{y_2}{z_2},\dots,\frac{y_s}{z_s}\right\},\ r+s=p-2,$$

where the fractions are irreducible and $(p,z_i)=1$ $(1\leq i\leq s)$. Clearly, $S\subseteq\left\{0,1,\frac{1}{p},\dots,\frac{p-1}{p}\right\}$ when $s=0$. If $r=0$, then $S\subseteq F_{p-1}$ and $\mathcal{Q}(S)\subseteq F_{p-1}$. By Theorem 1.3,

$$S=\left\{0,1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{p-1}\right\}\text{ or }S=\left\{0,1,\frac{1}{p-1},\frac{2}{p-1},\dots,\frac{p-2}{p-1}\right\}$$

except for $p=5$, where we have an additional set $S=\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{2}{3}\right\}$. Now, we consider the case $r\geq 1,s\geq 1$. Since $\frac{x_i}{p}\Big/\frac{y_j}{z_j}=\frac{x_iz_j}{py_j}\in\bar{F}_p$, we have

$$y_j\mid x_i,\ \frac{x_iz_j}{y_j}\leq p-1\ (1\leq i\leq r,\ 1\leq j\leq s).\tag{2.1}$$

Case 1. $(x_1,x_2,\dots,x_r)=1$. By (2.1), we have $y_j\mid(x_1,x_2,\dots,x_r)$ which implies $y_j=1$ for any $1\leq j\leq s$. And now, all z_j's are distinct and greater than 1. It follows from (2.1) that

$$r(s+1)\leq\max_{i\in[1,r]}x_i\cdot\max_{j\in[1,s]}z_j\leq p-1=r+s+1.\tag{2.2}$$

Thus, $r \le 2$. If $r = 1$, then $x_1 = 1$ by $(x_1, x_2, \ldots, x_r) = 1$. Therefore, $S \subseteq \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{p}\right\}$. If $r = 2$, then by (2.2), we have $s = 1$ and $p = 5$. Furthermore, by (2.2), we can get that $\{x_1, x_2\} = \{1, 2\}$ and $z_1 = 2$. Therefore, $S = \left\{0, 1, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}\right\}$.

Case 2. $(x_1, x_2, \ldots, x_r) > 1$. Without loss of generality, we may assume $x_r = \max_{i \in [1,r]} x_i$. Then $x_r \ge 2r$. Let $z'_j = \frac{x_r z_j}{y_j}$ $(1 \le j \le s)$. Then S can be rewritten as

$$S = \left\{0, 1, \frac{x_1}{p}, \frac{x_2}{p}, \ldots, \frac{x_r}{p}, \frac{x_r}{z'_1}, \frac{x_r}{z'_2}, \ldots, \frac{x_r}{z'_s}\right\}.$$

Since $S \subseteq F_p$, we have $x_r < z'_j$ and z'_j's are distinct. By (2.1),

$$2r + s \le x_r + s \le \max_{j \in [1,s]} z'_j \le p - 1 = r + s + 1.$$

It follows that $r = 1$, $x_1 = 2$ and $s = p - 3$. Therefore, $S = \left\{0, 1, \frac{2}{3}, \frac{2}{4} \ldots, \frac{2}{p}\right\}$. This completes the proof for the case $n = p$.

3. Proof of Theorem 1.6 for $n = 2p$

In this section, we prove Theorem 1.6 holds for $n = 2p$. Firstly, we give some lemmas which will be used in the the following proof.

Lemma 3.1. *Let p be odd prime and*

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_2}{2p}, \ldots, \frac{a_r}{2p}, \frac{b_1}{p}, \ldots, \frac{b_t}{p}, \frac{1}{2u_1}, \ldots, \frac{1}{2u_s}, \frac{1}{v_1}, \ldots, \frac{1}{v_k}\right\}$$

be a subset of F_{2p} with $r+s+t+k = 2p-2$, where these fractions are irreducible and $p \nmid u_i$ $(1 \le i \le s)$, $(v_j, 2p) = 1$ $(1 \le j \le k)$. If $\mathcal{Q}(S) \subseteq F_{2p}$, then

$$S \subseteq \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\} \text{ or } S \subseteq \left\{0, 1, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}.$$

Proof. We may assume that $a_1 < \cdots < a_r$, $b_1 < \cdots < b_t$, $u_1 < \cdots < u_s$ and $v_1 < \cdots < v_k$. Clearly, $r, s, t \le p - 1$ and $k \le p - 2$. We have $a_r \ge 2r - 1$ if $r \ge 1$ and $v_k \ge 2k + 1$ if $k \ge 1$. Moreover, by $a_i \ne p$ and $v_i \ne p$ we have $a_r \ge 2r + 1$ for $r \ge \frac{p+1}{2}$ and $v_k \ge 2k + 3$ for $k \ge \frac{p-1}{2}$.

Case 1. $r \ge 1$. We divide into two subcases $s + t \le p - 1$ and $s + t \ge p$.

Subcase 1.1. $s + t \le p - 1$. By $r + s + t + k = 2p - 2$ we have $r + k \ge p - 1$.

Subcase 1.1.1. $k \ge 1$. By $\frac{a_r}{2p} / \frac{1}{v_k} = \frac{a_r v_k}{2p} \in \bar{F}_{2p}$,

$$(2r - 1)(2k + 1) \le a_r v_k \le 2p - 1 \le 2r + 2k + 1.$$

This implies $r = 1$ and $k = p - 2$. Then we have $v_{p-2} = 2p - 1$ and $a_1 = 1$. If $t = 0$, then $S \subseteq \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$. If $t \ge 1$, then it follows from $\frac{b_t}{p} / \frac{1}{v_{p-2}} = \frac{b_t(2p-1)}{p} \in \bar{F}_{2p}$ that $b_t(2p - 1) \le 2p - 1$. Thus, $t = 1$ and $b_1 = 1$. Therefore, $S \subseteq \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$.

Subcase 1.1.2. $k = 0$. Note that $r + k \geq p - 1$ and $r \leq p - 1$. Then in this case, $r = p - 1$ and $s + t = p - 1$. We have $a_{p-1} = 2p - 1$. If $s = 0$, then $S \subseteq \left\{0, 1, \frac{1}{2p}, \dots, \frac{2p-1}{2p}\right\}$. If $s \geq 1$, then $\frac{a_{p-1}}{2p} / \frac{1}{2u_s} = \frac{(2p-1)u_s}{p} \in \bar{F}_{2p}$ implies $(2p - 1)u_s \leq 2p - 1$, and so $s = 1$ and $u_1 = 1$. Therefore, $S \subseteq \left\{0, 1, \frac{1}{2p}, \dots, \frac{2p-1}{2p}\right\}$.

Subcase 1.2. $s + t \geq p$. Noting that $s, t \leq p - 1$, we have $s, t \geq 1$. By $\frac{b_t}{p} / \frac{1}{2u_s} = \frac{2u_s b_t}{p} \in \bar{F}_{2p}$,

$$st \leq u_s b_t \leq p - 1. \tag{3.1}$$

If $s, t \geq 2$, then $s + t \leq st \leq p - 1$, a contradiction. Hence, $s = 1$ or $t = 1$, and so $\{s, t\} = \{1, p - 1\}$.

If $s = 1, t = p - 1$, then $u_1 = 1$ and $b_{p-1} = p - 1$ by (3.1). When $k \geq 1$, we can deduce from $\frac{b_{p-1}}{p} / \frac{1}{v_k} = \frac{(p-1)v_k}{p} \in \bar{F}_{2p}$ that $(p - 1)v_k \leq 2p - 1$. Thus, $v_k \leq 2$ which is impossible. Therefore, $k = 0$ and $S \subseteq \left\{0, 1, \frac{1}{2p}, \frac{2}{2p} \dots, \frac{2p-1}{2p}\right\}$.

If $s = p - 1, t = 1$, then $u_{p-1} = p - 1$ and $b_1 = 1$ by (3.1). Since $r \geq 1$, it follows from $\frac{a_r}{2p} / \frac{1}{2u_{p-1}} = \frac{(p-1)a_r}{p} \in \bar{F}_{2p}$ that $(p - 1)a_r \leq 2p - 1$. By $2 \nmid a_r$, we have $r = 1$ and $a_1 = 1$. Therefore, $S \subseteq \left\{0, 1, \frac{1}{2}, \frac{1}{3} \dots, \frac{1}{2p}\right\}$.

Case 2. $r = 0$. It is clear that $\mathcal{Q}(S) \subseteq F_{2p-1}$. By Theorem 1.3, we have $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2p-1}\right\}$ or $S = \left\{0, 1, \frac{1}{2p-1}, \frac{2}{2p-1}, \dots, \frac{2p-2}{2p-1}\right\}$. The latter form is impossible since $\frac{2}{2p-1} \notin S$. Therefore, $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2p-1}\right\} \subseteq \left\{0, 1, \frac{1}{2}, \frac{1}{3} \dots, \frac{1}{2p}\right\}$. This completes the proof. $\square$

Lemma 3.2. *Let p be odd prime and*

$$S = \left\{0, 1, \frac{p}{b}, \frac{X_1}{2b}, \frac{X_2}{2b}, \dots, \frac{X_k}{2b}, \frac{Y_1}{b}, \frac{Y_2}{b}, \dots, \frac{Y_l}{b}\right\}$$

be a subset of F_{2p} with $k + l = 2p - 3$, where $p < b \leq 2p - 1$, $(X_i, 2p) = 1$ $(1 \leq i \leq k)$ and $(Y_j, p) = 1$ $(1 \leq j \leq l)$. If $\mathcal{Q}(S) \subseteq F_{2p}$, then

$$S = \left\{0, 1, \frac{1}{2p-1}, \frac{2}{2p-1}, \dots, \frac{2p-2}{2p-1}\right\}$$

except for $p \in \{3, 5\}$. There is an additional set $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}\right\}$ for $p = 3$ and an additional set $S = \left\{0, 1, \frac{1}{6}, \frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}\right\}$ for $p = 5$.

Proof. If $k = 0$, then $l = 2p-3$. Since $p \notin \{Y_j : 1 \leq j \leq l\}$ and $Y_j < b \leq 2p-1$, we have $\{Y_j : 1 \leq j \leq l\} = \{1, 2, \dots, 2p - 2\} \setminus \{p\}$. Thus, $b = 2p - 1$ and

$$S = \left\{0, 1, \frac{1}{2p-1}, \frac{2}{2p-1}, \dots, \frac{2p-2}{2p-1}\right\}.$$

Now, we deal with the case $k > 0$. By $\frac{p}{b} / \frac{X_i}{2b} = \frac{2p}{X_i} \in \bar{F}_{2p}$ and $(X_i, 2p) = 1$, we have $X_i \leq 2p-1$. Since $b > p$, we have $\frac{X}{2b} \notin S$ for integers X with $(X, 2b) = 1$.

It follows that

$$\frac{1}{2b} \notin S,\ \frac{p}{2b} \notin S \tag{3.2}$$

and

$$\frac{b \pm 1}{2b} \notin S\ (2 \mid b),\ \frac{b \pm 2}{2b} \notin S\ (2 \nmid b). \tag{3.3}$$

Firstly, we suppose that $p \geq 7$. Let l_1 be the number of Y_j with $p < Y_j < b$. When b is even, we have $b+1 \in (p, 2p-1]$. By (3.2) and (3.3),

$$\Big\{\frac{X_1}{2b}, \frac{X_2}{2b}, \dots, \frac{X_k}{2b}\Big\} \subseteq \Big\{\frac{1}{2b}, \frac{3}{2b}, \dots, \frac{2p-1}{2b}\Big\} \setminus \Big\{\frac{1}{2b}, \frac{p}{2b}, \frac{b+1}{2b}\Big\}.$$

Hence, $k \leq p-3$. Similarly, we can get that $k \leq p-3$ when b is odd. It follows from $k+l = 2p-3$ that $l \geq p$, which indicates that $l_1 \geq 1$ and $b \geq p+2$. Let α_j be the integer such that $2^{\alpha_j} || Y_j$. For $p < Y_j < b$, by $\frac{X_i}{2b}/\frac{Y_j}{b} = \frac{X_i}{2Y_j} \in \bar{F}_{2p}$, we have $(X_i, Y_j) > 1$. From this, we get that, for $p < Y_j < b$,

$$\frac{Y_j/2^{\alpha_j} \pm 2}{2b} \notin S \tag{3.4}$$

since $(\frac{Y_j}{2^{\alpha_j}} \pm 2, 2Y_j) = 1$. And if $\frac{3}{2b} \in S$, then

$$3 \mid Y_j\ (p < Y_j < b). \tag{3.5}$$

By $2 \nmid X_i$, Y_j could not be of the form 2^{α}. Let

$$A = \Big\{\frac{Y_j/2^{\alpha_j} + 2}{2b} : 2 \mid Y_j, p < Y_j < b\Big\},\ B = \Big\{\frac{Y_j + 2}{2b} : 2 \nmid Y_j, p < Y_j < b\Big\}.$$

Hence, $\frac{5}{2b} \leq \min A \leq \max A \leq \frac{p}{2b}$, $\frac{p+4}{2b} \leq \min B \leq \max B \leq \frac{2p-1}{2b}$, and so $|A \cup B| = l_1$. Let

$$S' = \Big\{0, 1, \frac{p}{b}, \frac{3}{2b}, \frac{5}{2b}, \dots, \frac{2p-1}{2b}, \frac{Y_1}{b}, \frac{Y_2}{b}, \dots, \frac{Y_l}{b}\Big\} \setminus (A \cup B).$$

Then $S \subseteq S'$ and $|S'| = 3+l+(p-1-l_1) \leq 3+(p-1+l_1)+(p-1-l_1) = 2p+1$. The claim below will be usually used in the following proof.

Claim 1. Let x be an odd integer with $3 \leq x \leq 2p-1$. If $\frac{x}{2b} \notin S$ and $\frac{x}{2b} \notin A \cup B$, then

$$S = S' \setminus \Big\{\frac{x}{2b}\Big\}.$$

Proof of Claim 1. By $\frac{x}{2b} \notin S$ and $\frac{x}{2b} \notin A \cup B$, we have $S \subseteq S' \setminus \Big\{\frac{x}{2b}\Big\}$. It follows from

$$2p = |S| \leq |S' \setminus \Big\{\frac{x}{2b}\Big\}| = |S'| - 1 \leq 2p$$

that $S = S' \setminus \{\frac{x}{2b}\}$. $\square$

We return to the proof of Lemma 3.2 and divide into two cases according $2 \mid b$ and $2 \nmid b$.

Case 1. $2 \mid b$.

In this case, $p+3 \le b \le 2p-2$. When $|B| = 0$, by $\frac{b\pm 1}{2b} \notin A$ and (3.3), we have $S \subseteq S' \setminus \{\frac{b\pm 1}{2b}\}$, which implies that $|S| \le |S'|-2 \le 2p-1$, a contradiction. Hence, $|B| > 0$. Without loss of generality, we may assume that $p < Y_j < b$ for $j = 1, 2, \ldots, l_1$.

If $\frac{b+1}{2b} \notin B$, then $S = S' \setminus \{\frac{b+1}{2b}\}$ by Claim 1. This indicates that $\frac{3}{2b} \in S$, $\frac{p}{2b} \in A$ and $\frac{b-1}{2b} \in B$. By the definitions of A and B, we know that there are two integers $1 \le j', j'' \le l_1$ such that $Y_{j'} = 2p-4$, $Y_{j''} = b-3$. From $2 \mid b$ and $Y_{j'} < b$, we get that $b = 2p-2$, and so $Y_{j''} = 2p-5$. However, by (3.5), we have $3 \mid 2p-4$ and $3 \mid 2p-5$, which is impossible. Hence, $\frac{b+1}{2b} \in B$. Similarly, we can prove that $\frac{b-1}{2b} \in B$. By the definitions of A and B, there exist two integers $1 \le j_1, j_2 \le l_1$ such that $Y_{j_1} = b-1$, $Y_{j_2} = b-3$. From above argument, we know that $\frac{3}{2b} \notin S$. Let Y_{j_0} be the largest odd integer with $1 \le j_0 \le l_1$ such that $\frac{Y_{j_0}}{b} \in S$, $\frac{Y_{j_0}-2}{b} \notin S$. Define $Y_{j_0} = p+2$ if such integer does not exist. Then we have $\frac{Y_{j_0}}{2b} \notin B$. By $\left\{\frac{b-1}{b}, \frac{b-3}{b}\right\} \subseteq S$, we know $\frac{Y_{j_0}+2}{b} \in S$. It follows from (3.4) that $\frac{Y_{j_0}}{2b} \notin S$. Hence, $S \subseteq S' \setminus \{\frac{3}{2b}, \frac{Y_{j_0}}{2b}\}$. So $|S| \le |S'| - 2 \le 2p-1$, a contradiction.

Case 2. $2 \nmid b$.

In this case, $p+2 \le b \le 2p-1$. Firstly, we deal with the subcase $|B| = 0$. If $b = p+2$, then $p < b+2 < b+4 \le 2p-1$ by $p \ge 7$. It follows from $\frac{b+2}{2b} \notin S$ and $\frac{b+4}{2b} \notin S$ that $S \subseteq S' \setminus \left\{\frac{b+2}{2b}, \frac{b+4}{2b}\right\}$. Hence, $|S| \le |S'|-2 \le 2p-1$, a contradiction. If $b = 2p-1$, then $p < b-4 < b-2 \le 2p-1$. If $p+2 < b < 2p-1$, then $p < b-2 < b+2 \le 2p-1$. In both cases, by similar argument, we can get that $|S| \le |S'| - 2 \le 2p-1$, a contradiction.

Now, we deal with the subcase $|B| > 0$. At this point, $p+4 \le b \le 2p-1$.

For $p+4 \le b \le 2p-3$, we have $\frac{b+2}{2b} \notin B$. By (3.3) and Claim 1, we obtain $S = S' \setminus \{\frac{b+2}{2b}\}$, which indicates that $\frac{3}{2b} \in S$, $\frac{p}{2b} \in A$ and $\frac{b-2}{2b} \in B$. From the definition of A, there exists an integer $1 \le j' \le l_1$ such that $Y_{j'} = 2p-4$. And now, $b = 2p-3$ and $3 \mid 2p-4 = b-1$. Since $3 \nmid b-2$, we have $\frac{b-2}{b} \notin S$ and $\frac{(b-2)+2}{2b} = \frac{1}{2} \notin B$. Hence, $\frac{1}{2} \in S$. However, $\frac{1}{2}/\frac{2p-4}{2p-3} = \frac{2p-3}{2(2p-4)} \notin \bar{F}_{2p}$, a contradiction.

For $b = 2p-1$, we will prove that $\frac{2p-5}{2(2p-1)} \in B$ and $\frac{p}{2(2p-1)} \in A$. Clearly, $2p-5 > p$ and $\frac{2p-5}{2(2p-1)} \notin S$. If $\frac{2p-5}{2(2p-1)} \notin B$, then $S = S' \setminus \{\frac{2p-5}{2(2p-1)}\}$ by Claim 1, which indicates that $\frac{3}{2(2p-1)} \in S$, $\frac{p}{2(2p-1)} \in A$ and $\frac{2p-3}{2(2p-1)} \in B$. Hence, there are two integers $1 \le j', j'' \le l_1$ such that $Y_{j'} = 2p-4$ and $Y_{j''} = 2p-5$. By (3.5), $3 \mid 2p-4$ and $3 \mid 2p-5$ which is impossible. Therefore, $\frac{2p-5}{2(2p-1)} \in B$. Similarly, we can get that $\frac{p}{2(2p-1)} \in A$. By the definitions of A and B, $Y_{j_1} = 2p-4$ and

$Y_{j_2} = 2p-7$ for some $1 \le j_1, j_2 \le l_1$. From $p < Y_{j_2} = 2p-7$, we obtain $p > 7$, that is $p \ge 11$.

It is easy to see that $2p-9 > p$ and $\frac{2p-9}{2(2p-1)} \notin S$. If $\frac{2p-9}{2(2p-1)} \notin B$, then by Claim 1, we have $S = S' \setminus \{\frac{2p-9}{2(2p-1)}\}$, which indicates that $\frac{3}{2(2p-1)} \in S$ and $\frac{2p-3}{2(2p-1)} \in B$. Hence, there is an integer $1 \le j''' \le l_1$ such that $Y_{j'''} = 2p-5$. By (3.5), $3 \mid 2p-4$ and $3 \mid 2p-5$, which is impossible. Therefore, $\frac{2p-9}{2(2p-1)} \in B$. Similarly, we have $\frac{2p-3}{2(2p-1)} \in B$. By the definition of B, there are two integers $1 \le j_3, j_4 \le l_1$ such that $Y_{j_3} = 2p-5$ and $Y_{j_4} = 2p-11$. And now, $\frac{3}{2(2p-1)} \notin S$. Since

$$\frac{1}{2} / \frac{2p-5}{2p-1} = \frac{2p-1}{2(2p-5)} \notin \bar{F}_{2p},$$

we have $\frac{1}{2} = \frac{2p-1}{2(2p-1)} \notin S$. Hence, $\frac{2p-1}{2(2p-1)} \in B$, which implies that $Y_{j_5} = 2p-3$ for some integer $1 \le j_5 \le l_1$. Let Y_{j_0} be the largest odd integer with $1 \le j_0 \le l_1$ such that $\frac{Y_{j_0}}{2p-1} \in S$, $\frac{Y_{j_0}-2}{2p-1} \notin S$. Define $Y_{j_0} = p+2$ if such integer does not exist. By similar discussion to Case 1, we can get that $\frac{Y_{j_0}}{2(2p-1)} \notin S$ and $\frac{Y_{j_0}}{2(2p-1)} \notin B$. Hence, $S \subseteq S' \setminus \{\frac{3}{2(2p-1)}, \frac{Y_{j_0}}{2(2p-1)}\}$. So $|S| \le |S'| - 2 \le 2p-1$, a contradiction.

Now, we suppose that $p \in \{3, 5\}$. Note that $X_i \le 2p-1$ and $(X_i, 2p) = 1$ for $1 \le i \le k$. For $p = 3$, by $3 = p < b \le 2p-1 = 5$, we have $b \in \{4, 5\}$. If $b = 4$, then all of $\frac{1}{8}, \frac{3}{8}$ and $\frac{5}{8}$ do not belong to S, contradictory with $k > 0$. Hence, $b = 5$. By $\frac{1}{10} \notin S$, $\frac{3}{10} \notin S$ and $k > 0$, we have $\frac{1}{2} = \frac{5}{10} \in S$. It follows from $\frac{1}{2}/\frac{4}{5} = \frac{5}{8} \notin F_6$ that $\frac{4}{5} \notin S$. Thus, $S \subseteq \{0, 1, \frac{3}{5}, \frac{1}{2}, \frac{1}{5}, \frac{2}{5}\}$. $|S| = 2p = 6$ implies that $S = \{0, 1, \frac{3}{5}, \frac{1}{2}, \frac{1}{5}, \frac{2}{5}\}$. For $p = 5$, we have $b \in \{6, 7, 8, 9\}$. If $b = 6$, then $S \subseteq \{0, 1, \frac{3}{12}, \frac{9}{12}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\}$ since all of $\frac{1}{12}, \frac{5}{12}$ and $\frac{7}{12}$ do not belong to S. Hence, $|S| \le 9 < 10 = 2p$, a contradiction. If $b = 7$, then by $\frac{1}{14}, \frac{3}{14}, \frac{5}{14}$ and $\frac{9}{14}$ not belonging to S, we have $S \subseteq \{0, 1, \frac{7}{14}, \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}\}$. Hence $|S| \le 9 < 10 = 2p$, a contradiction. If $b = 8$, then $\frac{2i-1}{16} \notin S$ for $i \in [1, 5]$, contradictory with $k > 0$. If $b = 9$, then by $k > 0$ and $\frac{2i-1}{18} \notin S$ $(i = 1, 3, 4)$, we have $\frac{3}{18} \in S$ or $\frac{1}{2} = \frac{9}{18} \in S$. By (3.5), we have $\frac{7}{9} \notin S, \frac{8}{9} \notin S$ when $\frac{3}{18} \in S$. If $\frac{1}{2} \in S$, then by $\frac{1}{2}/\frac{j}{9} = \frac{9}{2j} \notin \bar{F}_{10}$ for $j = 7, 8$, we have $\frac{7}{9} \notin S, \frac{8}{9} \notin S$. Hence, $S \subseteq \{0, 1, \frac{1}{6}, \frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}\}$. $|S| = 10$ implies that $S = \{0, 1, \frac{1}{6}, \frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}\}$. This completes the proof. $\square$

Lemma 3.3. *Let S be a subset of F_{2p} with $|S| = 2p$ and $\mathcal{Q}(S) \subseteq F_{2p}$, where p is odd prime.*

(1) *If S contains no fractions whose denominators are $2p$ or p, then*

$$S = \left\{0, 1, \frac{1}{2p-1}, \frac{2}{2p-1} \ldots, \frac{2p-2}{2p-1}\right\}$$

except for $p \in \{3, 5\}$. There is an additional set $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}\right\}$ for $p = 3$ and an additional set $S = \left\{0, 1, \frac{1}{6}, \frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}\right\}$ for $p = 5$.

(2) *If S contains fractions whose denominators are $2p$ and also contains fractions whose denominators are p, then*

$$S \subseteq \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2p}\right\} \text{ or } S \subseteq \left\{0, 1, \frac{1}{2p}, \frac{2}{2p}, \dots, \frac{2p-1}{2p}\right\}.$$

Proof. (1) It is easy to see that $S \subseteq F_{2p-1}$. If S contains no fractions whose numerators are p, then $\mathcal{Q}(S) \subseteq F_{2p-1}$. By Theorem 1.3, we have $S = \left\{0, 1, \frac{1}{2}, \dots, \frac{1}{2p-1}\right\}$ or $S = \left\{0, 1, \frac{1}{2p-1}, \dots, \frac{2p-2}{2p-1}\right\}$. Both are impossible since $\frac{1}{p} \notin S$ and $\frac{p}{2p-1} \notin S$. If S contains fractions whose numerators are p, then we may assume that

$$S = \left\{0, 1, \frac{p}{b_1}, \dots, \frac{p}{b_t}, \frac{x_1}{y_1}, \dots, \frac{x_s}{y_s}\right\},$$

where the fractions are irreducible, $b_1 < b_2 < \dots < b_t$, $(x_i y_i, p) = 1$ $(1 \le i \le s)$ and $2 + t + s = 2p$. Since $\frac{p}{b_i} / \frac{x_j}{y_j} = \frac{p y_j}{b_i x_j} \in \bar{F}_{2p}$, we have $y_j \mid b_i$ or $\frac{y_j}{2} \mid b_i$. Let $d = (b_1, \dots, b_t)$. Then

$$y_j \mid d \quad \text{or} \quad \frac{y_j}{2} \mid d \ (y_j \nmid d).$$

We may assume that $\frac{y_i}{2} \mid d$ and $y_i \nmid d$ for $1 \le i \le k$ and $y_{k+j} \mid d$ for $1 \le j \le l$, where $k + l = s$. Let

$$X_i = \frac{x_i d}{y_i/2} \ (1 \le i \le k), \ Y_j = \frac{x_{k+j} d}{y_{k+j}} \ (1 \le j \le l).$$

Then S can be rewritten as

$$S = \left\{0, 1, \frac{p}{b_1}, \dots, \frac{p}{b_t}, \frac{X_1}{2d}, \dots, \frac{X_k}{2d}, \frac{Y_1}{d}, \dots, \frac{Y_l}{d}\right\},$$

where $(X_i, 2p) = 1, (Y_j, p) = 1$ and $2 + t + k + l = 2p$.

If $t \ge 2$, by $td \le b_t \le 2p - 1$, we have $d \le p - 1$. Since $b_i > p$ $(1 \le i \le t)$, we have $b_i \ge (i+1)d$. It follows that $t \le \lfloor \frac{2p-1}{d} \rfloor - 1$. By $p < b_i \le 2p - 1$, we can get that

$$t \le \min\left\{p - 1, \left\lfloor \frac{2p-1}{d} \right\rfloor - 1\right\}. \tag{3.6}$$

If $d = p - 1$, then $t \le 1$, a contradiction. So $d \le p - 2$. It is clear that $X_i < 2d$ and $Y_j < d$. Hence,

$$|S| \le 2 + t + d + d - 1 = t + 2d + 1.$$

When $d = 1$, it follows from (3.6) that $|S| \le p+2 < 2p$, a contradiction. Hence, $d \ge 2$. And now, $p \ge 5$ and $t \le \lfloor \frac{2p-1}{d} \rfloor - 1$. By $2 \le d \le p - 2$,

$$2p = |S| \le \left\lfloor \frac{2p-1}{d} \right\rfloor + 2d \le \max_{2 \le d \le p-2} \left\{ \left\lfloor \frac{2p-1}{d} \right\rfloor + 2d \right\} \le \max\{p+3, 2p-1\} < 2p,$$

a contradiction.

If $t = 1$, we may write $b_1 = b$. Then $d = b$. By Lemma 3.2, we know the result holds.

(2) Similar to the proof of Theorem 1.6, we have $S = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$ or $S = \left\{1, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}$ if $0 \notin S$ and $S = \left\{0, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$ or $S = \left\{0, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}$ if $1 \notin S$. Now, we assume that $\{0, 1\} \subseteq S$. Let

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_2}{2p}, \ldots, \frac{a_r}{2p}, \frac{b_1}{p}, \ldots, \frac{b_t}{p}, \frac{x_1}{2u_1}, \ldots, \frac{x_s}{2u_s}, \frac{y_1}{v_1}, \ldots, \frac{y_k}{v_k}\right\}$$

with $r+s+t+k = 2p-2$ and $r, t \geq 1$, where the fractions are irreducible and $(u_i, p) = 1$ $(1 \leq i \leq s)$, $(v_j, 2p) = 1$ $(1 \leq j \leq k)$. Since $\frac{b_i}{p} / \frac{x_j}{2u_j} = \frac{2u_j b_i}{p x_j} \in \bar{F}_{2p}$, we have

$$x_j \mid b_i, \quad \frac{2u_j b_i}{x_j} \leq 2p-2 \ (1 \leq i \leq t,\ 1 \leq j \leq s). \tag{3.7}$$

Similarly, we can obtain that $x_j \mid a_i$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$,

$$y_j \mid a_i,\ 2 \nmid y_j,\ \frac{a_i v_j}{y_j} \leq 2p-1 \ (1 \leq i \leq r,\ 1 \leq j \leq k) \tag{3.8}$$

and

$$y_j \mid b_i,\ \frac{b_i v_j}{y_j} \leq 2p-1 \ (1 \leq i \leq t,\ 1 \leq j \leq k). \tag{3.9}$$

If $(a_1, a_2, \ldots, a_r) = 1$ or $(b_1, b_2, \ldots, b_t) = 1$, then $x_i = 1$ $(1 \leq i \leq s)$ and $y_j = 1$ $(1 \leq j \leq k)$. By Lemma 3.1, we have $S \subseteq \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$ or $S \subseteq \left\{0, 1, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}$. Now, we assume $(a_1, a_2, \ldots, a_r) > 1$ and $(b_1, b_2, \ldots, b_t) > 1$. We will deduce a contradiction.

Without loss of generality, we may assume that $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_t$. Since $(a_1, a_2, \ldots, a_r) \geq 3$, we have $3(2r-1) \leq a_r \leq 2p-1$ and so $r \leq \frac{p+1}{3}$. Let $v_i' = \frac{v_i a_r}{y_i}$ and $v_i'' = \frac{v_i b_t}{y_i}$ for $i \in [1, k]$ and $u_j'' = \frac{2u_j b_t}{x_j}$ for $j \in [1, s]$. Then S can be rewritten as both

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_2}{2p}, \ldots, \frac{a_r}{2p}, \frac{b_1}{p}, \ldots, \frac{b_t}{p}, \frac{x_1}{2u_1}, \ldots, \frac{x_s}{2u_s}, \frac{a_r}{v_1'}, \ldots, \frac{a_r}{v_k'}\right\}$$

and

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_2}{2p}, \ldots, \frac{a_r}{2p}, \frac{b_1}{p}, \ldots, \frac{b_t}{p}, \frac{b_t}{u_1''}, \ldots, \frac{b_t}{u_s''}, \frac{b_t}{v_1''}, \ldots, \frac{b_t}{v_k''}\right\}.$$

By $S \subseteq F_{2p}$, all of v_i''s are distinct and larger than a_r, all of u_i'''s and v_j'''s are distinct and larger than b_t. If $2 \mid b_t$, then $2 \mid v_i''$ since $2 \nmid y_i$. By (3.7), (3.9) and $2 \mid u_j''$ we have

$$2t + 2s + 2k \leq b_t + 2s + 2k \leq \max(\{v_i'' : i \in [1, k]\} \cup \{u_j'' : j \in [1, s]\}) \leq 2p-2.$$

This implies that $t+s+k \leq p-1$, and so $r \geq p-1$, contradictory to $r \leq \frac{p+1}{3}$. Hence, $2 \nmid b_t$ and $2 \nmid v_i''$. From $(b_1, b_2, \ldots, b_t) > 1$ we have $b_t \geq 3t$. Since

$$3t+1+2(s-1) \leq b_t+1+2(s-1) \leq \max\{u_i'' : 1 \leq i \leq s\} \leq 2p-2 = r+s+t+k,$$

we have

$$2t + s \le r + k + 1. \tag{3.10}$$

It follows from $2 \nmid v_i'$ and (3.8) that

$$3r + 2k \le a_r + 2k \le \max\{v_i' : 1 \le i \le k\} \le 2p - 1 = r + s + t + k + 1,$$

and so

$$2r + k \le s + t + 1. \tag{3.11}$$

The inequalities (3.10) and (3.11) give us $r + t \le 2$. Hence, $r = t = 1$ and $b_1 \ge 3$. Note that $r + s + t + k = 2p - 2$. By (3.10) and (3.11), we can obtain that $s = k = p - 2$. Therefore, $\{v_1'', v_2'', \ldots, v_k''\} = \{3, 5, \ldots, 2p-1\} \setminus \{p\}$, which is impossible since $\frac{b_1}{3} \notin S$. This completes the proof. □

Now, we give the proof of Theorem 1.6 for the case $n = 2p$.

Proof of Theorem 1.6 for $n = 2p$. It is easy to verify that the sufficiency is true. Next, we prove the necessity. Firstly, we deal with the case $p = 2$. Since $\frac{1}{3}/\frac{3}{4} = \frac{4}{9} \notin F_4$, $\frac{1}{4}/\frac{2}{3} = \frac{3}{8} \notin \bar{F}_4$ and $\frac{2}{3}/\frac{3}{4} = \frac{8}{9} \notin \bar{F}_4$, S can not contain both x and y, where $(x, y) \in \{(\frac{1}{3}, \frac{3}{4}), (\frac{1}{4}, \frac{2}{3}), (\frac{2}{3}, \frac{3}{4})\}$. Hence, if $\frac{2}{3} \in S$, then $S \subseteq \{0, 1, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. If $\frac{1}{3} \in S$, then $S \subseteq \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$ or $S \subseteq \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. If neither $\frac{1}{3}$ nor $\frac{2}{3}$ belong to S, then $S \subseteq \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$. Therefore, Theorem 1.6 holds for $n = 4$. Now, we suppose that $p \ge 3$.

By Lemma 3.3, we just need to consider the following two cases:

(1) S contains fractions whose denominators are $2p$ but no fraction whose denominator is p;

(2) S contains fractions whose denominators are p but no fraction whose denominator is $2p$.

Firstly, we deal with the case (1). Similar to the discussion of Theorem 1.6, we have $S = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$ or $S = \left\{1, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}$ if $0 \notin S$ and $S = \left\{0, \frac{1}{2}, \ldots, \frac{1}{2p}\right\}$ or $S = \left\{0, \frac{1}{2p}, \ldots, \frac{2p-1}{2p}\right\}$ if $1 \notin S$. All forms are impossible since $\frac{1}{p} \notin S$. Hence, $\{0, 1\} \subseteq S$. Let

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_2}{2p}, \ldots, \frac{a_r}{2p}, \frac{x_1}{2u_1}, \ldots, \frac{x_s}{2u_s}, \frac{y_1}{v_1}, \ldots, \frac{y_k}{v_k}\right\}$$

with $r \ge 1$ and $r + s + k = 2p - 2$, where the fractions are irreducible and $(u_i, p) = 1$ $(1 \le i \le s)$, $(v_j, 2p) = 1$ $(1 \le j \le k)$. By $\frac{a_i}{2p}/\frac{x_j}{2u_j} = \frac{a_i u_j}{p x_j} \in \bar{F}_{2p}$, we have

$$x_j \mid a_i,\ \frac{a_i u_j}{x_j} \le 2p - 1\ (1 \le i \le r,\ 1 \le j \le s). \tag{3.12}$$

By $\frac{a_i}{2p}/\frac{y_j}{v_j} = \frac{a_i v_j}{2p y_j} \in \bar{F}_{2p}$, we have

$$y_j \mid a_i,\ \frac{a_i v_j}{y_j} \le 2p - 1\ (1 \le i \le r,\ 1 \le j \le k). \tag{3.13}$$

Let $(a_1, a_2, \dots, a_r) = d$. If $d = 1$, then $x_i = 1$ $(1 \le i \le s)$ and $y_j = 1$ $(1 \le j \le k)$. By Lemma 3.1, we have $S \subseteq \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2p}\right\}$ or $S \subseteq \left\{0, 1, \frac{1}{2p}, \frac{2}{2p}, \dots, \frac{2p-1}{2p}\right\}$. Now, we assume that $d > 1$.

If $r \ge 2$, then $2d < \max\limits_{1 \le i \le r} a_i \le 2p - 1$, and so $d < p$. Let $S_1 = S \cup \{\frac{d}{p}\}$. One can easily prove that $\mathcal{Q}(S_1) \subseteq F_{2p}$. By $|S_1| = 2p + 1$ and Theorem 1.3, we have $S_1 = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2p}\right\}$ or $S_1 = \left\{0, 1, \frac{1}{2p}, \frac{2}{2p}, \dots, \frac{2p-1}{2p}\right\}$. Both forms are impossible since $\frac{1}{p} \notin S_1$. Hence, $r = 1$ and $d = a_1 \ge 3$. Let $X_i = \frac{a_1 u_i}{x_i}$ $(1 \le i \le s)$ and $Y_j = \frac{a_1 v_j}{y_j}$ $(1 \le j \le k)$. Then S can be rewritten as

$$S = \left\{0, 1, \frac{a_1}{2p}, \frac{a_1}{2X_1}, \frac{a_1}{2X_2}, \dots, \frac{a_1}{2X_s}, \frac{a_1}{Y_1}, \frac{a_1}{Y_2}, \dots, \frac{a_1}{Y_k}\right\}.$$

Clearly, all of $2X_i$'s and Y_j's are distinct. By (3.12), (3.13) and $S \subseteq F_{2p}$, we have $a_1/2 < X_i \le 2p-1$, $(X_i, p) = 1$ and $a_1 < Y_j \le 2p-1$, $(Y_j, 2p) = 1$. Thus,

$$2p = |S| \le 2 + \left(2p - 1 - \frac{a_1 - 1}{2}\right) + \left(\frac{2p - 1 - a_1}{2}\right) = 3p + 1 - a_1.$$

So $a_1 \le p + 1$. By $(a_1, 2p) = 1$, we have $a_1 < p$. We observe that for $p < n \le 2p - 1$, if $(n, Y_j) = 1$ for some $j \in [1, k]$, then $\frac{a_1}{2n} \notin S$. Otherwise, $\frac{a_1}{2n} / \frac{a_1}{Y_j} = \frac{Y_j}{2n} \in \bar{F}_{2p}$ which is impossible since $2n > 2p$ and $(2n, Y_j) = 1$. Therefore,

$$\frac{a_1}{2(2^{\alpha_j} Y_j + 1)} \notin S \ (Y_j \in (a_1, p-2]), \ \frac{a_1}{2(Y_j - 1)} \notin S \ (Y_j \in [p+2, 2p-1]),$$

where α_j is an integer such that $2^{\alpha_j} Y_j \in [p+1, 2p-2]$. Furthermore, when $2^{\alpha_j} Y_j > p + 1$, we also get that

$$\frac{a_1}{2(2^{\alpha_j} Y_j - 1)} \notin S. \tag{3.14}$$

Let

$$A = \left\{\frac{a_1}{2(2^{\alpha_j} Y_j + 1)} : Y_j \in (a_1, p-2]\right\}, \ B = \left\{\frac{a_1}{2(Y_j - 1)} : Y_j \in [p+2, 2p-1]\right\}.$$

Then $|A| + |B| = k$. Since $a_1/2 < X_i \le 2p - 1$ and $a_1/2 \le (2p-1)/2 < p$, it follows from $(X_i, p) = 1$ that

$$\left\{\frac{a_1}{2X_1}, \dots, \frac{a_1}{2X_s}\right\} \subseteq \left\{\frac{a_1}{a_1+1}, \frac{a_1}{a_1+3}, \dots, \frac{a_1}{2(2p-1)}\right\} \setminus \left(A \cup B \cup \left\{\frac{a_1}{2p}\right\}\right).$$

Hence, $s \le 2p - 1 - \frac{a_1 - 1}{2} - k - 1$. Thus,

$$2p = |S| \le 3 + \left(2p - 2 - \frac{a_1 - 1}{2} - k\right) + k = 2p + 1 - \frac{a_1 - 1}{2},$$

which implies that $a_1 = 3$ and $p \ge 5$. At this point, one can easily get that

$$S = \left\{0, 1, \frac{3}{4}, \frac{3}{6}, \dots, \frac{3}{2(2p-1)}, \frac{3}{Y_1}, \frac{3}{Y_2}, \dots, \frac{3}{Y_k}\right\} \setminus (A \cup B). \tag{3.15}$$

By $2p-3>p$ and $(3,2p-3)=1$, we have $\frac{3}{2(2p-3)}\notin S$. Since the number of Y_j's with $Y_j\in(3,p-2]$ is $\le\frac{p-5}{2}$, there are at least two odd integers $p+2\le m_1,m_2\le 2p-1$ for which $\{\frac{3}{2m_1},\frac{3}{2m_2}\}\subseteq S$. Hence, there exists an odd integer $m\in[p+2,2p-5]$ such that $\frac{3}{2m}\in S$ and $p\ge 7$. Let

$$\max\left\{m\in[p+2,2p-5]:2\nmid m,\frac{3}{2m}\in S\right\}=m_0.$$

Then $\frac{3}{2(m_0+2)}\notin S$. By (3.15), we have $m_0+2=2^{\alpha_j}Y_j+1$ for some $Y_j\in(3,p-2]$. Since $2^{\alpha_j}Y_j=m_0+1>p+1$, it follows from (3.14) that $\frac{3}{2m_0}\notin S$, contradictory to the definition of m_0.

Now, we deal with the case (2). Similarly, we have $\{0,1\}\subseteq S$. Let

$$S=\left\{0,1,\frac{b_1}{p},\dots,\frac{b_t}{p},\frac{x_1}{2u_1},\dots,\frac{x_s}{2u_s},\frac{y_1}{v_1},\dots,\frac{y_k}{v_k}\right\}$$

with $t\ge 1$ and $t+s+k=2p-2$, where the fractions are irreducible and $(u_i,p)=1$ $(1\le i\le s)$, $(v_j,2p)=1$ $(1\le j\le k)$. By $\frac{b_i}{p}/\frac{x_j}{2u_j}=\frac{2b_iu_j}{px_j}\in\bar{F}_{2p}$, we have

$$x_j\mid b_i\ (1\le i\le t,\ 1\le j\le s).$$

By $\frac{b_i}{p}/\frac{y_j}{v_j}=\frac{b_iv_j}{py_j}\in\bar{F}_{2p}$, we have

$$y_j\mid b_i\quad\text{or}\quad\frac{y_j}{2}\mid b_i\ (1\le i\le t,\ 1\le j\le k).$$

Let $(b_1,\dots,b_t)=d$. Then $x_i\mid d$ for $1\le i\le s$ and $y_j\mid d$ or $\frac{y_j}{2}\mid d$ for $1\le j\le k$. Suppose that the number of y_j's with $y_j\mid d$ is k_1, and the number of y_j's with $y_j\nmid d$ and $\frac{y_j}{2}\mid d$ is m. Then $k_1+m=k$. Without loss of generality, we may assume that $y_j\mid d$ for $1\le j\le k_1$. Let $s+k_1=l$ and

$$X_i=\frac{2u_id}{x_i}\ (1\le i\le s),\ X_{s+i}=\frac{v_id}{y_i}\ (1\le i\le k_1)\text{ and }Y_j=\frac{2v_{k_1+j}d}{y_{k_1+j}}\ (1\le j\le m).$$

Clearly, we have $2\nmid Y_j$. Then S can be rewritten as

$$S=\left\{0,1,\frac{b_1}{p},\dots,\frac{b_t}{p},\frac{d}{X_1},\dots,\frac{d}{X_l},\frac{2d}{Y_1},\dots,\frac{2d}{Y_m}\right\}.$$

Assume that $b_1<b_2<\cdots<b_t$. Then $td\le b_t\le p-1$ and so $t\le\frac{p-1}{d}$. For $1\le i\le l$, by $\frac{b_t}{p}/\frac{d}{X_i}=\frac{b_tX_i}{pd}\in\bar{F}_{2p}$, we have

$$\frac{b_tX_i}{d}\le 2p-1\ (1\le i\le l).\tag{3.16}$$

For $1\le j\le m$, by $\frac{b_t}{p}/\frac{2d}{Y_j}=\frac{b_tY_j}{2pd}\in\bar{F}_{2p}$, we have that, if $2d\mid b_t$, then

$$\frac{b_tY_j}{2d}\le 2p-1,\tag{3.17}$$

if $2d \nmid b_t$, then

$$\frac{b_t Y_j}{d} \le 2p-1. \tag{3.18}$$

We distinguish into two cases according to $2 \le t \le \frac{p-1}{d}$ and $t = 1$.

Case 1. $2 \le t \le \frac{p-1}{d}$. By (3.16), (3.17), (3.18) and $S \subseteq F_{2p}$, we have

$$d < X_i \le \left\lfloor \frac{2p-1}{t} \right\rfloor \ (1 \le i \le l), \ 2d < Y_j \le \left\lfloor \frac{2(2p-1)}{t} \right\rfloor \ (1 \le j \le m). \tag{3.19}$$

It follows from $2 \nmid Y_j$ that

$$|S| \le 2 + t + \Big(\frac{2p-1}{t} - d\Big) + \Big(\frac{2p-1}{t} - d + \frac{1}{2}\Big) = \frac{2(2p-1)}{t} + t - 2d + \frac{5}{2}.$$

If $d \ge 2$, then $2 \le t \le \frac{p-1}{2}$ and $p \ge 5$. It follows that

$$|S| \le \frac{2(2p-1)}{t} + t - \frac{3}{2} \le \max\Big\{\frac{2(2p-1)}{t} + t : t \in \Big\{2, \frac{p-1}{2}\Big\}\Big\} - \frac{3}{2} < 2p,$$

a contradiction. Thus, $d = 1$. At this point, we have $2 \le t \le p-1$ and

$$2p = |S| \le \frac{2(2p-1)}{t} + t + \frac{1}{2}.$$

If $3 \le t \le p-1$, then $p \ge 5$ and $\frac{2(2p-1)}{t} + t + \frac{1}{2} < 2p$, a contradiction. Hence, $t = 2$. Since $Y_j \ne p$, it follows from (3.19) and $d = 1$ that

$$2p = |S| \le 4 + (p-1-d) + (p-d-1) = 4 + (p-2) + (p-2) = 2p.$$

This shows that $l = m = p-2$, and so $\{X_1, X_2, \ldots, X_l\} = \{2, 3, \ldots, p-1\}$ and $\{Y_1, Y_2, \ldots, Y_m\} = \{3, 5, \ldots, 2p-1\} \setminus \{p\}$. By (3.16), we get $b_2 \le 2$. So $\{b_1, b_2\} = \{1, 2\}$. Therefore,

$$S = \Big\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}, \frac{2}{3}, \frac{2}{5}, \ldots, \frac{2}{2p-1}\Big\} = \Big\{0, 1, \frac{2}{3}, \frac{2}{4}, \ldots, \frac{2}{2p}\Big\}.$$

Case 2. $t = 1$. In this case, $d = b_1$. By (3.16), (3.18) and $S \subseteq F_{2p}$, we have $b_1 < X_i \le 2p-1,\ X_i \ne p\ (1 \le i \le l)$ and $2b_1 < Y_j \le 2p-1,\ (2p, Y_j) = 1\ (1 \le j \le m)$. Note that for $p < n \le 2p-1$, if $(n, Y_j) = 1$ for some $j \in [1, m]$, then $\frac{b_1}{n} \notin S$. Otherwise, $\frac{b_1}{n} / \frac{2b_1}{Y_j} = \frac{Y_j}{2n} \in \bar{F}_{2p}$ which is impossible since $(2n, Y_j) = 1$ and $2n > 2p$. By similar discussion with case (1), we have $\frac{b_1}{2^{\alpha_j} Y_j + 1} \notin S$ for $2b_1 < Y_j < p$ and $\frac{b_1}{Y_j - 1} \notin S$ for $p < Y_j \le 2p-1$, where α_j is the integer such that $p < 2^{\alpha_j} Y_j < 2p$. Moreover, if $2^{\alpha_j} Y_j > p+1$, then

$$\frac{b_1}{2^{\alpha_j} Y_j - 1} \notin S \tag{3.20}$$

and if $p < Y_j + 1 < 2p$, then

$$\frac{b_1}{Y_j + 1} \notin S. \tag{3.21}$$

Let

$$A_1 = \left\{\frac{b_1}{2^{\alpha_j}Y_j+1} : 2b_1 < Y_j < p\right\},\ B_1 = \left\{\frac{b_1}{Y_j-1} : p < Y_j \le 2p-1\right\}$$

and

$$S_1 = \left\{0, 1, \frac{b_1}{b_1+1}, \frac{b_1}{b_1+2}, \ldots, \frac{b_1}{2p-1}, \frac{2b_1}{Y_1}, \frac{2b_1}{Y_2}, \ldots, \frac{2b_1}{Y_m}\right\} \setminus (A_1 \cup B_1).$$

Clearly, $S \subseteq S_1$. Since all Y_j's are odd, we have $A \cap B = \emptyset$, and so $|A_1|+|B_1| = m$. Hence,

$$2p = |S| \le |S_1| = 2 + (2p-1-b_1-m) + m = 2p+1-b_1.$$

This shows that $b_1 = 1$ and $l = 2p-3-m$. And now,

$$S = S_1 = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2p-1}, \frac{2}{Y_1}, \ldots, \frac{2}{Y_m}\right\} \setminus (A_1 \cup B_1). \tag{3.22}$$

For $p = 3$, by $2 = 2b_1 < Y_j \le 2p-1 = 5$ and $1 = (Y_j, 2p) = (Y_j, 6)$, we have $m \le 1$. When $m = 0$, (3.22) implies that $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$. When $m = 1$, we have $Y_1 = 5$. It follows from (3.22) that $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}\right\}$.

Let $p \ge 5$. Since $|A_1| = |\{Y_j : Y_j \in (2, p-2]\}| \le \frac{p-3}{2}$, there is at least one odd integer $a \in [p+2, 2p-1]$ such that $\frac{1}{a} \in S$. We will show that $\frac{1}{2p-1} \in S$. Assume that $\frac{1}{2p-1} \notin S$, then there is an odd integer $a \in [p+2, 2p-3]$ for which $\frac{1}{a} \in S$. Let

$$\max\left\{a \in [p+2, 2p-3] : 2 \nmid a,\ \frac{1}{a} \in S\right\} = a_0.$$

Then $\frac{1}{a_0+2} \notin S$. By the definition of A_1 and (3.22), we have $a_0+2 = 2^{\alpha_j}Y_j+1$ for some $Y_j \in (2, p-2]$. However, since $a_0 = 2^{\alpha_j}Y_j - 1 > p$, it follows from (3.20) that $\frac{1}{a_0} \notin S$, a contradiction with the definition of a_0. Thus, $\frac{1}{2p-1} \in S$. Similarly, we can prove that $\frac{1}{2p-3} \in S$ if $|\{Y_j : Y_j \in (2, p-2]\}| \le \frac{p-5}{2}$.

Subcase 2.1. $|\{Y_j : Y_j \in (2, p-2]\}| = \frac{p-3}{2}$. In this case,

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}, \frac{1}{2p-1}, \frac{2}{3}, \frac{2}{5}, \ldots, \frac{2}{p-2}\right\} \subseteq S.$$

By $\frac{1}{2p-1}/\frac{2}{2p-3} = \frac{2p-3}{2(2p-1)} \notin \bar{F}_{2p}$ and $\frac{1}{p+3}/\frac{2}{3} = \frac{3}{2(p+3)} \notin \bar{F}_{2p}$, we have $\frac{2}{2p-3} \notin S$ and $\frac{1}{p+3} \notin S$, respectively. By (3.22) and the definitions of A_1 and B_1, we obtain $\frac{1}{2p-4} \in S$ and $\frac{2}{p+4} \in S$. Thus,

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}, \frac{1}{2p-1}, \frac{1}{2p-4}, \frac{2}{3}, \frac{2}{5}, \ldots, \frac{2}{p-2}, \frac{2}{p+4}\right\} \subseteq S.$$

For $p = 5$, the cardinality of the left set above is $10 = 2p$. Therefore, $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{9}, \frac{2}{3}, \frac{2}{9}\right\}$.

By $\frac{2}{2p-3} \notin S$ and $\frac{2}{p+4} \in S$, we have $p \neq 7$. For $p \geq 11$, let

$$\max\{Y_j : Y_j \in [p+2, 2p-5]\} = Y.$$

One should notice that such Y does exist since $p+4 < 2p-5$ and $\frac{2}{p+4} \in S$. Then $\frac{2}{Y+2} \notin S$. By (3.22), we have $\frac{1}{Y+1} \in S$. However, it follows from $p < Y+1 < 2p$ and (3.21) that $\frac{1}{Y+1} \notin S$, a contradiction.

Subcase 2.2. $|\{Y_j : Y_j \in (2, p-2]\}| \leq \frac{p-5}{2}$. In this case, $\{\frac{1}{2p-1}, \frac{1}{2p-3}\} \subseteq S$. Since $\frac{1}{2p-1}/\frac{2}{2p-3} = \frac{2p-3}{2(2p-1)} \notin \bar{F}_{2p}$ and $\frac{1}{2p-3}/\frac{2}{2p-1} = \frac{2p-1}{2(2p-3)} \notin \bar{F}_{2p}$, both $\frac{2}{2p-3}$ and $\frac{2}{2p-1}$ do not belong to S.

Now, we prove that $m = 0$. If $|\{Y_j : Y_j \in [p+2, 2p-1]\}| \geq 1$, then we can deduce a contradiction by similar discussion with Subcase 2.1. Hence, $|\{Y_j : Y_j \in [p+2, 2p-1]\}| = 0$. By (3.22), we have $\frac{1}{p+3} \in S$. If $|\{Y_j : Y_j \in (2, p-2]\}| \geq 1$, let

$$\min\{2^{\alpha_j} Y_j + 1 : Y_j \in (2, p-2]\} = 2^{\alpha_{j_0}} Y_{j_0} + 1.$$

Then $2^{\alpha_{j_0}} Y_{j_0} + 1 = p+2$. Otherwise, $2^{\alpha_{j_0}} Y_{j_0} - 1 \geq p+2$. From (3.20), we deduce that $\frac{1}{2^{\alpha_{j_0}} Y_{j_0} - 1} \notin S$ which contradicts with (3.22). At this point, by $\frac{1}{p+3}/\frac{2}{Y_{j_0}} = \frac{Y_{j_0}}{2(p+3)} \in F_{2p}$, we have $(Y_{j_0}, p+3) > 1$, which is impossible since $(2^{\alpha_{j_0}} Y_{j_0}, p+3) = (p+1, p+3) = 2$. Thus, $|\{Y_j : Y_j \in (2, p-2]\}| = 0$, and so $m = 0$. Therefore, $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2p-1}\right\}$. This completes the proof for the case $n = 2p$. $\square$

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