Bull. Korean Math. Soc. **60** (2023), No. 4, pp. 1025–1034
https://doi.org/10.4134/BKMS.b220460
pISSN: 1015-8634 / eISSN: 2234-3016

ANNIHILATOR IDEALS OF SIMPLE MODULES OF RESTRICTED QUANTIZED ENVELOPING ALGEBRA

Yu Wang

Abstract. Let U be the restricted quantized enveloping algebra $\widetilde{U}_q(\mathfrak{sl}_2)$ over an algebraically closed field of characteristic zero, where q is a primitive l-th root of unity (with l being odd and greater than 1). In this paper we show that any indecomposable submodule of U under the adjoint action is generated by finitely many special elements. Using this result we describe all ideals of U. Moreover, we classify annihilator ideals of simple modules of U by generators.

1. Introduction

There has been momentous attention to the classification of ideals of algebras in history. For instance, Fisher et al. [3, 10, 11] discussed that under certain conditions a group algebra over an arbitrary field is a principal ideal ring. Subsequently Jespers and Okniński [4] gave a sufficient and necessary condition for a semigroup algebra to be a principal ideal ring. Catoiu et al. [1,8,12] showed that some (quantized) enveloping algebras are principal ideal rings. It is worth mentioning that Siciliano and Usefi in [12] raised an open question that under what conditions a Hopf algebra is a principal ideal ring. It seems that this problem is interesting and difficult to solve in the near future. Not long ago we in [16, 17] proved that finite dimensional pointed Hopf algebras of rank one are principal ideal rings and described all annihilator ideals of indecomposable modules. In particular, we [15,18] characterized all ideals and annihilator ideals of indecomposable modules of the Radford Hopf algebras. In this paper, we study the ideals of small quantum groups associated to $\mathfrak{sl}_2$ using the adjoint action.

We briefly introduce the history of restricted quantized enveloping algebra. The representation theory of restricted quantized enveloping algebra dates back to Suter [14] and Xiao [19]. They introduced different versions of restricted

Received July 3, 2022; Revised January 17, 2023; Accepted March 30, 2023.
2020 *Mathematics Subject Classification.* 16D25, 20G42.
Key words and phrases. Restricted quantized enveloping algebra, annihilator ideal, adjoint action.
This work was supported by the National Natural Science Foundation of China (Grant No. 11871063).

quantized enveloping algebras associated to $\mathfrak{sl}_2$, and classified all finite dimensional indecomposable modules over them. In the light of the work of Suter, Kondo and Saito [7] explicitly provided the decomposition rules of the tensor product of all indecomposable modules of $\overline{U}_q(\mathfrak{sl}_2)$, where q is a primitive $2p$-th root of unity with $p \geqslant 2$. Lately, Su and Yang [13] characterized the Green ring of $\overline{U}_q(\mathfrak{sl}_2)$ by generators and relations. We study another versions of small quantum groups $\widetilde{U}_q(\mathfrak{sl}_2)$, where q is a primitive l-th root of unity (with l being odd and greater than 1). All the finite dimensional indecomposable modules were completely determined by Chari and Premet [2]. Applying the results in [2], we first describe indecomposable submodules of $\widetilde{U}_q(\mathfrak{sl}_2)$ under adjoint action and obtain generators of them, and then characterize all two-sided ideals of $\widetilde{U}_q(\mathfrak{sl}_2)$. Moreover, we classify all annihilator ideals of simple modules of $\widetilde{U}_q(\mathfrak{sl}_2)$ by generators.

Throughout, we work over an algebraically closed field $\Bbbk$ of characteristic zero. Unless other stated, all algebras, Hopf algebras and modules are vector spaces over $\Bbbk$; all modules are finite dimensional; all maps are $\Bbbk$-linear; $\otimes$ means $\otimes_{\Bbbk}$. We assume that the reader has at least a passing familiarity with Hopf algebras and quantum groups; see [5, 9] for background.

2. Indecomposable modules

In this section, we recall the basic properties and the construction of indecomposable representations of $\widetilde{U}_q(\mathfrak{sl}_2)$. We assume that q is a primitive l-th root of unity with l being odd and greater than 1. Recall that the restricted quantized enveloping algebra $U = \widetilde{U}_q(\mathfrak{sl}_2)$ over $\Bbbk$ is generated by E, F, K associated with the following relations (see [2]):

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{2.1}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \tag{2.2}$$

$$K^l = 1, \quad E^l = 0, \quad F^l = 0. \tag{2.3}$$

It is well known that U admits a Hopf algebra structure determined by

$$\Delta(K) = K \otimes K, \qquad S(K) = K^{-1}, \qquad \varepsilon(K) = 1, \tag{2.4}$$

$$\Delta(E) = E \otimes K + 1 \otimes E, \qquad S(E) = -EK^{-1}, \qquad \varepsilon(E) = 0, \tag{2.5}$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \qquad S(F) = -KF, \qquad \varepsilon(F) = 0. \tag{2.6}$$

It is clear that U has a basis $\{E^iF^jK^s \mid 0 \leqslant i, j, s \leqslant l - 1\}$. In addition, the Casimir element

$$C_q = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} \tag{2.7}$$

is in the center $Z(U)$ of U, and the minimal annihilating polynomial of C_q is (see [6, equation (3.91)])

$$\psi(x) = \prod_{j=0}^{l-1}(x-\beta_j), \qquad \text{where } \beta_j = \frac{q^j+q^{-j}}{(q-q^{-1})^2}.$$

For $n \in \mathbb{N}$, set $[n] = [n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$. For $0 \leqslant r \leqslant l-1$, if $(n,r) \neq (0,0)$, let $V(n,r)$ denote the Weyl module of dimension $nl+r$ with a basis $v_0, v_1, \ldots, v_{nl+r-1}$, on which the action of U is given by

$$K \cdot v_i = q^{r-1-2i} v_i, \tag{2.8}$$

$$E \cdot v_i = [r-i] v_{i-1}, \tag{2.9}$$

$$F \cdot v_i = [i+1] v_{i+1}, \tag{2.10}$$

for $0 \leqslant i \leqslant nl+r-1$. Let

$$\Omega = FE + \frac{qK + q^{-1}K^{-1} - 2}{(q-q^{-1})^2} = C_q - \frac{2}{(q-q^{-1})^2} \in Z(U). \tag{2.11}$$

For the simple U-module, we have the following proposition.

Proposition 2.1 ([2, Proposition 3.2])**.**

(1) *Each simple U-module is isomorphic either to $V(0,r)$ for some $1 \leqslant r \leqslant l-1$ or to $V(1,0)$.*

(2)

$$\left(\Omega - \left[\frac{r}{2}\right]^2\right) \cdot V(n,r) = 0,$$

where $[\frac{r}{2}] = [\frac{(l+1)r}{2}]$.

The dual M^* of a U-module M is defined by using the antipode:

$$(u \cdot f)(m) := f(S(u)\cdot m) \quad \text{for all } u \in U,\ f \in M^* \text{ and } m \in M.$$

Fix a basis of M. Then the action of $u \in U$ on M^* in the dual basis is the transpose of the action of $S(u)$ on M in the original basis. It is clear that the dual of an indecomposable representation is also indecomposable. Therefore, the dual Weyl modules form another class of indecomposable modules for U. Now we write the dual module of $V(n,r)$ explicitly. $V(n,r)^*$ has a basis v_0^*, $v_1^*, \ldots, v_{nl+r-1}^*$ on which the action of K, E, F is given by

$$K \cdot v_i^* = q^{-(r-1-2i)} v_i^*, \tag{2.12}$$

$$E \cdot v_i^* = -q^{-(r-3-2i)}[r-i-1] v_{i+1}^*, \tag{2.13}$$

$$F \cdot v_i^* = -q^{r-1-2i}[i] v_{i-1}^*, \tag{2.14}$$

for $0 \leqslant i \leqslant nl+r-1$. By [2, Lemma 3.4], we have that the modules $V(0,r)$ and $V(1,0)$ are self-dual.

Let $X(r)$ be a $2l$-dimensional module with basis $\{x_0, x_1, \ldots, x_{l-1}, w_0, w_1, \ldots, w_{l-1}\}$ on which the generators of U act as follows:

$$\begin{aligned} K \cdot x_i &= q^{l-r-2i-1} x_i, \\ K \cdot w_i &= q^{r-2i-1} w_i, \\ F \cdot x_i &= [i+1] x_{i+1}, \\ F \cdot w_i &= [i+1] w_{i+1}, \\ E \cdot x_i &= [l-r-i] x_{i-1}, \\ E \cdot w_i &= [r-i] w_{i-1} + \begin{bmatrix} l-r-1+i \\ i \end{bmatrix} x_{i-r-1+l}, \end{aligned}$$

for $0 \leqslant i \leqslant l-1$. And we assume that x_i and w_i are zero if $i < 0$ or $i > l-1$. By [2, Corollary 3.9], we have that the action of Ω on $X(r)$ is not semisimple and $X(r)^* \cong X(r)$.

For $n \geqslant 0$, $0 < r \leqslant l-1$ and $\lambda \in \mathbb{C}$, let $V^\lambda(n,r)$ be an $(nl+r)$-dimensional vector space with basis $\{v_0, v_1, \ldots, v_{nl+r-1}\}$ and the action of U on which given by

$$\begin{aligned} K \cdot v_i &= q^{r-1-2i} v_i, \\ F \cdot v_i &= [i+1] v_{i+1}, \\ E \cdot v_i &= [r-i] v_{i-1}, & i &\not\equiv 0 \pmod{l}, \\ E \cdot v_i &= [r] v_{i-1} + [r](1+\lambda) v_{i+l-1}, & i &\equiv 0 \pmod{l}, \end{aligned}$$

where $0 \leqslant i \leqslant nl+r-1$. Let $V(\lambda, n, r)$ be the submodule of $V^\lambda(n,r)$ spanned by $\{v_0, v_1, \ldots, v_{nl-1}\}$ and $V(\infty, n, r)$ the submodule spanned by $\{v_r, v_{r+1}, \ldots, v_{nl+r-1}\}$.

By definition, it is clear that $V^\lambda(n,r)^*$ is an $(nl+r)$-dimensional vector space with basis $\{v_0^*, v_1^*, \ldots, v_{nl+r-1}^*\}$ and the action of U on which given by

$$\begin{aligned} K \cdot v_i^* &= q^{2i+1-r} v_i^*, \\ F \cdot v_i^* &= -[i] q^{r-1-2i} v_{i-1}^*, \\ E \cdot v_i^* &= -q^{2i+3-r} [r-i-1] v_{i+1}^*, & i &\not\equiv -1 \pmod{l}, \\ E \cdot v_i^* &= -q^{1-r} [r] v_{i+1}^* - q^{1-r} [r](1+\lambda) v_{i+1-l}^*, & i &\equiv -1 \pmod{l}, \end{aligned}$$

for $0 \leqslant i \leqslant nl+r-1$. It is clear that $V(\lambda, n, r)^*$ is the submodule of $V^\lambda(n,r)^*$ spanned by $\{v_0^*, v_1^*, \ldots, v_{nl-1}^*\}$ and $V(\infty, n, r)^*$ is the submodule of $V^\lambda(n,r)^*$ spanned by $\{v_r^*, v_{r+1}^*, \ldots, v_{nl+r-1}^*\}$.

For the basic properties of the indecomposable representations of U, we have the following proposition.

Proposition 2.2 ([2, Proposition 4.1]). *Let M be an indecomposable U-module. Then there exists $0 \leqslant r \leqslant \frac{l-1}{2}$ such that*

$$\left(\Omega - \left[\frac{r}{2}\right]^2\right)^2 \cdot M = 0.$$

If $r = 0$, Ω is zero on M.

For $0 \leqslant r \leqslant \frac{l-1}{2}$, let $\mathcal{C}_r$ denote the category of U-modules M with the property

$$\left(\Omega - \left[\frac{r}{2}\right]^2\right)^2 \cdot M = 0.$$

For the characterization of all indecomposable U-modules, we have the following proposition.

Proposition 2.3 ([2, Theorem 4.2]). *Let M be an indecomposable object of $\mathcal{C}_r$.*

(1) *If $r = 0$, then M is isomorphic to $V(1, 0)$.*
(2) *If $r > 0$ and Ω is semisimple, then M or M^* is isomorphic to precisely one of $V(n, i), V(\lambda, m, i)$ and $V(\infty, m, i)$, where $i = l - r$ or r, $n \geqslant 0$ and $m > 0$.*
(3) *If Ω is not semisimple on M, then $r > 0$ and M is isomorphic to $X(r)$.*

We end this section by giving the following result which will be used in the next section.

Proposition 2.4. *If M is a finite dimensional indecomposable U-module, then M is generated by finitely many elements $v_1, v_2, \ldots, v_t$ such that $K \cdot v_i = q^{m_i} v_i$ for $m_i \in \mathbb{N}$ and $1 \leqslant i \leqslant t$.*

Proof. According to the constructions of $V(n, r), V(\lambda, m, r), V(\infty, m, r)$ and $X(r)$, it is clear that the basis of them are eigenvectors of the action of K. And so is their dual modules. □

3. Annihilator ideals of simple modules

In this section, we shall describe all ideals of U using the adjoint action. And then we determine the generators of annihilator ideals of simple U-modules. For any $a \in U$, we write $\Delta(a) = \Sigma a_1 \otimes a_2$. We can consider U as a U-module under the following action:

$$(\mathrm{ad} a)b = \sum a_1 b S(a_2) \quad \text{for } a, b \in U.$$

For any $a \in U$, we denote $[a]$ the submodule of U generated by a under the adjoint action and (a) the two-sided ideal of U generated by a. It is obvious that $[a] \in (a)$.

Proposition 3.1. *Let V be any indecomposable U-submodule of U under the adjoint action. Then there exist integers $n_1, \ldots, n_s, m_1, \ldots, m_k$ with $0 \leqslant n_i, m_j \leqslant l-1$ and polynomials $f_i(\Omega, K), g_j(\Omega, K)$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant k$ such that*

$$V = [E^{n_1} f_1(\Omega, K), \ldots, E^{n_s} f_s(\Omega, K), F^{m_1} g_1(\Omega, K), \ldots, F^{m_k} g_k(\Omega, K)].$$

Proof. According to Proposition 2.4, we have that each indecomposable U-submodule of U under the adjoint action is generated by finitely many eigenvectors of $\mathrm{ad}K$. Let $u = \sum_{i,j,p} a_{ijp} E^i F^j K^p \in V$ be one of the generators with weight q^t for $0 \leqslant t \leqslant l-1$. It is clear that

$$(\mathrm{ad}K)u = KuK^{-1} = \sum_{i,j,p} q^{2i-2j} a_{ijp} E^i F^j K^p = q^t \sum_{i,j,p} a_{ijp} E^i F^j K^p.$$

It follows that $q^{2i-2j} = q^t$ if $a_{ijp} \neq 0$. Hence when $a_{ijp} \neq 0$, we have $2i - 2j = t + dl$, where $d \in \{0, 1, -1, -2\}$. If $d \in \{0, 1\}$, then we can rewrite u as the form $\sum_j E^{j+\frac{t+dl}{2}} F^j h_j(K)$ for some polynomials $h_j(K)$. If $d \in \{-1, -2\}$, then we can rewrite u as the form $\sum_i E^i F^{i-\frac{t+dl}{2}} h_i(K)$ for some polynomials $h_i(K)$. By (2.7) and (2.11), we can replace each factor EF by the element Ω modulo a polynomial in K. Thus, u can be rewritten in the form of $E^{\frac{t+dl}{2}} f(\Omega, K)$ ($d \in \{0, 1\}$) or $F^{-\frac{t+dl}{2}} g(\Omega, K)$ ($d \in \{-1, -2\}$) for polynomials $f(\Omega, K)$ and $g(\Omega, K)$. Hence we complete the proof. □

Proposition 3.2. *Let I be any non-zero two-sided ideal of U. Then there exist integers $0 \leqslant n_1, \ldots, n_s, m_1, \ldots, m_k \leqslant l-1$ and polynomials $f_i(\Omega, K), g_j(\Omega, K)$ for $1 \leqslant i \leqslant s, 1 \leqslant j \leqslant k$ such that*

$$I = (E^{n_1} f_1(\Omega, K), \ldots, E^{n_s} f_s(\Omega, K), F^{m_1} g_1(\Omega, K), \ldots, F^{m_k} g_k(\Omega, K)).$$

Proof. Let I be a non-zero two-sided ideal of U. Then I is a submodule of U under the adjoint action. It follows from Proposition 3.1 that

$$I = \sum_{i=1}^{s} [E^{n_i} f_i(\Omega, K)] + \sum_{j=1}^{k} [F^{m_j} g_j(\Omega, K)].$$

Noting that each $E^{n_i} f_i(\Omega, K), F^{m_j} g_j(\Omega, K) \in I$, we obtain that

$$(E^{n_1} f_1(\Omega, K), \ldots, E^{n_s} f_s(\Omega, K), F^{m_1} g_1(\Omega, K), \ldots, F^{m_k} g_k(\Omega, K)) \subseteq I.$$

On the other hand,

$$\begin{aligned} I &= \sum_{i=1}^{s} [E^{n_i} f_i(\Omega, K)] + \sum_{j=1}^{k} [F^{m_j} g_j(\Omega, K)] \\ &\subseteq (E^{n_1} f_1(\Omega, K), \ldots, E^{n_s} f_s(\Omega, K), F^{m_1} g_1(\Omega, K), \ldots, F^{m_k} g_k(\Omega, K)). \end{aligned}$$

It follows that

$$I = (E^{n_1} f_1(\Omega, K), \ldots, E^{n_s} f_s(\Omega, K), F^{m_1} g_1(\Omega, K), \ldots, F^{m_k} g_k(\Omega, K)).$$ □

Theorem 3.3. *For $1 \leqslant r \leqslant l-1$, the annihilator ideal of $V(0,r)$ is*

$$\left(\Omega - \left[\frac{r}{2}\right]^2, (K-q^{r-1})(K-q^{r-3})\cdots(K-q^{-(r-1)})\right).$$

Proof. Let

$$I = (E^{n_1} f_1(\Omega,K), \dots, E^{n_s} f_s(\Omega,K), F^{m_1} g_1(\Omega,K), \dots, F^{m_k} g_k(\Omega,K))$$

be the annihilator ideal of $V(0,r)$. By (2.8), (2.9) and (2.10), we have that

$$\begin{aligned} \Omega \cdot v_i &= FE \cdot v_i + \frac{qK + q^{-1}K^{-1} - 2}{(q-q^{-1})^2} \cdot v_i \\ &= [i][r-i]v_i + \frac{q^{r-2i} + q^{2i-r} - 2}{(q-q^{-1})^2} v_i = \left[\frac{r}{2}\right]^2 v_i \end{aligned}$$

for $0 \leqslant i \leqslant r-1$ and

$$(K-q^{r-1}) \cdot v_0 = 0,\ (K-q^{r-3}) \cdot v_1 = 0,\ \dots,\ (K-q^{-(r-1)}) \cdot v_{r-1} = 0.$$

Hence it follows that

$$\left(\Omega - \left[\frac{r}{2}\right]^2, (K-q^{r-1})(K-q^{r-3})\cdots(K-q^{-(r-1)})\right) \subseteq I.$$

Since I is the annihilator ideal of $V(0,r)$, we obtain that

$$E^{n_j} f_j(\Omega,K) \cdot v_i = 0, \quad F^{m_p} g_p(\Omega,K) \cdot v_i = 0$$

for $0 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant s$ and $1 \leqslant p \leqslant k$. If $n_j \geqslant r$, then $E^{n_j} f_j(\Omega,K) \in (E^r)$. If $n_j \leqslant r-1$, noting that $f_j(\Omega,K)$ acts on v_i as multiplied by a scalar and $E^{n_j} \cdot v_i \neq 0$ for $n_j \leqslant i \leqslant r-1$, we have $f_j(\Omega,K) \cdot v_i = 0$ for $n_j \leqslant i \leqslant r-1$. Since l is the degree of the minimum annihilator polynomial of Ω, we may assume that

$$f_j(\Omega,K) = \left(\Omega - \left[\frac{r}{2}\right]^2\right)^{l-1} h_{j,1}(K) + \cdots + \left(\Omega - \left[\frac{r}{2}\right]^2\right) h_{j,l-1}(K) + h_{j,l}(K),$$

where $h_{j,a}(K)$ are polynomials of K for $1 \leqslant a \leqslant l$. Hence $f_j(\Omega,K) \cdot v_i = h_{j,l}(K) \cdot v_i = 0$ for $n_j \leqslant i \leqslant r-1$. Since $K \cdot v_i = q^{r-1-2i} v_i$, we have $h_{j,l}(q^{r-1-2i}) = 0$ for $n_j \leqslant i \leqslant r-1$. Hence

$$y - q^{r-1-2n_j} \mid h_{j,l}(y),\ y - q^{r-1-2(n_j+1)} \mid h_{j,l}(y),\ \dots,\ y - q^{r-1-2(r-1)} \mid h_{j,l}(y).$$

So we suppose

$$h_{j,l}(y) = (y - q^{r-1-2n_j})(y - q^{r-1-2(n_j+1)}) \cdots (y - q^{r-1-2(r-1)}) a_j(y)$$

for some $a_j(y) \in \Bbbk[y]$. Hence for $1 \leqslant j \leqslant s$, we have

$$\begin{aligned} &E^{n_j} f_j(\Omega,K) \\ &= E^{n_j} \left(\Omega - \left[\frac{r}{2}\right]^2\right)^{l-1} h_{j,1}(K) + \cdots + E^{n_j} \left(\Omega - \left[\frac{r}{2}\right]^2\right) h_{j,l-1}(K) \\ &\quad + E^{n_j} (K - q^{r-1-2n_j})(K - q^{r-1-2(n_j+1)}) \cdots (K - q^{r-1-2(r-1)}) a_j(K) \end{aligned}$$

$$\in \left(\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}), \ldots,\ E^{r-1}(K - q^{-(r-1)})\right).$$

We shall show that

$$\left(\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}), \ldots,\ E^{r-1}(K - q^{-(r-1)}), E^r\right)$$
$$= \left(\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})\right).$$

Note that

$$\begin{aligned} &E(K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \\ &\quad - (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})E \\ &= (1 - q^{2r})EK(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \\ &\in ((K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})). \end{aligned}$$

Since $1 - q^{2r} \neq 0$ and K is invertible, we have

$$\begin{aligned} &E(K - q^{r-3})(K - q^{r-5}) \cdots (K - q^{-(r-1)}) \\ &\in ((K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})). \end{aligned}$$

Similarly, we can prove

$$\begin{aligned} &E^b(K - q^{r-1-2b})(K - q^{r-1-2(b+1)}) \cdots (K - q^{r-1-2(r-1)}) \\ &\in ((K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})) \end{aligned}$$

for $1 \leqslant b \leqslant r$. Hence for $1 \leqslant j \leqslant s$,

$$E^{n_j} f_j(\Omega, K) \in \left(\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})\right).$$

In a similar way, we can prove that

$$F^{m_p} g_p(\Omega, K) \in \left(\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})\right)$$

for $1 \leqslant p \leqslant k$. Hence $I = (\Omega - \left[\frac{r}{2}\right]^2,\ (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}))$. □

Proposition 3.4. *The annihilator ideal of* $V(1,0)$ *is*

$$(\Omega, (K - q^{l-1})(K - q^{l-3}) \cdots (K - q^{-(l-1)})).$$

Proof. The proof is similar to that of Theorem 3.3. □

Proposition 3.5. *For* $1 \leqslant r \leqslant l - 1$,

$$\left(\Omega - \left[\frac{r}{2}\right]^2, (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)})\right)$$

can not be any of the annihilator ideals of indecomposable modules except $V(0, r)$.

Proof. Set $I = (\Omega - \left[\frac{r}{2}\right]^2, (K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}))$. Consider the basis element v_r in $V(n, r)$ with $n \geqslant 1$. Note that $K \cdot v_r = q^{-r-1} v_r$. Since

$$\begin{aligned}&(K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \cdot v_r \\ = &(q^{-r-1} - q^{r-1})(q^{-r-1} - q^{r-3}) \cdots (q^{-r-1} - q^{-(r-1)}) v_r \neq 0,\end{aligned}$$

it follows that I can not be the annihilator ideal of $V(n, r)$ with $n \geqslant 1$. Consider $v_0 \in V(1, 0)$. Note that $K \cdot v_0 = q^{-1} v_0$. Since

$$\begin{aligned}&(K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \cdot v_0 \\ = &(q^{-1} - q^{r-1})(q^{-1} - q^{r-3}) \cdots (q^{-1} - q^{-(r-1)}) v_0 \neq 0,\end{aligned}$$

it follows that I can not be the annihilator ideal of $V(1, 0)$. Consider $x_0 \in X(r)$ and v_r in $V(\lambda, m, r)$ or $V(\infty, m, r)$. Note that $K \cdot x_0 = q^{-r-1} x_0$ and $K \cdot v_r = q^{-r-1} v_r$. Similarly,

$$(K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \cdot x_0 \neq 0$$

and

$$(K - q^{r-1})(K - q^{r-3}) \cdots (K - q^{-(r-1)}) \cdot v_r \neq 0.$$

Hence I can not be the annihilator ideal of $X(r)$, $V(\lambda, m, r)$ or $V(\infty, m, r)$. Using the same technique, we prove that I can not be any of the annihilator ideals of their dual modules. □

Proposition 3.6. *The following ideal*

$$(\Omega, (K - q^{l-1})(K - q^{l-3}) \cdots (K - q^{-(l-1)}))$$

can not be any of the annihilator ideals of indecomposable modules except $V(1, 0)$.

Proof. The proof is similar to that of Proposition 3.5. □

Acknowledgements. The author would like to express gratitude to Professor Libin Li for his helpful suggestions and constructive comments. The author also wishes to thank the reviewer for his or her careful review.

References

[1] S. Catoiu, *Ideals of the enveloping algebra* $U(\mathrm{sl}_2)$, J. Algebra **202** (1998), no. 1, 142–177. https://doi.org/10.1006/jabr.1997.7284

[2] V. Chari and A. A. Premet, *Indecomposable restricted representations of quantum* sl_2, Publ. Res. Inst. Math. Sci. **30** (1994), no. 2, 335–352. https://doi.org/10.2977/prims/1195166137

[3] J. L. Fisher and S. K. Sehgal, *Principal ideal group rings*, Comm. Algebra **4** (1976), no. 4, 319–325. https://doi.org/10.1080/00927877608822109

[4] E. Jespers and J. Okniński, *Semigroup algebras that are principal ideal rings*, J. Algebra **183** (1996), no. 3, 837–863. https://doi.org/10.1006/jabr.1996.0241

[5] C. Kassel, *Quantum Groups*, Springer, New York, 1995. https://doi.org/10.1007/978-1-4612-0783-2

[6] T. Kerler, *Mapping class group actions on quantum doubles*, Comm. Math. Phys. **168** (1995), no. 2, 353–388. http://projecteuclid.org/euclid.cmp/1104272363

[7] H. Kondo and Y. Saito, *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to sl_2*, J. Algebra **330** (2011), 103–129. https://doi.org/10.1016/j.jalgebra.2011.01.010
[8] L. B. Li and P. Zhang, *Weight property for ideals of $U_q(\mathrm{sl}(2))$*, Comm. Algebra **29** (2001), no. 11, 4853–4870.
[9] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. https://doi.org/10.1090/cbms/082
[10] K. Morita, *On group rings over a modular field which possess radicals expressible as principal ideals*, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A **4** (1951), 177–194.
[11] D. S. Passman, *Observations on group rings*, Comm. Algebra **5** (1977), no. 11, 1119–1162. https://doi.org/10.1080/00927877708822213
[12] S. Siciliano and H. Usefi, *Enveloping algebras that are principal ideal rings*, J. Pure Appl. Algebra **221** (2017), no. 10, 2573–2581. https://doi.org/10.1016/j.jpaa.2017.01.003
[13] D. Su and S. Yang, *Representation rings of small quantum groups $\overline{U}_q(sl_2)$*, J. Math. Phys. **58** (2017), no. 9, 091704, 24 pp. https://doi.org/10.1063/1.4986839
[14] R. Suter, *Modules over $U_q(sl_2)$*, Comm. Math. Phys. **163** (1994), no. 2, 359–393. http://projecteuclid.org/euclid.cmp/1104270468
[15] Y. Wang, *Classification of ideals of 8-dimensional Radford Hopf algebra*, Czechoslovak Math. J. **72(147)** (2022), no. 4, 1019–1028. https://doi.org/10.21136/CMJ.2022.0313-21
[16] Y. Wang, *Annihilator ideals of indecomposable modules of finite-dimensional pointed Hopf algebras of rank one*, Algebra Colloq., to appear. Available at https://arxiv.org/abs/2205.14603v2 (2022), 21 pages.
[17] Y. Wang, Z. H. Wang, and L. B. Li, *Ideals of finite-dimensional pointed Hopf algebras of rank one*, Algebra Colloq. **28** (2021), no. 2, 351–360. https://doi.org/10.1142/S1005386721000274
[18] Y. Wang, Y. Zheng, and L. B. Li, *On the ideals of the Radford Hopf algebras*, Comm. Algebra **49** (2021), no. 10, 4109–4122. https://doi.org/10.1080/00927872.2021.1914073
[19] J. Xiao, *Finite-dimensional representations of $U_t(\mathrm{sl}(2))$ at roots of unity*, Canad. J. Math. **49** (1997), no. 4, 772–787. https://doi.org/10.4153/CJM-1997-038-4

Yu Wang
School of Mathematics and Physics
Jiangsu University of Technology
Changzhou, Jiangsu 213001, P. R. China
Email address: yu.wang@mail.sdu.edu.cn