Bull. Korean Math. Soc. **60** (2023), No. 4, pp. 1003–1016
https://doi.org/10.4134/BKMS.b220452
pISSN: 1015-8634 / eISSN: 2234-3016

CLASSIFICATION OF TWISTED PRODUCT LIGHTLIKE SUBMANIFOLDS

Sangeet Kumar and Megha Pruthi

Abstract. In this paper, we introduce the idea of twisted product lightlike submanifolds of semi-Riemannian manifolds and provide non-trivial examples of such lightlike submanifolds. Then, we prove the non-existence of proper isotropic or totally lightlike twisted product submanifolds of a semi-Riemannian manifold. We also show that for a twisted product lightlike submanifold of a semi-Riemannian manifold, the induced connection ∇ is not a metric connection. Further, we prove that a totally umbilical SCR-lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$ does not admit any twisted product SCR-lightlike submanifold of the type $M_{\perp} \times_{\phi} M_T$, where $M_{\perp}$ is a totally real submanifold and M_T is a holomorphic submanifold of $\tilde{M}$. Consequently, we obtain a geometric inequality for the second fundamental form of twisted product SCR-lightlike submanifolds of the type $M_T \times_{\phi} M_{\perp}$ of an indefinite Kaehler manifold $\tilde{M}$, in terms of the gradient of $\ln \phi$, where ϕ stands for the twisting function. Subsequently, the equality case of this inequality is discussed. Finally, we construct a non-trivial example of a twisted product SCR-lightlike submanifold in an indefinite Kaehler manifold.

1. Introduction

For a general investigation of totally umbilical submanifolds and extrinsic spheres in Riemannian geometry, Chen [4] introduced the idea of twisted product manifolds as:

Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be two Riemannian manifolds and $\phi > 0$ be a differentiable function on $M_1 \times M_2$. Then, the twisted product $M_1 \times_{\phi} M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric given by

$$g = g_{M_1} + \phi^2 g_{M_2}. \tag{1}$$

If Y is tangent to $M = M_1 \times_{\phi} M_2$ at (x, y), then

$$||Y||^2 = ||d\pi_1(Y)||^2 + \phi^2(x, y)||d\pi_2(Y)||^2,$$

Received June 30, 2022; Revised January 5, 2023; Accepted February 10, 2023.
2020 *Mathematics Subject Classification.* 53B30, 53B25, 53B35.
Key words and phrases. Twisted product lightlike submanifold, indefinite Kaehler manifold, SCR-lightlike submanifold.
This work was financially supported by SERB, GoI vide File No. ECR/2017/000786.

where π_i $(i = 1, 2)$, respectively, denote the canonical projections of $M_1 \times M_2$ onto M_1 and M_2 with $d\pi_i$'s and ϕ being the differential maps and the twisting function, respectively. In case, the twisting function ϕ depends on M_1 only, then the twisted product manifold reduces to a warped product manifold (c.f., [3]).

The concept of twisted product manifolds has been employed to study several geometric properties of hypersurfaces in different ambient space settings, namely, in hypersurfaces of complex space forms, Lagrangian submanifolds and curvature netted hypersurfaces (c.f., [6, 13, 16]). Moreover, the curvature properties of twisted product manifolds have been extensively explored in semi-Riemannian geometry (c.f., [2,12,17]). The relationship among twisted product and warped product manifolds in semi-Riemannian geometry has been investigated by Fernández-López et al. in [11] and by Ponge and Reckziegel in [18].

In [5], Chen considered CR-submanifolds as twisted products of the type $M_\perp \times_\phi M_T$ and $M_T \times_\phi M_\perp$ in Kaehler manifolds such that $M_\perp$ represents a totally real submanifold and M_T represents a holomorphic submanifold of a Kaehler manifold $\tilde{M}$. Precisely, he proved the non-existence of twisted product CR-submanifolds of the type $M_\perp \times_\phi M_T$ in Kaehler manifolds and established a geometric inequality for twisted product CR-submanifolds of the type $M_T \times_\phi M_\perp$ in Kaehler manifolds in terms of the second fundamental form. In [20], Sahin discussed the non-existence of doubly warped product CR-submanifolds and doubly twisted product CR-submanifolds in Kaehler manifolds.

One may note that the majority of the available work on twisted products and warped products emphasizes on manifolds with positive definite metric. Thus, the available results may not be suitable to study those topics of mathematical physics and relativity, where indefinite metrics are employed, thereby limiting the application area of available work. The relativity theory led to development and investigation of semi-Riemannian manifolds, which in turn provides a broad set up for the examination of twisted products and warped products and may prompt some striking applications. In this context, Duggal [7] introduced warped product lightlike manifolds, where he discussed two classes of warped product lightlike manifolds. Further, Sahin [19] initiated the idea of warped product lightlike submanifolds in a semi-Riemannian manifold and proved various characterization results on this class of warped products. In this continuation, the warped product lightlike submanifolds are studied by Kumar in indefinite Kaehler and nearly Kaehler manifolds (c.f., [14, 15]). But, till date, no endeavours have been made to study twisted product lightlike submanifolds of semi-Riemannian manifolds. Therefore in this paper, we study twisted product lightlike submanifolds of semi-Riemannian manifolds. After defining a twisted product lightlike submanifold of a semi-Riemannian manifold, we present two non-trivial examples of such lightlike submanifolds. Then, we show that there does not exist any proper isotropic or totally lightlike twisted product submanifolds of a semi-Riemannian manifold. Further, we investigate twisted product SCR-lightlike submanifolds in indefinite Kaehler manifolds and prove that a totally umbilical SCR-lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$ does not admit any twisted product SCR-lightlike submanifold of the type $M_{\perp} \times_{\phi} M_T$, where $M_{\perp}$ represents a totally real submanifold and M_T represents a holomorphic submanifold of an indefinite Kaehler manifold $\tilde{M}$. Moreover, we obtain a geometric inequality for the second fundamental form of twisted product SCR-lightlike submanifolds of the type $M_T \times_{\phi} M_{\perp}$ in $\tilde{M}$, in terms of the gradient of $\ln \phi$, where ϕ stands for the twisting function. Consequently, we discuss the equality case of this inequality. Finally, we present a non-trivial example of a twisted product SCR-lightlike submanifold of an indefinite Kaehler manifold.

2. Preliminaries

2.1. Geometry of lightlike submanifolds

Assume that (M_m, g) is an immersed submanifold of a semi-Riemannian manifold $(\tilde{M}_{m+n}, \tilde{g})$ with constant index q, (provided, $m, n \geq 1$ and $1 \leq q \leq m+n-1$) and g is the induced metric of $\tilde{g}$ on M. Then M is known as a lightlike submanifold of $\tilde{M}$ if $\tilde{g}$ becomes degenerate on the tangent bundle TM of M. For a degenerate metric g on M, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces, but no longer complementary. Thus, there exists a radical (null) subspace $Rad(T_xM)$ such that $Rad(T_xM) = T_xM \cap T_xM^{\perp}$. The submanifold M of $\tilde{M}$ is said to be an r-lightlike submanifold [8] if the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$ defines a smooth distribution on M with rank $r > 0$, $1 \leq r \leq m$. While the radical distribution $Rad(TM)$ of TM is defined as

$$Rad(TM) = \cup_{x \in M}\{\xi \in T_xM \mid g(u, \xi) = 0, \ \forall\, u \in T_xM, \ \xi \neq 0\}.$$

Moreover, $S(TM)$ is the screen distribution in TM such that

$$TM = Rad(TM) \perp S(TM).$$

On the other hand, $S(TM^{\perp})$ denotes a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$ such that $TM^{\perp} = Rad(TM) \perp S(TM^{\perp})$. Moreover, there exists a local null frame $\{N_i\}$ of null sections with values in orthogonal complementary subspace of $S(TM^{\perp})$ in $S(TM^{\perp})^{\perp}$ satisfying

$$\tilde{g}(N_i, N_j) = 0, \quad \tilde{g}(N_i, \xi_j) = \delta_{ij} \ \text{ for } \ i, j \in \{1, 2, \ldots, r\},$$

where $\{\xi_1, \ldots, \xi_r\}$ is a local basis of $\Gamma(Rad(TM))$. This implies that $tr(TM)$ and $ltr(TM)$, respectively, are the vector bundles in $T\tilde{M} \mid_M$ and $S(TM^{\perp})^{\perp}$ with the property

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}) \tag{2}$$

and

$$T\tilde{M} \mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{3}$$

In view of decomposition (3), the Gauss and Weingarten formulae are

$$\tilde{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h(Y_1, Y_2), \quad \tilde{\nabla}_{Y_1} V = -A_V Y_1 + \nabla^t_{Y_1} V$$

for $V \in \Gamma(tr(TM))$ and $Y_1, Y_2 \in \Gamma(TM)$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$. According to Eq. (2), the Gauss and Weingarten formulae become

$$\tilde{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{4}$$

$$\tilde{\nabla}_{Y_1} W = -A_W Y_1 + \nabla^s_{Y_1} W + D^l(Y_1, W), \tag{5}$$

$$\tilde{\nabla}_{Y_1} N = -A_N Y_1 + \nabla^l_{Y_1} N + D^s(Y_1, N),$$

where $N \in \Gamma(ltr(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $Y_1, Y_2 \in \Gamma(TM)$. Further, using Eqs. (4) and (5), one has

$$g(A_W Y_1, Y_2) = \tilde{g}(h^s(Y_1, Y_2), W) + \tilde{g}(D^l(Y_1, W), Y_2) \tag{6}$$

for $W \in \Gamma(S(TM^{\perp}))$ and $Y_1, Y_2 \in \Gamma(TM)$.

It may be noted that the induced connection ∇ on M is not a metric connection. As $\tilde{\nabla}$ is a metric connection on $\tilde{M}$, thus employing Eq. (4), one has

$$(\nabla_{Y_1} g)(Y_2, Y_3) = \tilde{g}(h^l(Y_1, Y_3), Y_2) + \tilde{g}(h^l(Y_1, Y_2), Y_3) \tag{7}$$

for $Y_1, Y_2, Y_3 \in \Gamma(TM)$.

Definition 1 ([9])**.** Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold. Then, a lightlike submanifold (M, g) of $(\tilde{M}, \tilde{g})$ is said to be totally umbilical if there exist a smooth transversal curvature vector field $H \in \Gamma(tr(TM))$ on M such that for $Y_1, Y_2 \in \Gamma(TM)$,

$$h(Y_1, Y_2) = H\tilde{g}(Y_1, Y_2).$$

According to Eqs. (4) and (5), M is called a totally umbilical lightlike submanifold if and only if there exist smooth vector fields $H^s \in \Gamma(S(TM^{\perp}))$ and $H^l \in \Gamma(ltr(TM))$ satisfying

$$D^l(Y_1, W) = 0, \quad h^s(Y_1, Y_2) = H^s\tilde{g}(Y_1, Y_2), \quad h^l(Y_1, Y_2) = H^l\tilde{g}(Y_1, Y_2)$$

for $Y_1, Y_2 \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Moreover, a lightlike submanifold M of a semi-Riemannian manifold $\tilde{M}$ is said to be mixed geodesic if and only if $h(X, Y) = 0$ for $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Definition 2. An indefinite almost Hermitian manifold $\tilde{M}$ with an indefinite Hermitian metric $\tilde{g}$ and an almost complex structure $\tilde{J}$ is said to be an indefinite Kaehler manifold (c.f., [1]) if

$$\tilde{J}^2 = -I, \quad \tilde{g}(\tilde{J}Y_1, \tilde{J}Y_2) = \tilde{g}(Y_1, Y_2), \quad (\tilde{\nabla}_{Y_1}\tilde{J})Y_2 = 0, \ \forall\ Y_1, Y_2 \in \Gamma(TM).$$

Definition 3 ([10])**.** A real lightlike submanifold $(M, g, S(TM))$ of an indefinite Kaehler manifold $(\tilde{M}, \tilde{g}, \tilde{J})$ is known as a Screen Cauchy-Riemann (SCR)-lightlike submanifold if

(A) There exists a real non-null distribution $D \subset S(TM)$ satisfying

$$S(TM) = D \oplus D^{\perp}, \quad \tilde{J}D = D, \quad \tilde{J}D^{\perp} \subset S(TM^{\perp}),$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

(B) $Rad(TM)$ is invariant with respect to $\tilde{J}$.

In view of Definition 3, we consider $D' = D \perp Rad(TM)$.

3. Twisted product lightlike submanifolds of semi-Riemannian manifolds

In the present segment, firstly, we define a twisted product lightlike submanifolds of a semi-Riemannian manifold following the approach of Sahin [19] as follows:

Definition 4. Let (M_1^r, g_1) and (M_2^m, g_2) be a totally lightlike submanifold and a semi-Riemannian submanifold, respectively, of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$. Then, the twisted product lightlike submanifold is defined as the product manifold $M = M_1 \times_{\phi} M_2$ of $\tilde{M}$ with the degenerate metric g defined by

$$g(Y_1, Y_2) = g_1(\pi_* Y_1, \pi_* Y_2) + \phi^2 g_2(\eta_* Y_1, \eta_* Y_2)$$

for every $Y_1, Y_2 \in \Gamma(TM)$ and $*$ denotes the tangent map. Here, $\pi : M_1 \times M_2 \to M_1$ and $\eta : M_1 \times M_2 \to M_2$ represent projection maps satisfying $\pi(x, y) = x$ and $\eta(x, y) = y$ for $(x, y) \in M_1 \times M_2$.

Remark 3.1. In view of above definition, we conclude

(i) If $M_1 \neq \{0\}$, $M_2 \neq \{0\}$ and ϕ is non-constant on M, then M becomes a proper twisted product lightlike submanifold.

(ii) If ϕ depends only on M_1, then M becomes a warped product lightlike submanifold.

(iii) M becomes an r-lightlike submanifold of $\tilde{M}$ if $Rad(TM)$ and $S(TM)$ have rank r and m, respectively.

Next, we give two examples of twisted product lightlike submanifolds of a semi-Riemannian manifold.

Example 3.2. Let M be a submanifold of a semi-Riemannian manifold $\tilde{M} = (R_1^8, \tilde{g})$ with

$$x^1 = \sqrt{2}u^1, \quad x^2 = u^1 \sin u^2, \quad x^3 = u^1 \cos u^2, \quad x^4 = u^1 \sin u^3,$$
$$x^5 = u^1 \cos u^3, \quad x^6 = \tfrac{(u^2)^2}{2}, \quad x^7 = u^2 u^3, \quad x^8 = \tfrac{(u^3)^2}{2},$$

where $u^2, u^3 \in R - \{\frac{n\pi}{2}, n \in Z\}$. Then TM is spanned by Z_1, Z_2, Z_3 such that

$$Z_1 = \sqrt{2}\partial x_1 + \sin u^2 \partial x_2 + \cos u^2 \partial x_3 + \sin u^3 \partial x_4 + \cos u^3 \partial x_5,$$
$$Z_2 = u^1 \cos u^2 \partial x_2 - u^1 \sin u^2 \partial x_3 + u^2 \partial x_6 + u^3 \partial x_7,$$
$$Z_3 = u^1 \cos u^3 \partial x_4 - u^1 \sin u^3 \partial x_5 + u^2 \partial x_7 + u^3 \partial x_8.$$

It is clear that M is a 1-lightlike submanifold with $Rad(TM) = Span\{Z_1\}$ and $S(TM) = Span\{Z_2, Z_3\}$. Further, $S(TM^\perp) = Span\{W = \sin u^2\partial x_2 + \cos u^2\partial x_3 - \sin u^3\partial x_4 - \cos u^3\partial x_5\}$ and $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{4}(-\sqrt{2}\partial x_1 + \sin u^2\partial x_2 + \cos u^2\partial x_3 + \sin u^3\partial x_4 + \cos u^3\partial x_5).$$

Here, clearly $S(TM)$ and $Rad(TM)$ are integrable. If M_1 and M_2 represent the leaves of $Rad(TM)$ and $S(TM)$, respectively, then the induced metric tensor on M is given by

$$\begin{aligned} ds^2 &= 0(du_1^2) + ((u^1)^2 + (u^2)^2 + (u^3)^2)(du_2^2 + du_3^2) \\ &= ((u^1)^2 + (u^2)^2 + (u^3)^2)(du_2^2 + du_3^2). \end{aligned}$$

Hence, M is a proper twisted product lightlike submanifold $M_1 \times_\phi M_2$ of R_1^8, with the twisting function $\phi = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}$.

Example 3.3. Let M be a submanifold of a semi-Riemannian manifold $\tilde{M} = (R_2^{10}, \tilde{g})$ with

$$\begin{aligned} &x^1 = \sqrt{2}u^1, \quad x^2 = u^2, \quad x^3 = u^1 \sin u^3, \quad x^4 = u^1 \sin u^3, \\ &x^5 = u^1 \sin u^4, \quad x^6 = u^1 \cos u^4, \quad x^7 = u^2, \quad x^8 = \tfrac{(u^3)^2}{2}, \\ &x^9 = u^3u^4, \quad x^{10} = \tfrac{(u^4)^2}{2}, \quad \text{where } \ u^3, u^4 \in R - \{\tfrac{n\pi}{2}, \ n \in Z\}. \end{aligned}$$

Then TM is spanned by Z_1, Z_2, Z_3, Z_4 such that

$$\begin{aligned} Z_1 &= \sqrt{2}\partial x_1 + \sin u^3\partial x_3 + \cos u^3\partial x_4 + \sin u^4\partial x_5 + \cos u^4\partial x_6, \\ Z_2 &= \partial x_2 + \partial x_7, \\ Z_3 &= u^1 \cos u^3\partial x_3 - u^1 \sin u^3\partial x_4 + u^3\partial x_8 + u^4\partial x_9, \\ Z_4 &= u^1 \cos u^4\partial x_5 - u^1 \sin u^4\partial x_6 + u^3\partial x_9 + u^4\partial x_{10}. \end{aligned}$$

It is clear that M is a 2-lightlike submanifold with $Rad(TM) = Span\{Z_1, Z_2\}$ and $S(TM) = Span\{Z_3, Z_4\}$. Further, $S(TM^\perp) = Span\{W = \sin u^3\partial x_3 + \cos u^3\partial x_4 - \sin u^4\partial x_5 - \cos u^4\partial x_6\}$ and $ltr(TM)$ is spanned by

$$\begin{aligned} N_1 &= \frac{1}{2}(-\sqrt{2}\partial x_1 + \sin u^3\partial x_3 + \cos u^3\partial x_4 + \sin u^4\partial x_5 + \cos u^4\partial x_6), \\ N_2 &= \frac{1}{2}(-\partial x_2 + \partial x_7). \end{aligned}$$

Here, clearly $S(TM)$ and $Rad(TM)$ are integrable. If M_1 and M_2 represent the leaves of $Rad(TM)$ and $S(TM)$, respectively, then induced metric tensor on M is given by

$$\begin{aligned} ds^2 &= 0(du_1^2 + du_2^2) + ((u^1)^2 + (u^3)^2 + (u^4)^2)(du_3^2 + du_4^2) \\ &= ((u^1)^2 + (u^3)^2 + (u^4)^2)(du_3^2 + du_4^2). \end{aligned}$$

Hence, M is a proper twisted product lightlike submanifold $M_1 \times_\phi M_2$ of R_2^{10}, with the twisting function $\phi = \sqrt{(u^1)^2 + (u^3)^2 + (u^4)^2}$.

Now, from Proposition 1 of [11], for a twisted product manifold, we have the following result.

Lemma 3.4 ([11])**.** *Let $M = M_1 \times_\phi M_2$ be a twisted product manifold. Then*

$$\nabla_{Y_1} Y_2 \in \Gamma(TM_1),$$

$$\nabla_{Y_1} Z = \nabla_Z Y_1 = \left(\frac{Y_1\phi}{\phi}\right) Z \tag{8}$$

for $Y_1, Y_2 \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$.

Proposition 3.5. *There do not exist any proper isotropic or totally lightlike twisted product submanifolds of a semi-Riemannian manifold $\tilde{M}$.*

Proof. Assume that M is an isotropic twisted product lightlike submanifold. Then $S(TM) = 0$ which implies that $M_2 = 0$. Next, if we consider M to be a totally lightlike submanifold, then one has $S(TM) = 0$ which further gives $M_2 = 0$ and hence the proof follows. □

In general, Eq. (7) implies that the induced connection ∇ on M is not a metric connection. In this context, we have the following result.

Theorem 3.6. *Consider a twisted product lightlike submanifold $M = M_1 \times_\phi M_2$ of a semi-Riemannian manifold $\tilde{M}$. Then, the induced connection ∇ defined on M is not a metric connection.*

Proof. If possible, suppose that ∇ is a metric connection on M, then from Eq. (7), we get $h^l = 0$. As $\tilde{\nabla}$ is a metric connection on $\tilde{M}$, therefore for $Z_1, Z_2 \in \Gamma(S(TM))$ and $Y_1 \in \Gamma(Rad(TM))$, we have $\tilde{g}(\tilde{\nabla}_{Z_1} Z_2, Y_1) = -\tilde{g}(Z_2, \tilde{\nabla}_{Z_1} Y_1)$, further using Eqs. (4) and (8), we obtain

$$\tilde{g}(h^l(Z_1, Z_2), Y_1) = -Y_1(ln\phi)g(Z_1, Z_2). \tag{9}$$

Since $h^l = 0$, therefore Eq. (9) becomes $Y_1(ln\phi)g(Z_1, Z_2) = 0$, which implies that either $g(Z_1, Z_2) = 0$ or $Y_1(ln\phi) = 0$, but this leads to a contradiction as $S(TM)$ is non-degenerate and M is a proper twisted product lightlike submanifold, thus the result follows. □

Note. In the forthcoming part of the paper, M_T represents a holomorphic submanifold, $M_\perp$ represents a totally real submanifold and $\tilde{M}$ represents an indefinite Kaehler manifold, unless otherwise stated.

4. Twisted product SCR-lightlike submanifolds of the type $M_\perp \times_\phi M_T$ and $M_T \times_\phi M_\perp$ in indefinite Kaehler manifolds

In this part, we will investigate SCR-lightlike submanifolds of an indefinite Kaehler manifold $\tilde{M}$, which are twisted products of the type $M_\perp \times_\phi M_T$ and $M_T \times_\phi M_\perp$.

Theorem 4.1. *Suppose that M is a totally umbilical SCR-lightlike submanifold of $\tilde{M}$. If M is a twisted product SCR-lightlike submanifold of the type $M = M_\perp \times_\phi M_T$ in $\tilde{M}$, then M is an SCR-lightlike product.*

Proof. For $Z \in \Gamma(D^\perp)$ and $Y_1, Y_2 \in \Gamma(D')$, from Eq. (1), we obtain

$$\begin{aligned} Zg(Y_1, Y_2) &= (2\phi)(Z\phi)g_{M_T}(Y_1, Y_2) \\ &= 2\left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2). \end{aligned} \tag{10}$$

As M is totally umbilical in $\tilde{M}$ and $\tilde{\nabla}$ is a metric connection on $\tilde{M}$, therefore, employing Eqs. (4) and (8), for $Y_1, Y_2 \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, we find

$$\begin{aligned} Zg(Y_1, Y_2) &= g(\nabla_Z Y_1, Y_2) + g(Y_1, \nabla_Z Y_2) \\ &= \left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) + g(Y_1, \nabla_{Y_2} Z) \\ &= \left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) + \tilde{g}(Y_1, \tilde{\nabla}_{Y_2} Z) \\ &= \left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) - \tilde{g}(\tilde{\nabla}_{Y_2} Y_1, Z) \\ &= \left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) - g(\nabla_{Y_2} Y_1, Z). \end{aligned} \tag{11}$$

Using Eqs. (10) and (11), we get

$$\left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) = -g(\nabla_{Y_2} Y_1, Z). \tag{12}$$

Now, let h^T and A^T denote the second fundamental form and the shape operator of M_T in M, respectively. Then according to Gauss formula and Eq. (12), we obtain

$$\left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_2) = -g(h^T(Y_1, Y_2), Z). \tag{13}$$

For the second fundamental form h' of M_T in $\tilde{M}$, we acquire

$$h'(Y_1, Y_2) = h^T(Y_1, Y_2) + h^l(Y_1, Y_2) + h^s(Y_1, Y_2) \tag{14}$$

for any $Y_1, Y_2 \in \Gamma(D')$. From Eqs. (13) and (14), we can write

$$\begin{aligned} \tilde{g}(h'(Y_1, Y_1), Z) &= g(h^T(Y_1, Y_1), Z) \\ &= -\left(\frac{Z\phi}{\phi}\right) g(Y_1, Y_1). \end{aligned} \tag{15}$$

On the other hand, since M_T is a holomorphic submanifold in $\tilde{M}$, therefore one has

$$h'(Y_1, \tilde{J}Y_2) = h'(\tilde{J}Y_1, Y_2) = \tilde{J}h'(Y_1, Y_2). \tag{16}$$

Here, by combining Eqs. (15) and (16), we attain

$$\begin{aligned}\tilde{g}(h'(Y_1,Y_1),Z) &= \tilde{g}(h'(\tilde{J}Y_1,Y_1),\tilde{J}Z)\\ &= \left(\frac{Z\phi}{\phi}\right) g(Y_1,Y_1).\end{aligned}$$

Therefore, we obtain $Z(\ln\phi)g(Y_1,Y_1) = 0$ for $Z \in \Gamma(D^{\perp})$ and $Y_1 \in \Gamma(D')$. In particular, using the non-degeneracy of D, we derive $Z(\ln\phi) = 0$ for $Y_1 \in \Gamma(D)$. This gives that the twisting function ϕ depends only on M_T, which further reduces the twisted product SCR-lightlike submanifold of the type $M = M_{\perp} \times_{\phi} M_T$ into an SCR-lightlike product $M = M_{\perp} \times M_T$ with the new metric $g = g_{M_{\perp}} + \tilde{g}_{M_T}$, where $\tilde{g}_{M_T} = \phi^2 g_{M_T}$ in $\tilde{M}$. □

Next, we investigate twisted product SCR-lightlike submanifolds of the type $M_T \times_{\phi} M_{\perp}$ in $\tilde{M}$ and establish a geometric characteristic for the second fundamental form of twisted product SCR-lightlike submanifolds in $\tilde{M}$, in terms of the gradient of twisting function ϕ. At first, we establish an essential lemma for later use.

Lemma 4.2. *For a twisted product SCR-lightlike submanifold $M = M_T \times_{\phi} M_{\perp}$ of $\tilde{M}$, we have*

$$\tilde{g}(h^s(\tilde{J}Y_1,Z_1),\tilde{J}Z_2) = Y_1(\ln\phi)g(Z_1,Z_2) \tag{17}$$

for $Z_1, Z_2 \in \Gamma(D^{\perp})$ and $Y_1 \in \Gamma(D')$.

Proof. For $Z_1, Z_2 \in \Gamma(D^{\perp})$ and $Y_1 \in \Gamma(D')$, from Eq. (1), we can write

$$\begin{aligned}Y_1 g(Z_1,Z_2) &= (2\phi)(Y_1\phi)g_{M_{\perp}}(Z_1,Z_2)\\ &= 2\left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2).\end{aligned} \tag{18}$$

Since $\tilde{\nabla}$ is a metric connection on $\tilde{M}$, therefore from Eqs. (4) and (8), we can write

$$\begin{aligned}Y_1 g(Z_1,Z_2) &= g(\nabla_{Y_1}Z_1,Z_2) + g(Z_1,\nabla_{Y_1}Z_2)\\ &= \left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2) + g(Z_1,\nabla_{Z_2}Y_1)\\ &= \left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2) + \tilde{g}(Z_1,\tilde{\nabla}_{Z_2}Y_1)\\ &= \left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2) - \tilde{g}(\tilde{\nabla}_{Z_2}Z_1,Y_1)\\ &= \left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2) - g(\nabla_{Z_2}Z_1,Y_1) - \tilde{g}(h^l(Z_2,Z_1),Y_1).\end{aligned} \tag{19}$$

Using Eqs. (18) and (19), we acquire

$$\left(\frac{Y_1\phi}{\phi}\right) g(Z_1,Z_2) = -g(\nabla_{Z_2}Z_1,Y_1) - \tilde{g}(h^l(Z_2,Z_1),Y_1) \tag{20}$$

and from Eqs. (4) and (5), we derive

$$\begin{aligned}(21)\quad \tilde{g}(\tilde{J}A_{\tilde{J}Z_2}Z_1, Y_1) &= -g(A_{\tilde{J}Z_2}Z_1, \tilde{J}Y_1)\\ &= \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Z_2 - D^l(Z_1, \tilde{J}Z_2) - \nabla^s_{Z_1}\tilde{J}Z_2, \tilde{J}Y_1)\\ &= \tilde{g}(\tilde{\nabla}_{Z_1}\tilde{J}Z_2, \tilde{J}Y_1) - \tilde{g}(D^l(Z_1, \tilde{J}Z_2), \tilde{J}Y_1)\\ &= g(\nabla_{Z_1}Z_2, Y_1) + \tilde{g}(h^l(Z_1, Z_2), Y_1) - \tilde{g}(D^l(Z_1, \tilde{J}Z_2), \tilde{J}Y_1).\end{aligned}$$

Then, from Eq. (6), we have

$$(22)\qquad \tilde{g}(\tilde{J}A_{\tilde{J}Z_2}Z_1, Y_1) = -\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_2) - \tilde{g}(D^l(Z_1, \tilde{J}Z_2), \tilde{J}Y_1).$$

Hence, the result follows from Eqs. (20), (21) and (22). □

Theorem 4.3. *Let $M = M_T \times_\phi M_\perp$ be a twisted product SCR-lightlike submanifold of $\tilde{M}$. Then one has*

(i) *The squared norm of second fundamental form of M in $\tilde{M}$ satisfies*

$$(23)\qquad ||h||^2 \geq 2q||\nabla^T(\ln\phi)||^2,$$

where q denotes the dimension of $M_\perp$ and $\nabla^T(\ln\phi)$ denotes the M^T-component of the gradient of $\ln\phi$.

(ii) *If $||h||^2 = 2q||\nabla^T(\ln\phi)||^2$ holds identically for $Y_1 \in \Gamma(D)$, then M_T is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\tilde{M}$.*

Proof. Consider a local orthonormal frame of vector fields $\{X_1, X_2, X_3, \ldots, X_p, X_{p+1} = \tilde{J}X_1, X_{p+2} = \tilde{J}X_2, \ldots, X_{2p} = \tilde{J}X_p, X_{2p+1} = \xi_1, X_{2p+2} = \xi_2, \ldots, X_{2p+r} = \xi_r, X_{2p+r+1} = \tilde{J}\xi_1, X_{2p+r+2} = \tilde{J}\xi_2, \ldots, X_{2p+2r} = \tilde{J}\xi_r\}$ on M_T and local orthonormal frame of vector fields $\{Z_1, Z_2, Z_3, \ldots, Z_q\}$ on $M_\perp$. Then, we acquire

$$||h||^2 = ||h(D^\perp, D^\perp)||^2 + 2||h(D', D^\perp)||^2 + ||h(D', D')||^2,$$

which further gives

$$||h||^2 = ||h^s(D^\perp, D^\perp)||^2 + 2||h^s(D', D^\perp)||^2 + ||h^s(D', D')||^2.$$

From which, we acquire

$$\begin{aligned}||h||^2 = &\sum_{i,j=1}^{2p+2r}\tilde{g}(h^s(X_i, X_j), h^s(X_i, X_j)) + \sum_{m,n=1}^{q}\tilde{g}(h^s(Z_m, Z_n), h^s(Z_m, Z_n))\\ &+ 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q}\tilde{g}(h^s(X_i, Z_m), h^s(X_i, Z_m)).\end{aligned}$$

Thus, we have

$$(24)\qquad ||h||^2 \geq 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q}\tilde{g}(h^s(X_i, Z_m), h^s(X_i, Z_m)).$$

Now, using Lemma 4.2, Eq. (24) yields

$$\begin{aligned}||h||^2 &\geq 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q}(X_i(\ln\phi))^2\tilde{g}(Z_m,Z_m)\\ &\geq 2q||\nabla^T(\ln\phi)||^2,\end{aligned}$$

which proves part (i) of the assertion.

Now, for $Y_1 \in \Gamma(D)$ and $Z_1, Z_2 \in \Gamma(D^{\perp})$, from Eqs. (21) and (22), we have

$$g(\nabla_{Z_1}Z_2, Y_1) = \tilde{g}(\tilde{J}A_{\tilde{J}Z_2}Z_1, Y_1) = -\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_2). \tag{25}$$

Hence, using Eqs. (17) and (25), we obtain

$$g(\nabla_{Z_1}Z_2, Y_1) = -Y_1(\ln\phi)g(Z_1, Z_2). \tag{26}$$

Moreover, let $h^{\perp}$ denotes the second fundamental form of $M_{\perp}$ in M. Then we attain

$$g(h^{\perp}(Z_1, Z_2), Y_1) = g(\nabla_{Z_2}Z_1, Y_1). \tag{27}$$

Employing Eqs. (26) and (27) together with the non-degeneracy of D, we derive

$$h^{\perp}(Z_1, Z_2) = -\nabla^T(\ln\phi)g(Z_1, Z_2). \tag{28}$$

Now, let us consider the equality case of Eq. (23) holds identically. Then from Eq. (17), we have

$$h^s(D^{\perp}, D^{\perp}) = 0, \quad h^s(D', D') = 0, \quad h(D', D^{\perp}) \subset \tilde{J}D^{\perp}. \tag{29}$$

By hypothesis, M_T is totally geodesic and further using the first condition of Eq. (29), we obtain M_T is a totally geodesic submanifold in $\tilde{M}$. On the other hand, employing Eq. (28) together with second condition of Eq. (29), we conclude that $M_{\perp}$ is a totally umbilical submanifold in $\tilde{M}$, which completes the proof. □

Theorem 4.4. *Let $M = M_T \times_{\phi} M_{\perp}$ be a twisted product SCR-lightlike submanifold of $\tilde{M}$. If M is mixed geodesic, then one has*

1. *The twisting function ϕ is a function defined on $M_{\perp}$.*
2. *$M_T \times M_{\perp}^{\phi}$ is a SCR-lightlike product, where $M_{\perp}^{\phi}$ denotes the manifold $M_{\perp}$ equipped with the metric $g_{M_{\perp}}^{\phi} = \phi^2 g_{M_{\perp}}$.*

Proof. Employing Eq. (17), we acquire

$$\tilde{g}(h^s(\tilde{J}Y_1, Z_1), \tilde{J}Z_2) = Y_1(\ln\phi)g(Z_1, Z_2) \tag{30}$$

for $Y_1 \in \Gamma(D')$ and $Z_1, Z_2 \in \Gamma(D^{\perp})$. As M is mixed geodesic, therefore Eq. (30) gives

$$Y_1(\ln\phi)g(Z_1, Z_2) = 0.$$

Further, using the non-degeneracy of $D^{\perp}$, we derive $Y_1(\ln\phi) = 0$ for any vector $Y_1 \in \Gamma(TM_T)$. Thus, the twisting function ϕ of the twisted product SCR-lightlike submanifold $M_T \times_{\phi} M_{\perp}$ depends on $M_{\perp}$ only, which in turn yields that the twisted product $M = M_T \times_\phi M_\perp$ becomes isomorphic to the semi-Riemannian product $M_T \times M_\perp^\phi$. Hence, $M_T \times M_\perp^\phi$ is an SCR-lightlike product submanifold in $\tilde{M}$, thus the proof follows. □

5. Example of a non-trivial twisted product *SCR*-lightlike submanifold

Finally, we construct a non-trivial example of a twisted product SCR-lightlike submanifold of $\tilde{M}$ as follows.

Example 5.1. Let M be an 5-dimensional submanifold of $(R_2^{10}, \tilde{g})$ given by

$$x^1 = u^1, \quad x^2 = u^2, \quad x^3 = u^1 \sin u^3,$$
$$x^4 = u^2 \sin u^3, \quad x^5 = u^1 \cos u^3, \quad x^6 = u^2 \cos u^3,$$
$$x^7 = u^4, \quad x^8 = u^5, \quad x^9 = x^{10} = e^{u^3}, \quad \text{where } u^3 \in R - \{\tfrac{n\pi}{2},\ n \in Z\}.$$

Then TM is spanned by Z_1, Z_2, Z_3, Z_4, Z_5, where

$$Z_1 = \partial x_1 + \sin u^3 \partial x_3 + \cos u^3 \partial x_5,$$
$$Z_2 = \partial x_2 + \sin u^3 \partial x_4 + \cos u^3 \partial x_6,$$
$$Z_3 = u^1 \cos u^3 \partial x_3 + u^2 \cos u^3 \partial x_4 - u^1 \sin u^3 \partial x_5 - u^2 \sin u^3 \partial x_6$$
$$+ e^{u^3} \partial x_9 + e^{u^3} \partial x_{10},$$
$$Z_4 = \partial x_7, \quad Z_5 = \partial x_8.$$

Clearly, M is a 2-lightlike submanifold with $Rad(TM) = Span\{Z_1, Z_2\}$. As $\tilde{J}Z_4 = Z_5$ gives that $S(TM) = Span\{Z_4, Z_5\}$. Further, by direct calculations, $S(TM^\perp) = Span\{W = -u^2 \cos u^3 \partial x_3 + u^1 \cos u^3 \partial x_4 + u^2 \sin u^3 \partial x_5 - u^1 \sin u^3 \partial x_6 - e^{u^3} \partial x_9 + e^{u^3} \partial x_{10}\}$ and $\tilde{J}Z_3 = W$. Moreover, $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\partial x_1 + \sin u^3 \partial x_3 + \cos u^3 \partial x_5),$$
$$N_2 = \frac{1}{2}(-\partial x_2 + \sin u^3 \partial x_4 + \cos u^3 \partial x_6),$$

where $Span\{N_1, N_2\}$ is invariant with respect to $\tilde{J}$. Hence,

$$ltr(TM) = Span\{N_1, N_2\} \text{ and } D' = Span\{Z_1, Z_2, Z_4, Z_5\}.$$

Thus M is a proper SCR-lightlike submanifold of R_2^{10}. Here it is clear that D' is integrable. Now M_T and $M_\perp$, respectively, denote the leaves of D' and $D^\perp$. Then, the induced metric tensor of $M = M_T \times_\phi M_\perp$ is given by

$$ds^2 = 2(du_4^2 + du_5^2) + ((u^1)^2 + (u^2)^2 + 2(e^{u^3})^2)du_3^2.$$

Hence, M is a twisted product SCR-lightlike submanifold of the type $M_T \times_\phi M_\perp$ of R_2^{10}, with the twisting function $\phi = \sqrt{(u^1)^2 + (u^2)^2 + 2(e^{u^3})^2}$.

Acknowledgment. The authors would like to express their sincere gratitude to the referees for their valuable suggestions, which definitely improved the manuscript.

References

[1] M. Barros and A. Romero, *Indefinite Kähler manifolds*, Math. Ann. **261** (1982), no. 1, 55–62. https://doi.org/10.1007/BF01456410

[2] J. K. Beem and P. E. Ehrlich, *Global Lorentzian geometry*, Monographs and Textbooks in Pure and Applied Mathematics, 67, Marcel Dekker, Inc., New York, 1981.

[3] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49. https://doi.org/10.2307/1995057

[4] B.-Y. Chen, *Totally umbilical submanifolds*, Soochow J. Math. **5** (1979), 9–37.

[5] B.-Y. Chen, *Twisted product CR-submanifolds in Kaehler manifolds*, Tamsui Oxf. J. Math. Sci. **16** (2000), no. 2, 105–121.

[6] B.-Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\widetilde{M}^n(4c)$*, Math. Proc. Cambridge Philos. Soc. **124** (1998), no. 1, 107–125. https://doi.org/10.1017/S030500419700217X

[7] K. L. Duggal, *Warped product of lightlike manifolds*, Nonlinear Anal. **47** (2001), no. 5, 3061–3072. https://doi.org/10.1016/S0362-546X(01)00425-4

[8] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and its Applications, 364, Kluwer Acad. Publ., Dordrecht, 1996. https://doi.org/10.1007/978-94-017-2089-2

[9] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. **26** (2003), no. 1, 49–68. https://doi.org/10.2996/kmj/1050496648

[10] K. L. Duggal and B. Sahin, *Screen Cauchy Riemann lightlike submanifolds*, Acta Math. Hungar. **106** (2005), no. 1-2, 137–165. https://doi.org/10.1007/s10474-005-0011-7

[11] M. Fernández-López, E. García-Río, D. N. Kupeli, and B. Ünal, *A curvature condition for a twisted product to be a warped product*, Manuscripta Math. **106** (2001), no. 2, 213–217. https://doi.org/10.1007/s002290100204

[12] G. Ganchev and V. Mihova, *Riemannian manifolds of quasi-constant sectional curvatures*, J. Reine Angew. Math. **522** (2000), 119–141. https://doi.org/10.1515/crll.2000.034

[13] N. Koike, *The decomposition of curvature netted hypersurfaces*, Geom. Dedicata **54** (1995), no. 1, 1–11. https://doi.org/10.1007/BF01265295

[14] S. Kumar, *Geometry of warped product lightlike submanifolds of indefinite nearly Kaehler manifolds*, J. Geom. **109** (2018), no. 1, Paper No. 21, 18 pp. https://doi.org/10.1007/s00022-018-0425-3

[15] S. Kumar, *Geometry of semi-transversal lightlike warped product submanifolds of indefinite Kaehler manifolds*, Asian-Eur. J. Math. **14** (2021), no. 1, Paper No. 2150001, 12 pp. https://doi.org/10.1142/S1793557121500017

[16] M. Lohnherr and H. Reckziegel, *On ruled real hypersurfaces in complex space forms*, Geom. Dedicata **74** (1999), no. 3, 267–286. https://doi.org/10.1023/A:1005000122427

[17] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

[18] R. Ponge and H. Reckziegel, *Twisted products in pseudo-Riemannian geometry*, Geom. Dedicata **48** (1993), no. 1, 15–25. https://doi.org/10.1007/BF01265674

[19] B. Sahin, *Warped product lightlike submanifolds*, Sarajevo J. Math. **1(14)** (2005), no. 2, 251–260.

[20] B. Sahin, *Notes on doubly warped and doubly twisted product CR-submanifolds of Kaehler manifolds*, Mat. Vesnik **59** (2007), no. 4, 205–210.

Sangeet Kumar
Department of Mathematics
Sri Guru Teg Bahadur Khalsa College
Sri Anandpur Sahib 140118, Punjab, India
Email address: sp7maths@gmail.com

Megha Pruthi
Department of Mathematics
Sri Guru Teg Bahadur Khalsa College
Sri Anandpur Sahib 140118, Punjab, India
Email address: meghapruthi4194@gmail.com