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TOEPLITZ-TYPE OPERATORS ON THE FOCK SPACE F_α^2

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Abstract. Let j be a nonnegative integer. We define the Toeplitz-type operators $T_a^{(j)}$ with symbol $a \in L^\infty(C)$, which are variants of the traditional Toeplitz operators obtained for $j = 0$. In this paper, we study the boundedness of these operators and characterize their compactness in terms of its Berezin transform.

1. Introduction

For any positive parameter α, we consider the Gaussian measure

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z),$$

where dA is the Euclidean area measure on the complex plane $\mathbb{C}$, it's easy to show that $d\lambda_\alpha$ is a probability measure. The Fock space F_α^2 consists of all entire functions on C that are also in $L^2(C, d\lambda_\alpha)$. F_α^2 is a closed subspace of $L^2(C, d\lambda_\alpha)$ and F_α^2 is a Hilbert space with inner product inherited from $L^2(C, d\lambda_\alpha)$:

$$\langle f, g\rangle = \int_C f(z)\overline{g(z)} d\lambda_\alpha(z).$$

Let $L^\infty(C)$ be the space of the functions f on C such that

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in C\} < \infty.$$

For $z, w \in C$, let $K_z(w) = e^{\alpha \bar{z} w}$ be the reproducing kernel of F_α^2, and let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel in F_α^2, where $|| \cdot ||$ denote the norm of F_α^2. Each operator S on F_α^2 induces a function $\widetilde{S}$ on C, namely,

$$\widetilde{S}(z) = \langle S k_z, k_z\rangle, \ z \in C.$$

We called $\widetilde{S}$ the Berezin transform of S. For more information about Berezin transform one refers to [16, 17].

For any $z \in C$, define the operator U_z on F_α^2 by $U_z f = (f \circ \varphi_z) k_z$, where $\varphi_z(w) = z - w$ for $w \in C$, then U_z is unitary and self-adjoint. For any S a

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bounded operator on F_α^2, define S_z to be the bounded operator on F_α^2 given by conjugation with U_z:

$$S_z = U_z S U_z.$$

It's easy to check that $\widetilde{S}\circ\varphi_z = \widetilde{S_z}$. For any $f, g \in F_\alpha^2$, let $f \otimes g$ be the rank-one operator on F_α^2 which is defined by

$$(f \otimes g)h = \langle h, g\rangle f, \ \forall h \in F_\alpha^2.$$

Let $e_j(z) = \sqrt{\frac{\alpha^j}{j!}} z^j (j \geq 0)$. Then $\{e_j\}_{j\geq 0}$ is an orthonormal basis of F_α^2. The operator $E_j := e_j \otimes e_j$ is in fact the orthogonal projection onto the subspace generated by e_j. For any $z \in C$, it is easy to check that

$$\langle U_z E_0 U_z f, g\rangle = f(z)\overline{g(z)} e^{-\alpha|z|^2}, \ \forall f, g \in F_\alpha^2. \tag{1.1}$$

By (1.1), the traditional Toeplitz operator T_a on F_α^2 with the symbol $a \in L^\infty(C)$ can be written as

$$T_a = \frac{\alpha}{\pi}\int_C U_z E_0 U_z a(z) dA(z),$$

where the integral converges in the weak operator topology.

Let $\mathbb{D}$ be the unit disk in the complex plane. The Bergman space $A^2(\mathbb{D})$ consists of all analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$, where $L^2(\mathbb{D}, dA)$ be the complex valued measurable functions on $\mathbb{D}$ such that

$$\|f\|_2 := \left[\int_{\mathbb{D}} |f(z)|^2 dA(z)\right]^{\frac{1}{2}} < +\infty.$$

For $z \in \mathbb{D}$, let W_z on $A^2(\mathbb{D})$ by $W_z f = (f\circ\varphi)\varphi'$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. Then W_z is unitary and self-adjoint on $A^2(\mathbb{D})$. Engliš [8] considered the more general operator on $A^2(\mathbb{D})$ defined as

$$R_a := \int_{\mathbb{D}} W_z R W_z a(z) d\widetilde{A}(z), \ a \in L^\infty(\mathbb{D}), \tag{1.2}$$

where $d\widetilde{A}(z) = \frac{dA(z)}{(1-|z|^2)^2}$. Meanwhile, he showed that if R is a radial operator in the trace class, then $||R_a|| \leq ||R||_{tr}||a||_\infty$ and R_a is bounded on $A^2(\mathbb{D})$. Since the operator R is an ℓ^1 linear combination of the projections $\zeta_j = (\sqrt{j+1}z^j)\otimes(\sqrt{j+1}z^j)$, with the trace norm of R given by the correspondent ℓ^1-norm of its eigenvalues, the above result is equivalent to

$$T_a^{(j)} := \int_{\mathbb{D}} W_z \zeta_j W_z a(z) d\widetilde{A}(z) \ \text{ and } \ ||T_a^{(j)}|| \leq ||a||_\infty \tag{1.3}$$

for every integer $j \geq 0$. More generally, let μ be a finite Borel measure on $\mathbb{D}$, and for $j \geq 0$, Suárez defined the following Toeplitz-type operator with symbol μ on the Bergman space (see [13]):

$$T_\mu^{(j)} := \int_{\mathbb{D}} W_z \zeta_j W_z (1-|z|^2)^{-2} d\mu(z). \tag{1.4}$$

Suárez, using Carleson measure conditions, characterized the boundedness and compactness of the operator $T^{(j)}_\mu$ on $A^2(\mathbb{D})$.

Based on the research of the above scholars, we consider similar operator on the Fock space, and later found that the operator we defined is essentially a localized operator. Let j be a nonnegative integer, we now define the following Toeplitz-type operators $T^{(j)}_a$ on the Fock space F^2_α:

$$\langle T^{(j)}_a f, g\rangle = \frac{\alpha}{\pi}\int_C \langle U_z f, e_j\rangle\langle e_j, U_z g\rangle a(z)dA(z),\ f,g\in F^2_\alpha. \tag{1.5}$$

Toeplitz operators have been widely studied in the contexts of Hardy and Bergman spaces on various domains, and a large number of techniques and methods have been developed over the past twenty years or so; see [12, 15, 16]. In [14], the authors give sufficient conditions for a densely-defined operator on Fock space to be bounded or compact, under the boundedness condition. They characterize the compactness of the operator in terms of its Berezin transform. It is obvious that T_f is bounded when $f\in L^\infty(C)$. However, the boundedness of Toeplitz-type operators $T^{(j)}_f$ is not so obvious when $f\in L^\infty(C)$ and $j\geq 1$. In this paper, we will continue study the boundedness and compactness of the operator $T^{(j)}_f$, where $f\in L^\infty(C)$.

2. Boundedness of Toeplitz-type operators

The definition of the Weyl operator on $L^2(C, d\lambda_\alpha)$ is as follows:

$$W_z w(\xi) = e^{\alpha\xi\cdot\bar{z}-\frac{\alpha}{2}|z|^2}w(\xi - z).$$

Let $w\in F^2_\alpha$, and $f\in L^\infty(C)$, the Gabor-Daubechies localization operator $L^{(w)}_f$ on F^2_α defined by

$$\langle L^{(w)}_f u, \xi\rangle = \int_C f(z)\langle u, W_z w\rangle\langle W_z w, \xi\rangle dv(z),\ \forall\ u,\ \xi\in F^2_\alpha.$$

So, the operator $T^{(j)}_f$ defined in (1.5) is essentially a localization operator. Let $BC^\infty(C)$ be the space of all $C^\infty(C)$ functions whose partial derivatives are bounded. For $w\in F^2_\pi$, and $f\in BC^\infty(C)$, Abreu, Luís and Faustino [1] showed that

$$L^{(w)}_f = T_{D(w)f}.$$

On the right side of the equation above, Toeplitz operator whose symbol is obtained from the symbol of the localization operator by action of a differential operator $D(w)$, whose coefficients are constants explicitly determined by w. For more information one refers to [4, 6, 7, 9–11].

Engliš [9] characterized the boundedness of the localization operator $L^{(w)}_f$ by Bargmann transform. In this section, we use another interesting method to prove the boundedness of Toeplitz-type operators.

Theorem 2.1. *If $a\in L^\infty(C)$, then $T^{(j)}_a$ is bounded on F^2_α.*

Proof. For any $f, g \in F_\alpha^2$, we have

$$
\begin{aligned}
(2.1)\qquad & |\langle T_a^{(j)} f, g\rangle| \\
&\leq \frac{\alpha}{\pi}\int_C |\langle U_z f, e_j\rangle||\langle U_z g, e_j\rangle||a(z)|dA(z) \\
&\leq \frac{\alpha}{\pi}||a||_\infty \left(\int_C |\langle U_z f, e_j\rangle|^2 dA(z)\right)^{\frac{1}{2}} \left(\int_C |\langle U_z g, e_j\rangle|^2 dA(z)\right)^{\frac{1}{2}}.
\end{aligned}
$$

We just need to prove that there exists a positive constant M such that

$$\int_C |\langle U_z h, e_j\rangle|^2 dA(z) \leq M||h||^2,\ \ h \in F_\alpha^2.$$

Case 1: Let $m, n \geq j$. We can calculate that

$$
\begin{aligned}
(2.2)\qquad & \int_C \langle U_z w^m, e_j\rangle \overline{\langle U_z w^n, e_j\rangle} dA(z) \\
&= \int_C \langle (z-w)^m e^{\alpha \overline{z} w}, e_j(w)\rangle \overline{\langle (z-w)^n e^{\alpha \overline{z} w}, e_j(w)\rangle} e^{-\alpha|z|^2} dA(z) \\
&= \int_C \left\langle \sum_{k_1=0}^{m} C_m^{k_1}(-1)^{k_1} w^{k_1} z^{m-k_1} \sum_{l_1=0}^{\infty} \frac{\alpha^{l_1}\overline{z}^{l_1} w^{l_1}}{l_1!}, e_j(w)\right\rangle \\
&\quad \cdot \overline{\left\langle \sum_{k_2=0}^{n} C_n^{k_2}(-1)^{k_2} w^{k_2} z^{n-k_2} \sum_{l_2=0}^{\infty} \frac{\alpha^{l_2}\overline{z}^{l_2} w^{l_2}}{l_2!}, e_j(w)\right\rangle} e^{-\alpha|z|^2} dA(z) \\
&= \int_C \left\langle \sum_{k_1=0}^{j} C_m^{k_1}(-1)^{k_1} w^{k_1} z^{m-k_1} \frac{\alpha^{j-k_1}\overline{z}^{j-k_1} w^{j-k_1}}{(j-k_1)!}, e_j(w)\right\rangle \\
&\quad \cdot \overline{\left\langle \sum_{k_2=0}^{j} C_n^{k_2}(-1)^{k_2} w^{k_2} z^{n-k_2} \frac{\alpha^{j-k_2}\overline{z}^{j-k_2} w^{j-k_2}}{(j-k_2)!}, e_j(w)\right\rangle} e^{-\alpha|z|^2} dA(z) \\
&= \frac{j!}{\alpha^j} \sum_{k_1,k_2=0}^{j} C_m^{k_1} C_n^{k_2} (-1)^{k_1+k_2} \frac{\alpha^{j-k_1}\alpha^{j-k_2}}{(j-k_1)!(j-k_2)!} \\
&\quad \times \int_C z^{m+j-k_1-k_2} \overline{z}^{n+j-k_1-k_2} e^{-\alpha|z|^2} dA(z).
\end{aligned}
$$

If $m \neq n$, by (2.2), we get

$$\int_C \langle U_z w^m, e_j\rangle \overline{\langle U_z w^n, e_j\rangle} dA(z) = 0.$$

If $m = n \geq j$, by the proof process of (2.2),

$$\begin{aligned}
&\int_C |\langle U_z w^n, e_j\rangle|^2 dA(z)\\
&= \frac{\pi j!}{\alpha^{j+1}} \sum_{k_1,k_2=0}^{j} (-1)^{k_1+k_2} C_n^{k_1} C_n^{k_2} \frac{\alpha^{j-k_1}\alpha^{j-k_2}}{(j-k_1)!(j-k_2)!} \frac{(n+j-k_1-k_2)!}{\alpha^{n+j-k_1-k_2}}\\
&= \frac{\pi j!}{\alpha^{n+1}} \sum_{k_1,k_2=0}^{j} (-1)^{k_1+k_2} C_n^{k_1} C_n^{k_2} \frac{(n+j-k_1-k_2)!}{(j-k_1)!(j-k_2)!}\\
&= \frac{\pi n!}{\alpha^{n+1}} \sum_{k_1,k_2=0}^{j} (-1)^{k_1+k_2} C_n^{k_2} C_{n+j-k_1-k_2}^{j-k_2} C_j^{k_1}.
\end{aligned}$$

Let

$$I_{j,n} = \sum_{k_1,k_2=0}^{j} (-1)^{k_1+k_2} C_n^{k_2} C_{n+j-k_1-k_2}^{j-k_2} C_j^{k_1}.$$

We prove next that $I_{j,n} = 1$. Construct a function

$$F(x,y) = (y-x)^n (y-1)^j \sum_{k=0}^{n+j} y^k (1+x)^{n+j-k},$$

where $x, y \in (0,1)$.

It is easy to calculate that $I_{j,n}$ is the coefficient of $x^j y^{n+j}$. The coefficient of y^{n+j} is

$$\begin{aligned}
&\sum_{p=0}^{n}\sum_{q=0}^{j} C_n^p (-x)^p C_j^q (-1)^q (1+x)^{n+j-p-q}\\
&= \sum_{p=0}^{n} C_n^p (-x)^p (1+x)^{n-p} \left(\sum_{q=0}^{j} C_j^q (1+x)^{j-q} (-1)^q \right)\\
&= x^j.
\end{aligned}$$

Then

$$I_{j,n} = 1.$$

Case 2: If $m > j$, $n < j$, or $n > j$, $m < j$, or $m, n < j$ and $m \neq n$, by the proof process of (2.2), we have

$$\int_C \langle U_z w^m, e_j\rangle \overline{\langle U_z w^n, e_j\rangle} dA(z) = 0.$$

Case 3: If $m = n < j$, by a proof process similar to (2.2), we get

$$\int_C |\langle U_z w^n, e_j\rangle|^2 dA(z) = \frac{\pi n!}{\alpha^{n+1}}.$$

Hence,

$$\int_C \langle U_z w^m, e_j\rangle \overline{\langle U_z w^n, e_j\rangle} dA(z) = \begin{cases} \frac{\pi n!}{\alpha^{n+1}}, & m=n; \\ 0, & m \neq n. \end{cases}$$

Write $h(z) = \sum_{n=0}^{\infty} a_n z^n$, where $||h||^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha^n} |a_n|^2$. Then

$$\begin{aligned} \int_C |\langle U_z h, e_j\rangle|^2 dA(z) &= \int_C \left| \left\langle \sum_{n=0}^{\infty} a_n U_z w^n, e_j \right\rangle \right|^2 dA(z) \\ &= \int_C \sum_{n=0}^{\infty} |a_n|^2 |\langle U_z w^n, e_j\rangle|^2 dA(z) \\ &= \sum_{n=0}^{\infty} |a_n|^2 \int_C |\langle U_z w^n, e_j\rangle|^2 dA(z) \\ &= \sum_{n=0}^{\infty} |a_n|^2 \frac{\pi n!}{\alpha^{n+1}} = \frac{\pi}{\alpha} ||h||^2. \end{aligned} \tag{2.3}$$

This completes the proof. □

Through this theorem, we can observe that for all $h \in F_\alpha^2$, we have

$$\frac{\alpha}{\pi} \int_C |\langle U_z h, e_j\rangle|^2 dA(z) = ||h||^2.$$

3. Compactness of Toeplitz-type operators

Let $a \in L^\infty(C)$. Then we can see that the Berezin transform of the Toeplitz-type operators $T_a^{(j)}$ is

$$\begin{aligned} \widetilde{T_a^{(j)}}(w) &= \langle T_a^{(j)} k_w, k_w\rangle \\ &= \frac{\alpha}{\pi} \int_C |\langle U_z e_j, k_w\rangle|^2 a(z) dA(z) \\ &= \frac{\alpha^{j+1}}{\pi j!} \int_C |(U_z \xi^j)(w)|^2 e^{-\alpha|w|^2} a(z) dA(z) \\ &= \frac{\alpha^{j+1}}{\pi j!} \int_C |\varphi_z(w)^j|^2 |k_z(w)|^2 e^{-\alpha|w|^2} a(z) dA(z) \\ &= \frac{\alpha^{j+1}}{\pi j!} \int_C |\varphi_w(z)^j|^2 e^{-\alpha|\varphi_w(z)|^2} a(z) dA(z). \end{aligned} \tag{3.1}$$

In particular, $\widetilde{T_a^{(j)}} = \widetilde{T_a} = \widetilde{a}$ when $j = 0$. If $a \in L^\infty(C)$, then T_a is a compact operator if and only if $\widetilde{a}$ vanishing at infinity, see [3]. We next use the Berezin transformation of the Toeplitz-type operator $T_a^{(j)}$ to characterize the compactness of this operator.

In order to prove the compactness of Toeplitz-type operators, we need some preparations.

Lemma 3.1. *For* $w, z \in C$, *let* $t = e^{iIm(\alpha z\overline{w})}$. *Then we have* $U_z U_w = U_{\varphi_z(w)} V_t$, *where* $V_t f(u) = t f(-u)$ *for* $f \in F^2_\alpha$.

Proof. Since $\varphi_w \circ \varphi_z \circ \varphi_{\varphi_z(w)} = -I$, where I is identity, for any $f \in F^2_\alpha$,

$$\begin{aligned} U_w U_z f(\xi) &= f \circ \varphi_z \circ \varphi_w(\xi) k_z(\varphi_w(\xi)) k_w(\xi) \\ &= f \circ \varphi_{\varphi_z(w)}(-\xi) e^{\alpha \overline{z} \varphi_w(\xi) - \frac{\alpha}{2}|z|^2 + \alpha \overline{w} \xi - \frac{\alpha}{2}|w|^2} \\ &= f \circ \varphi_{\varphi_z(w)}(-\xi) k_{\varphi_z(w)}(-\xi) e^{-iIm(\alpha z \overline{w})} \\ &= V_{\overline{t}} U_{\varphi_z(w)} f(\xi). \end{aligned}$$

It is easy to show that $V_t^* = V_{\overline{t}}$, therefore, $U_z U_w = U_{\varphi_z(w)} V_t$, where $t = e^{iIm(\alpha z\overline{w})}$. □

Lemma 3.2. *For* $f \in L^\infty(C)$, *we have*

$$(T_f^{(j)} K_z)(u) = K_z(u)(T^{(j)}_{f\circ\varphi_z} 1)(\varphi_z(u)). \tag{3.2}$$

Proof. For $f \in L^\infty(C)$, we have

$$\begin{aligned} & K_z(u)(T^{(j)}_{f\circ\varphi_z} 1)(\varphi_z(u)) \\ &= K_z(u) \langle T^{(j)}_{f\circ\varphi_z} 1, K_{\varphi_z(u)} \rangle \\ &= K_z(u) \int_C \langle U_\varsigma 1, e_j \rangle \langle e_j, U_\varsigma K_{\varphi_z(u)} \rangle f \circ \varphi_z(\varsigma) dA(\varsigma) \\ &= K_z(u) \int_C \langle 1, U_{\varphi_z(\eta)} e_j \rangle \langle U_{\varphi_z(\eta)} e_j, K_{\varphi_z(u)} \rangle f(\eta) dA(\eta). \end{aligned}$$

Since $\overline{K_u(\varphi_z(v)) k_z(v)} = \overline{K_{\varphi_z(u)}(v)} k_z(u)$, we get $\overline{k_z(u)} U_z K_{\varphi_z(u)} = K_u$. Taking $\lambda = e^{iIm(\alpha z\overline{\eta})}$, using Lemma 3.1, we have $U_z U_\eta = U_{\varphi_z(\eta)} V_\lambda$, where $V_\lambda e_j = \lambda(-1)^j e_j$, then

$$\begin{aligned} & K_z(u)(T^{(j)}_{f\circ\varphi_z} 1)(\varphi_z(u)) \\ &= \frac{\alpha}{\pi} K_z(u) \int_C \langle U_z 1, U_\eta e_j \rangle \langle U_\eta e_j, U_z K_{\varphi_z(u)} \rangle f(\eta) dA(\eta) \\ &= \frac{\alpha}{\pi} \int_C \langle K_z, U_\eta e_j \rangle \langle U_\eta e_j, \overline{k_z(u)} U_z K_{\varphi_z(u)} \rangle f(\eta) dA(\eta) \\ &= \frac{\alpha}{\pi} \int_C \langle K_z, U_\eta e_j \rangle \langle U_\eta e_j, K_u \rangle f(\eta) dA(\eta) \\ &= (T_f^{(j)} K_z)(u). \end{aligned}$$

□

Lemma 3.3. *Let* S *be a bounded operator on* F^2_α, *and let* $\widetilde{S}(z) \to 0$ *as* $z \to \infty$. *Then* $S_z 1 \to 0$ *weakly in* F^2_α *as* $z \to \infty$.

Proof. The proof is similar to the corresponding result in [2] or [5]. For completeness, we present an elementary proof in some details here. It suffices to show that $\langle S_z 1, w^s \rangle \to 0$ as $z \to \infty$ for every nonnegative integer s. So fix a nonnegative integer s. For $z,\eta\in C$, by the definition of the Berezin transform of operator, we have

$$\widetilde{S}\circ\varphi_z(\eta)=\widetilde{S_z}(\eta)=e^{-\alpha|\eta|^2}\sum_{j,m=0}^{\infty}\frac{\alpha^{j+m}}{j!m!}\overline{\eta}^j\eta^m\langle S_zw^j,w^m\rangle.$$

Now fix $r\in(0,1)$, multiply both sides of the last equation by $\overline{\eta}^se^{\alpha|\eta|^2}$, and then integrate over M_r, where $M_r=\{z\in C:|z|\le r\}$, to obtain

$$\begin{aligned}&\int_{M_r}\widetilde{S}\circ\varphi_z(\eta)\overline{\eta}^se^{\alpha|\eta|^2}d\lambda_\alpha(\eta)\\
=&\sum_{j,m=0}^{\infty}\frac{\alpha^{j+m}}{j!m!}\langle S_zw^j,w^m\rangle\int_{M_r}\overline{\eta}^{j+s}\eta^m d\lambda_\alpha(\eta)\\
=&\sum_{j=0}^{\infty}\frac{\alpha^{2j+s+1}}{j!(j+s)!}\langle S_zw^j,w^{j+s}\rangle\int_0^{r^2}x^{j+s}e^{-\alpha x}dx\\
=&\frac{\alpha^{s+1}}{s!}\int_0^{r^2}x^se^{-\alpha x}dx\left(\langle S_z1,w^s\rangle+\sum_{j>0}^{\infty}\frac{\alpha^{2j}s!}{j!(j+s)!}\langle S_zw^j,w^{j+s}\rangle\frac{\int_0^{r^2}x^{j+s}e^{-\alpha x}dx}{\int_0^{r^2}x^se^{-\alpha x}dx}\right).\end{aligned}$$

Let now us examine the infinite sum above. It is easy to check that

$$\frac{\int_0^{r^2}x^{j+s}e^{-\alpha x}dx}{\int_0^{r^2}x^se^{-\alpha x}dx}\le r^{2j}$$

and

$$\left|\sum_{j>0}^{\infty}\frac{\alpha^{2j}s!}{j!(j+s)!}\langle S_zw^j,w^{j+s}\rangle\frac{\int_0^{r^2}x^{j+s}e^{-\alpha x}dx}{\int_0^{r^2}x^se^{-\alpha x}dx}\right|\le s!\alpha^{-\frac{s}{2}}||S||(e^{\alpha r^2}-1).$$

Thus given $\varepsilon>0$, we can choose $r\in(0,1)$ such that

$$\left|\sum_{j>0}^{\infty}\frac{\alpha^{2j}s!}{j!(j+s)!}\langle S_zw^j,w^{j+s}\rangle\frac{\int_0^{r^2}x^{j+s}e^{-\alpha x}dx}{\int_0^{r^2}x^se^{-\alpha x}dx}\right|\le\varepsilon$$

for all $z\in C$. Hence we get $\lim_{z\to\infty}|\langle S_z1,w^s\rangle|\le\varepsilon$. The proof of lemma is completed. □

We are now ready to prove the main result of this section.

Theorem 3.4. *Let $a\in L^\infty(\mathbb{C}^n)$. Then Topeplitz-type operator $T_a^{(j)}$ is compact if and only if $\lim_{|z|\to\infty}\widetilde{T_a^{(j)}}(z)=0$.*

Proof. The necessity is obvious. We will prove the sufficiency in several steps.

Step 1. Let $g_z = T^{(j)}_{a\circ\varphi_z}1 \in F^2_\alpha$. We have $\sup_{z\in C}||g_z||_{F^2_\alpha} \le ||a||_\infty$. We first observe that $T_{|g_z|} = PM_{|g_z|}P = (M_{|g_z|^{\frac{1}{2}}}P)^*(M_{|g_z|^{\frac{1}{2}}}P)$, where P is the projection from $L^2(C, d\lambda_\alpha)$ onto F^2_α. Using Cauchy-Schwarz inequality, we have $||M_{|g_z|^{\frac{1}{2}}}P||^2 = ||T_{|g_z|}||$. Then

$$
\begin{aligned}
(3.3)\qquad |(T^{(j)}_{a\circ\varphi_z}1)(u)| &\le \int_C \left||g_z(\sigma)|^{\frac{1}{2}}|K_u(\sigma)|^{\frac{1}{2}}\right|^2 d\lambda_\alpha(\sigma)\\
&\le ||M_{|g_z|^{\frac{1}{2}}}||^2||K_u^{\frac{1}{2}}||^2 = e^{\frac{\alpha|u|^2}{4}}||M_{|g_z|^{\frac{1}{2}}}P||^2\\
&= e^{\frac{\alpha|u|^2}{4}}||T_{|g_z|}|| \le ||a||_\infty e^{\frac{\alpha|u|^2}{4}}.
\end{aligned}
$$

Step 2. For $\lim_{|z|\to\infty}\widetilde{T^{(j)}_a}(z) = 0$, we have $T^{(j)}_{a\circ\varphi_z}1 \to 0$ weakly as $|z|\to\infty$. In particular, $(T^{(j)}_{a\circ\varphi_z}1)(u) = \langle T^{(j)}_{a\circ\varphi_z}1, K_u\rangle \to 0$ pointwisely as $|z|\to\infty$.

Proof. Let $z, w \in C$ and $t = e^{iIm(\alpha z\overline{\varphi_z(w)})}$. Then, using Lemma 3.1, we have

$$U_zU_{\varphi_z(w)} = U_wV_t,$$

where $V_te_j = t(-1)^je_j$. So, for $g, h \in F^2_\alpha$,

$$
\begin{aligned}
\langle U_zT^{(j)}_aU_zg, h\rangle &= \frac{\alpha}{\pi}\int_C \langle U_wU_zg, e_j\rangle\langle e_j, U_wU_zh\rangle a(w)dA(w)\\
&= \frac{\alpha}{\pi}\int_C \langle U_{\varphi_z(\eta)}U_zg, e_j\rangle\langle e_j, U_{\varphi_z(\eta)}U_zh\rangle a(\varphi_z(\eta))dA(\eta)\\
&= \frac{\alpha}{\pi}\int_C \langle U_\eta g, e_j\rangle\langle e_j, U_\eta h\rangle a\circ\varphi_z(\eta)dA(\eta)\\
&= \langle T^{(j)}_{a\circ\varphi_z}g, h\rangle,
\end{aligned}
$$

then $U_zT^{(j)}_aU_z = T^{(j)}_{a\circ\varphi_z}$. Hence, by Lemma 3.3, we get $\widetilde{T^{(j)}_{a\circ\varphi_z}1} \to 0$ weakly as $z\to\infty$, since $\lim_{|z|\to\infty}\widetilde{T^{(j)}_a}(z) = 0$. □

Step 3. Let $a \in L^\infty(C)$ and $\lim_{|z|\to\infty}\widetilde{T^{(j)}_a}(z) = 0$. For $R > 0$, let $B(0, R)$ denote the open disk of radius R centered at 0. Let $T^{(j)}_{a,R}$ denote the operator from F^2_α into $L^2(C, d\lambda_\alpha)$ defined by

$$T^{(j)}_{a,R} = M_{B(0,R)}T^{(j)}_a,$$

where $M_{B(0,R)}$ is the multiplication operator with respect to the characteristic function $\chi_{B(0,R)}(\cdot)$ on $B(0, R)$. Then $T^{(j)}_{a,R}$ is compact and

$$\lim_{R\to\infty}||T^{(j)}_a - T^{(j)}_{a,R}|| = 0,$$

so that $T^{(j)}_a$ is compact.

Proof. By a simple normal family argument, $M_{B(0,R)} : F^2_\alpha \to L^2(C, d\lambda_\alpha)$ is compact, and so each $T^{(j)}_{a,R}$ is compact. It is easy to show that $(T^{(j)}_a)^* = T^{(j)}_{\overline{a}}$. For $g \in F^2_\alpha$, we get that

$$\begin{aligned}(T^{(j)}_a - T^{(j)}_{a,R})g(z) &= \left((1-\chi_{B(0,R)})T^{(j)}_a g\right)(z)\\ &= (1-\chi_{B(0,R)}(z))\langle T^{(j)}_a g, K_z\rangle\\ &= (1-\chi_{B(0,R)}(z))\langle g, T^{(j)}_{\overline{a}} K_z\rangle\\ &= \int_C g(u)(1-\chi_{B(0,R)}(z))\overline{(T^{(j)}_{\overline{a}} K_z)}(u)d\lambda_\alpha(u),\end{aligned}$$

so $T^{(j)}_a - T^{(j)}_{a,R}$ is the restriction to F^2_α of an integral operator on $L^2(C, d\lambda_\alpha)$ with kernel $K_{a,R}(z,u) = (1-\chi_{B(0,R)}(z))\overline{(T^{(j)}_{\overline{a}} K_z)}(u)$. By Schur's test, see [16, 17], whenever there exist a positive measurable function h on C and constants $C_{1,R}$ and C_2 such that

$$\begin{aligned}&\int_C |K_{a,R}(z,u)|h(z)d\lambda_\alpha(z) \le C_2 h(u),\ \forall u \in C,\\ &\int_C |K_{a,R}(z,u)|h(u)d\lambda_\alpha(u) \le C_{1,R}h(z),\ \forall z \in C.\end{aligned}$$

We get that $||T^{(j)}_a - T^{(j)}_{a,R}||^2 \le C_{1,R}C_2$.

Let $h(z) = e^{\frac{\alpha|z|^2}{2}}$. By (3.2), we have that

$$\begin{aligned}\int_C |K_{a,R}(z,u)|h(z)d\lambda_\alpha(z) &\le \int_C |T^{(j)}_{\overline{a}} K_z)(u)|e^{\frac{\alpha|z|^2}{2}}d\lambda_\alpha(z)\\ &= \frac{\alpha}{\pi}\int_C |K_z(u)||(T^{(j)}_{\overline{a}\circ\varphi_z}1)(\varphi_z(u))|e^{\frac{-\alpha|z|^2}{2}}dA(z)\\ &= \frac{\alpha}{\pi}\int_C |K_{z+u}(u)||(T^{(j)}_{\overline{a}\circ\varphi_{z+u}}1)(z)|e^{\frac{-\alpha|z+u|^2}{2}}dA(z).\end{aligned}$$

However

$$|K_{z+u}(u)|e^{\frac{-\alpha|z+u|^2}{2}} = e^{\frac{\alpha|u|^2}{2}-\frac{\alpha|z|^2}{2}},$$

and by (3.3), we have that

$$|(T^{(j)}_{\overline{a}\circ\varphi_{z+u}}1)(z)| \le ||a||_\infty e^{\frac{\alpha|z|^2}{4}}.$$

Hence

$$\int_C |K_{a,R}(z,u)|h(z)d\lambda_\alpha(z) \le \frac{\alpha}{\pi}||a||_\infty e^{\frac{\alpha|u|^2}{2}}\int_C e^{-\frac{\alpha|z|^2}{2}}dA(z) \le C_2 e^{\frac{\alpha|u|^2}{2}}.$$

For $|z| \ge R$, we have

$$\int_C |K_{a,R}(z,u)|h(u)d\lambda_\alpha(u) = \frac{\alpha}{\pi}\int_C |K_{a,R}(z,u)|e^{-\frac{\alpha|u|^2}{2}}dA(u),$$

which by (3.2) is equal to

$$\frac{\alpha}{\pi}\int_C |K_z(u)||(T^{(j)}_{\overline{a}\circ\varphi_z}1)(\varphi_z(u))|e^{-\frac{\alpha|u|^2}{2}}dA(u).$$

By the change of variables $u \to \varphi_z(u)$, this is equal to

$$\frac{\alpha}{\pi}\int_C |K_z(\varphi_z(u))||(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)|e^{-\frac{\alpha|z-u|^2}{2}}dA(u)$$
$$= \frac{\alpha}{\pi}\int_C |(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)||e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}dA(u).$$

If $1 < p < \infty$ with conjugate exponent q, then by Hölder's inequality, we get that

$$\frac{\alpha}{\pi}\int_C |(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)||e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}dA(u)$$
$$= \frac{\alpha}{\pi}\int_C \left(|(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)|e^{-\frac{2\alpha|u|^2}{5}}\right)\left(e^{\frac{2\alpha|u|^2}{5}}|e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}\right)dA(u)$$
$$= \frac{\alpha}{\pi}\left(\int_C \left(|(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)|e^{-\frac{2\alpha|u|^2}{5}}\right)^p dA(u)\right)^{\frac{1}{p}}$$
$$\times \left(\int_C \left(e^{\frac{2\alpha|u|^2}{5}}|e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}\right)^q dA(u)\right)^{\frac{1}{q}}.$$

Let

$$C'_{1,R} = \sup_{|z|\geq R}\left(\int_C \left(|(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)|e^{-\frac{2\alpha|u|^2}{5}}\right)^p dA(u)\right)^{\frac{1}{p}}.$$

From (3.3), we have that $\left(|(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u)|e^{-\frac{2\alpha|u|^2}{5}}\right)^p \leq ||a||^p_\infty e^{-\frac{3\alpha p|u|^2}{20}}$, and from Step 2, we have that $\lim_{R\to\infty}(T^{(j)}_{\overline{a}\circ\varphi_z}1)(u) = 0$ pointwisely in u. Thus, by the dominated convergence theorem, $C'_{1,R} \to 0$ as $R \to \infty$. Moreover,

$$\left(e^{\frac{2\alpha|u|^2}{5}}|e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}\right)^q = e^{\frac{\alpha q|z|^2}{2}}e^{\frac{-\alpha q|u|^2}{10}},$$

so that

$$\frac{\alpha}{\pi}\left(\int_C \left(e^{\frac{2\alpha|u|^2}{5}}|e^{\alpha(z-u)\overline{z}}|e^{-\frac{\alpha|z-u|^2}{2}}\right)^q dA(u)\right)^{\frac{1}{q}} = \frac{\alpha}{\pi}e^{\frac{\alpha|z|^2}{2}}\left(\int_C e^{\frac{-\alpha q|u|^2}{10}}dA(u)\right)^{\frac{1}{q}}.$$

Finally, we get that

$$\int_C |K_{a,R}(z,u)|h(u)d\lambda_\alpha(u) \leq C_{1,R}h(z),\ \forall z \in C,$$

where $C_{1,R} \to 0$ as $R \to \infty$. By Schur's test, this proves Step 3. □

This completes the proof. □

In this paper, we only solve the boundedness and compactness of Toeplitz-type operators induced by bounded symbols. Recall that BMO as the vector space of locally integrable f on C such that

$$\sup \int_C |f \circ \varphi_z - \widetilde{f}(z)| d\lambda_\alpha < \infty.$$

Coburn and Isralowitz [5] studied that the Toeplitz operator induced by BMO is bounded (compact) if and only if it's Berezin transform is bounded (vanishing at infinity). However, the boundedness and compactness of Toeplitz-type operators induced by BMO are not clear. So, we raise the following conjecture.

Conjecture 3.5. *Let $f \in BMO$. Then the Toeplitz-type operator induced by BMO is bounded* (*compact*) *if and only if its Berezin transform is bounded* (*vanishing at infinity*).

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