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A STUDY OF TWO SPECIES MODEL WITH HOLLING TYPE RESPONSE FUNCTION USING TRIANGULAR FUZZY NUMBERS

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Abstract. In this paper, we developed three theoretical models based on prey and predator that exhibit holling-type response functions. In both a fuzzy and a crisp environment, we have provided a mathematical formulation for the prey predator concept. We used the signed distance method to defuzzify the triangular fuzzy numbers using the *alpha*-cut function. We can identify equilibrium points for all three theoretical models using the defuzzification technique. Utilizing a variational matrix, stability is also performed with the two species model through three theoretical models. Results are presented, followed by discussion. MATLAB software is used to provide numerical simulations.

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Key words and phrases : Fuzzy number, prey–predator, stability, holling type, signed distance, equilibrium points.

1. Introduction

Alfred J. Lotka introduced a type of predator-prey model in the domain of autocatalytic chemical processes in 1910, that is later called the Lotka - Volterra Predator - Prey model [1, 6, 7]. In 1926, Vito Volterra, a mathematician, and physicist interested in mathematical biology published his results. The Rosenzweig–MacArthur model was eventually expanded to include density-dependent prey development and functional responses, similar to those described by C. S. Holling [16]. The Lotka–Volterra [9] and Rosenzweig–MacArthur models have both been used to represent realistic predator-prey ecosystem dynamics. Holling (1959) [13, 14, 15, 16] classified functional responses into three categories. Prey density and the greatest number of prey killed have a linear relationship in Type I, whereas Type II has a monotonic relationship between prey density and the

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fraction of prey devoured. Type III is characterized by a sigmoid relationship, in which the proportion of prey consumed is positively density-dependent over specific prey density regions.
Zadeh (1965) [21, 22, 25] was the first to use a fuzzy set to reflect imprecision or vagueness in ordinary life. In this paper, we used a Holling-type response function to investigate a Prey Predator Model in a fuzzy environment. We made the biological parameters fuzzy sense and expressed them using triangular fuzzy numbers to make the model more interesting. We used the signed distance approach to construct the defuzzification procedure for the fuzzy prey-predator model. Differential inclusion is a notion used by Hullermeier (1997). Using α - cut at the beginning value and the solution, the given differential equation is transformed into a differential inclusion, and the answer is identified as the α - cut of the fuzzy solution. Many studies have been conducted on this subject, but we continued the work with some additional inputs and concepts that improve the model's acceptability and reliability. Finally, we have deduced the model's dynamical behavior in a fuzzy sense.

2. Preliminaries

Definition 2.1. A fuzzy set $\widetilde{A}$ defined on X and expressed as a collection of ordered pairs, $\widetilde{A} = \{(x, \mu_A(x)),\, x \in X\}$and if X is a universe of discourse and x is a specific element of X, where $\mu_{\widetilde{A}}(x)$ is a membership function is associated with a fuzzy set $\widetilde{A}$such that every element of the universe of discourse X is mapped to the interval [0,1] by this function. The mapping is written as $\mu_{\widetilde{A}}(x) : x \to [0, 1]$.

Definition 2.2. A triangular membership function is specified by three parameters a_1, a_2, a_3; $a_1, a_2, a_3 \in R$;$a_1 < a_2 < a_3$; with membership function $\mu_{\widetilde{A}}(x)$is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 \quad if\ x \le a_1 \\ \dfrac{x - a_1}{a_2 - a_1}\ if\ a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}\ if\ a_2 \le x \le a_3 \\ 0 \quad if\ x \ge a_3 \end{cases}$$

Definition 2.3. Operation on fuzzy numbers
(i) Addition:

$$[\widetilde{A}_1 + \widetilde{B}_1](\alpha) = [A_{1L}(\alpha) + B_{1L}(\alpha), A_{1U}(\alpha) + B_{1U}(\alpha)]$$

(ii) Substraction: $[\widetilde{A}_1 - \widetilde{B}_1](\alpha) = [A_{1L}(\alpha) - B_{1R}(\alpha), A_{1R}(\alpha) - B_{1L}(\alpha)]$

(iii) Multiplication:

$$[\widetilde{A_1}.\widetilde{B_1}](\alpha)$$
$$= [min\{A_{1L}(\alpha)B_{1L}(\alpha), A_{1L}(\alpha)B_{1U}(\alpha), A_{1U}(\alpha)B_{1L}(\alpha), A_{1U}(\alpha)B_{1U}(\alpha)\},$$
$$max\{A_{1L}(\alpha)B_{1L}(\alpha), A_{1L}(\alpha)B_{1U}(\alpha), A_{1U}(\alpha)B_{1L}(\alpha), A_{1U}(\alpha)B_{1U}(\alpha)\}]$$

Definition 2.4. For $\widetilde{A} \in \Phi$, defined the signed distance of $\widetilde{A}$ to $\widetilde{0}$ as,

$$d(\widetilde{A}, \widetilde{0}) = \int_0^1 d_0([A_L(\alpha), A_U(\alpha); \alpha], \widetilde{0})d\alpha$$
$$d(\widetilde{A}, \widetilde{0}) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_U(\alpha)]d\alpha$$

By solving we get, $d(\widetilde{A}, \widetilde{0}) = a + \frac{1}{4}(\Delta_2 - \Delta_1)$

Definition 2.5. Let $\mu \in$ F(X) and $\alpha \in$ [0,1] then the sets $[\mu]_\alpha$={$x \in X/\mu(x) \geq \alpha$}, $[\mu]_\alpha$={$x \in X/\mu(x) > \alpha$} are called the $\alpha-$ cut and strict $\alpha-$ cut of μ.

3. Basic Expression of Prey – Predator Mathematical Model

Crisp Model – I:

Assume a Lotka–Volterra system with the Prey–Predator Model.
(i). The rate of prey growth is logistic in this model I.
(ii). The catch rate function pEx will be based on the premise of catch per unit.
(iii). Predation is related to the prey density.

$$\frac{dx}{dt} = \alpha\left(1 - \frac{x}{k}\right)x - \beta xy - pEx$$
$$\frac{dy}{dt} = -\gamma y + \delta xy$$

Crisp Model – II:

Now we consider the prey-predator system below, which includes a catch rate function and a Holling-II type response function

$$\frac{dx}{dt} = \alpha\left(1 - \frac{x}{k}\right)x - \frac{\beta xy}{a + x} - pEx$$
$$\frac{dy}{dt} = -\gamma y + \frac{\delta xy}{a + x}$$

Crisp Model – III: Now we consider the prey-predator system below, which includes a catch rate function and a Holling-III type response function.

$$\frac{dx}{dt} = \alpha\left(1 - \frac{x}{k}\right)x - \frac{\beta x^2 y}{a^2 + x^2} - pEx$$
$$\frac{dy}{dt} = -\gamma y + \frac{\delta x^2 y}{a^2 + x^2}$$

4. Mathematical Formulation of Fuzzy Prey – Predator Model

By using the idea of a fuzzy set to extend the crisp model which includes biological parameters that are incorrect. All of the parameters are supposed to be constant or time dependant in the crisp model, but in reality, they vary around certain average values owing to the changing environment. As an outcome, the parameters are not assumed to be constants. To deal with this kind of uncertainty, a fuzzy model is needed [17, 18, 19, 24], in which biological parameters are considered to be fuzzy numbers represented by Triangular Fuzzy Numbers [3]. The fuzzy parameters are defuzzified using the signed distance approach [10], and the dynamical behavior of the Holling type response function is discussed in a fuzzy manner. If the biological parameters are fuzzy, the crisp model becomes a fuzzy prey-predator model, resulting in the fuzzy differential equations below:

Model - I

$$\frac{\widetilde{dx}}{dt} = \widetilde{\alpha}\left(1 - \frac{x}{k}\right)x \ominus \widetilde{\beta}xy \ominus pEx \qquad (4.1)$$
$$\frac{\widetilde{dy}}{dt} = -\widetilde{\gamma}y \oplus \widetilde{\delta}xy \qquad (4.2)$$

Model - II

$$\frac{\widetilde{dx}}{dt} = \widetilde{\alpha}\left(1 - \frac{x}{k}\right)x \ominus \frac{\widetilde{\beta}xy}{a + x} \ominus pEx \qquad (4.3)$$
$$\frac{\widetilde{dy}}{dt} = -\widetilde{\gamma}y \oplus \frac{\widetilde{\delta}xy}{a + x} \qquad (4.4)$$

Model - III

$$\frac{\widetilde{dx}}{dt} = \widetilde{\alpha}\left(1 - \frac{x}{k}\right)x \ominus \frac{\widetilde{\beta}x^2 y}{a^2 + x^2} \ominus pEx \qquad (4.5)$$
$$\frac{\widetilde{dy}}{dt} = -\widetilde{\gamma}y \oplus \frac{\widetilde{\delta}x^2 y}{a^2 + x^2} \qquad (4.6)$$

where

x − show the total number of prey species

y − show the total number of predator species (biomass).

$\widetilde{\alpha} = (\alpha - \Delta_1, \alpha, \alpha + \Delta_1)$, $0 < \Delta_1 < \alpha$ denote the fuzzy inherent growth rate of prey species in absence of predator

$$\widetilde{\beta} = (\beta - \Delta_2, \beta, \beta + \Delta_2),\ 0 < \Delta_2 < \beta$$

The greatest values of the per capita decrease rate x owing to y are denoted.
$\widetilde{\gamma} = (\gamma - \Delta_3, \gamma, \gamma + \Delta_3),\ 0 < \Delta_3 < \gamma$ indicate the fuzzy mortality rate of predator species
$\widetilde{\delta} = (\delta - \Delta_4, \delta, \delta + \Delta_4),\ 0 < \Delta_4 < \delta$ indicate fuzzy conversion biomass rates

5. Defuzzification Process:

Applying defuzzification method for triangular fuzzy numbers using signed distance method for an inherent growth rate of prey species, Ratio of prey mortality per capita, the death rate of predator, and conversion biomass rate. Applying both the left and right sides of the α - cut [0,1] of $\widetilde{\frac{dx_i}{dt}}$ in (4.1) to (4.6) we get the idea of

$$\left[\left(\frac{dx_i}{dt}\right)_L(\alpha), \left(\frac{dx_i}{dt}\right)_R(\alpha)\right]$$

, i=1,2 are as follows:

Model - I

$$\left(\frac{dx}{dt}\right)_L(\alpha) = \alpha_L(\alpha)x\left(1 - \frac{x}{k}\right) - \beta_R(\alpha)xy - pEx$$
$$\left(\frac{dx}{dt}\right)_R(\alpha) = \alpha_R(\alpha)x\left(1 - \frac{x}{k}\right) - \beta_L(\alpha)xy - pEx$$
$$\left(\frac{dy}{dt}\right)_L(\alpha) = -\gamma_L(\alpha)y + \delta_R(\alpha)xy$$
$$\left(\frac{dy}{dt}\right)_R(\alpha) = -\gamma_R(\alpha)y + \delta_L(\alpha)xy$$

Model - II

$$\left(\frac{dx}{dt}\right)_L(\alpha) = \alpha_L(\alpha)x\left(1 - \frac{x}{k}\right) - \frac{\beta_R(\alpha)xy}{a+x} - pEx$$
$$\left(\frac{dx}{dt}\right)_R(\alpha) = \alpha_R(\alpha)x\left(1 - \frac{x}{k}\right) - \frac{\beta_L(\alpha)xy}{a+x} - pEx$$
$$\left(\frac{dy}{dt}\right)_L(\alpha) = -\gamma_L(\alpha)y + \frac{\delta_R(\alpha)xy}{a+x}$$
$$\left(\frac{dy}{dt}\right)_R(\alpha) = -\gamma_R(\alpha)y + \frac{\delta_L(\alpha)xy}{a+x}$$

Model - III

$$\left(\frac{dx}{dt}\right)_L(\alpha) = \alpha_L(\alpha)x\left(1-\frac{x}{k}\right) - \frac{\beta_R(\alpha)x^2y}{a^2+x^2} - pEx$$
$$\left(\frac{dx}{dt}\right)_R(\alpha) = \alpha_R(\alpha)x\left(1-\frac{x}{k}\right) - \frac{\beta_L(\alpha)x^2y}{a^2+x^2} - pEx$$
$$\left(\frac{dy}{dt}\right)_L(\alpha) = -\gamma_L(\alpha)y + \frac{\delta_R(\alpha)x^2y}{a^2+x^2}$$
$$\left(\frac{dy}{dt}\right)_R(\alpha) = -\gamma_R(\alpha)y + \frac{\delta_L(\alpha)x^2y}{a^2+x^2}$$

where

$\alpha_L(\alpha) = \alpha - \Delta_1 + \alpha\Delta_1, \alpha_R(\alpha) = \alpha + \Delta_2 - \alpha\Delta_2, \beta_L(\alpha) = \beta - \Delta_3 + \alpha\Delta_3, \beta_R(\alpha) = \beta + \Delta_4 - \alpha\Delta_4,$
$\gamma_L(\alpha) = \gamma - \Delta_5 + \alpha\Delta_5, \gamma_R(\alpha) = \gamma + \Delta_6 - \alpha\Delta_6, \delta_L(\alpha) = \delta - \Delta_7 + \alpha\Delta_7, \delta_R(\alpha) = \delta + \Delta_8 - \alpha\Delta_8$

For the fuzzy prey-predator model (4.1) to (4.6) signed distance method will be used. Hence signed distance $\widetilde{\frac{dx}{dt}}$ to $\widetilde{0}$ represented as $d\left(\widetilde{\frac{dx}{dt}}, \widetilde{0}\right)$ are as follows:

Model - I

$$d\left(\widetilde{\frac{dx}{dt}}, \widetilde{0}\right) = d(\widetilde{\alpha}, \widetilde{0})x\left(1-\frac{x}{k}\right) - d(\widetilde{\beta}, \widetilde{0})xy - pEx \tag{5.1}$$
$$d\left(\widetilde{\frac{dy}{dt}}, \widetilde{0}\right) = -d(\widetilde{\gamma}, \widetilde{0})y + d(\widetilde{\delta}, \widetilde{0})xy \tag{5.2}$$

Model - II

$$d\left(\widetilde{\frac{dx}{dt}}, \widetilde{0}\right) = d(\widetilde{\alpha}, \widetilde{0})x\left(1-\frac{x}{k}\right) - \frac{d(\widetilde{\beta}, \widetilde{0})xy}{a+x} - pEx \tag{5.3}$$
$$d\left(\widetilde{\frac{dy}{dt}}, \widetilde{0}\right) = -d(\widetilde{\gamma}, \widetilde{0})y + \frac{d(\widetilde{\delta}, \widetilde{0})xy}{a+x} \tag{5.4}$$

Model - III

$$d\left(\widetilde{\frac{dx}{dt}}, \widetilde{0}\right) = d(\widetilde{\alpha}, \widetilde{0})x\left(1-\frac{x}{k}\right) - \frac{d(\widetilde{\beta}, \widetilde{0})x^2y}{a^2+x^2} - pEx \tag{5.5}$$
$$d\left(\widetilde{\frac{dy}{dt}}, \widetilde{0}\right) = -d(\widetilde{\gamma}, \widetilde{0})y + \frac{d(\widetilde{\delta}, \widetilde{0})x^2y}{a^2+x^2} \tag{5.6}$$

By using signed distance, we have

$d\left(\widetilde{\frac{dx}{dt}}, \widetilde{0}\right) = \frac{1}{2}\int\limits_0^1 \left[\left(\frac{dx}{dt}\right)_L(\alpha) + \left(\frac{dx}{dt}\right)_R(\alpha)\right] d\alpha = \frac{dx}{dt}$

The above system (5.1) to (5.6) reduces to

Model - I

$$\frac{dx}{dt} = B_1 x\left(1 - \frac{x}{k}\right) - B_2 xy - pEx \tag{5.7}$$

$$\frac{dy}{dt} = -B_3 y + B_4 xy \tag{5.8}$$

Model - II

$$\frac{dx}{dt} = B_1 x\left(1 - \frac{x}{k}\right) - \frac{B_2 xy}{a+x} - pEx \tag{5.9}$$

$$\frac{dy}{dt} = -B_3 y + \frac{B_4 xy}{a+x} \tag{5.10}$$

Model - III

$$\frac{dx}{dt} = B_1 x\left(1 - \frac{x}{k}\right) - \frac{B_2 x^2 y}{a^2+x^2} - pEx \tag{5.11}$$

$$\frac{dy}{dt} = -B_3 y + \frac{B_4 x^2 y}{a^2+x^2} \tag{5.12}$$

6. Equilibrium Points

Model - I: We have the following equations to calculate the equilibrium points of the fuzzy system equations (5.7) and (5.8):

$$\begin{aligned} B_1 x\left(1 - \frac{x}{k}\right) - B_2 xy - pEx &= 0 \\ -B_3 y + B_4 xy &= 0 \end{aligned}$$

Solving the above two equations we get the equilibrium points such as (0,0), $\left(k\left(1 - \frac{pE}{B_1}\right), 0\right)$, (x^*, y^*) where $x^* = \frac{B_3}{B_4}$ and $y^* = \frac{1}{B_2}\left(B_1\left(1 - \frac{x^*}{k}\right) - pE\right)$

Model - II: To identify the equilibrium points of the fuzzy system equations (5.9) and (5.10), we have

$$\begin{aligned} B_1 x\left(1 - \frac{x}{k}\right) - \frac{B_2 xy}{a+x} - pEx &= 0 \\ -B_3 y + \frac{B_4 xy}{a+x} &= 0 \end{aligned}$$

Solving the above two equations we get the equilibrium points such as (0,0), $\left(k\left(1 - \frac{pE}{B_1}\right), 0\right)$ and (x^*, y^*)

where $x^* = \frac{B_3 a}{B_4 - B_3}$ and $y^* = \frac{1}{B_2}(a + x^*)\left(B_1\left(1 - \frac{x^*}{k}\right) - pE\right)$

Model - III: We have the following equations to calculate the Equilibrium points of the fuzzy system equations (5.11) and (5.12):

$$\begin{aligned} B_1 x\left(1 - \frac{x}{k}\right) - \frac{B_2 x^2 y}{a^2+x^2} - pEx &= 0 \\ -B_3 y + \frac{B_4 x^2 y}{a^2+x^2} &= 0 \end{aligned}$$

Solving the above two equations we get the equilibrium points such as (0,0), $\left(k\left(1-\frac{pE}{B_1}\right),0\right)$ and (x^*,y^*) where $x^* = \left(\frac{B_3 a^2}{B_4-B_3}\right)^{\frac{1}{2}}$ and $y^* = \frac{1}{B_2 x^*}(a^2 + x^{*2})\left(B_1\left(1-\frac{x^*}{k}\right)-pE\right)$

6.1. Variational matrix for Model – I:.

$$v(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \end{pmatrix}$$

where

$$\begin{aligned} f_1(x,y) &= B_1 x\left(1-\frac{x}{k}\right) - B_2 xy - pEx \\ g_1(x,y) &= -B_3 y + B_4 xy \end{aligned}$$

$$v(x,y) = \begin{bmatrix} B_1\left(1-\frac{2x}{k}\right) - B_2 y - pE & -B_2 x \\ B_4 y & -B_3 + B_4 x \end{bmatrix} \tag{I}$$

At (0,0) in I

$$v(0,0) = \begin{bmatrix} B_1 - pE & 0 \\ 0 & -B_3 \end{bmatrix}$$

The characteristic equation of the above matrix is given as follows:

$$\begin{vmatrix} B_1 - pE - \lambda & 0 \\ 0 & -B_3 - \lambda \end{vmatrix} = 0$$

$\lambda = B_1 - pE > 0;\ \lambda = -B_3 < 0$
The equilibrium point is a saddle since the eigenvalues are different with opposite signs so the system is unstable.
At $\left(k\left(1-\frac{pE}{B_1}\right),0\right)$ in I

$$v\left(k\left(1-\frac{pE}{B_1}\right),0\right) = \begin{bmatrix} B_1\left(1-\frac{2\left(k\left(1-\frac{pE}{B_1}\right)\right)}{k}\right) - pE & -B_2 k\left(1-\frac{pE}{B_1}\right) \\ 0 & -B_3 + B_4 k\left(1-\frac{pE}{B_1}\right) \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1\left(1-2\left(1-\frac{pE}{B_1}\right)\right) - pE - \lambda & -B_2 k\left(1-\frac{pE}{B_1}\right) \\ 0 & -B_3 + B_4 k\left(1-\frac{pE}{B_1}\right) - \lambda \end{vmatrix} = 0$$

$$\begin{aligned} f(\lambda) &= \lambda^2 B_1 - \lambda(B_1 B_4 - B_1 B_3 - B_4 kpE - pEB_1 - B_1^2) \\ &\quad + (B_1^2 B_3 - B_1^2 B_4 + B_1 B_4 kpE + pE(B_3 B_1 - B_4 B_1 + B_4 kpE)) = 0 \end{aligned}$$

By using Descartes's rule there are two changes of a sign so we have two negative roots. Hence, we say the system is said to be a stable node.

At (x^*, y^*) in I
where $x^* = \frac{B_3}{B_4}$, $y^* = \frac{1}{B_2}\left(B_1\left(1 - \frac{x^*}{k}\right) - pE\right)$

$$v(x^*, y^*) = \begin{bmatrix} B_1\left(1 - \frac{2x^*}{k}\right) - B_2y^* - pE & -B_2x^* \\ B_4y^* & -B_3 + B_4x^* \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1\left(1 - \frac{2x^*}{k}\right) - B_2y^* - pE - \lambda & -B_2x^* \\ B_4y^* & -\lambda \end{vmatrix} = 0$$

$$f(\lambda) = \lambda^2 - \lambda\left(B_1\left(1 - \frac{2x^*}{k}\right) - B_2y^* - pE\right) + B_2x^*(B_4y^*) = 0$$

By using Descartes's rule there are two changes of a sign so we have two negative roots. Hence, we say the system is said to be a stable node.

6.2. Variational matrix for Model – II:.

$$f_1(x, y) = B_1x\left(1 - \frac{x}{k}\right) - \frac{B_2xy}{a + x} - pEx$$

$$g_1(x, y) = -B_3y + \frac{B_4xy}{a + x}$$

$$v(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \end{pmatrix}$$

$$v(x, y) = \begin{bmatrix} B_1\left(1 - \frac{2x}{k}\right) - \frac{B_2ya}{(a+x)^2} - pE & -\frac{B_2x}{a+x} \\ \frac{B_4ya}{(a+x)^2} & -B_3 + \frac{B_4x}{a+x} \end{bmatrix} \tag{II}$$

At (0,0) in II

$$v(0, 0) = \begin{bmatrix} B_1 - pE & 0 \\ 0 & -B_3 \end{bmatrix}$$

The characteristic equation of the above matrix is given as follows:

$$\begin{vmatrix} B_1 - pE - \lambda & 0 \\ 0 & -B_3 - \lambda \end{vmatrix} = 0$$

$\lambda = B_1 - pE > 0$; $\lambda = -B_3 < 0$
The equilibrium point is a saddle since the eigenvalues are different with opposite signs so the system is unstable.
At $\left(k\left(1 - \frac{pE}{B_1}\right), 0\right)$ in II

$$v\left(k\left(1 - \frac{pE}{B_1}\right), 0\right) = \begin{bmatrix} B_1\left(1 - \frac{2\left(k\left(1 - \frac{pE}{B_1}\right)\right)}{k}\right) - pE & -\frac{B_2k\left(1 - \frac{pE}{B_1}\right)}{a + k\left(1 - \frac{pE}{B_1}\right)} \\ 0 & -B_3 + \frac{B_4k\left(1 - \frac{pE}{B_1}\right)}{a + k\left(1 - \frac{pE}{B_1}\right)} \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1\left(1-2\left(1-\frac{pE}{B_1}\right)\right)-pE-\lambda & -B_2k\left(1-\frac{pE}{B_1}\right) \\ 0 & -B_3+\frac{B_4k\left(1-\frac{pE}{B_1}\right)}{a+k\left(1-\frac{pE}{B_1}\right)}-\lambda \end{vmatrix}=0$$

$$\begin{aligned} &f(-\lambda) \\ &=\lambda^2-\lambda\left(B_1+pE-\frac{B_4k\left(1-\frac{pE}{B_1}\right)}{a+k\left(1-\frac{pE}{B_1}\right)}+B_3\right)+\frac{B_4k\left(1-\frac{pE}{B_1}\right)}{a+k\left(1-\frac{pE}{B_1}\right)}-B_1-pE-B_3 \\ &=0 \end{aligned}$$

Here we have 2 changes of sign. By using Descarte's rule we say that there are 2 negative roots when $\frac{B_4k}{a+k}\left(1-\frac{pE}{B_1}\right)>B_1+pE+B_3$. Then equilibrium point is said to be node since the roots are different or complex with a negative real part. Hence we say the system is asymptotically stable when $\frac{B_4k}{a+k}\left(1-\frac{pE}{B_1}\right)>B_1+pE+B_3$

At (x^*,y^*) in II

where $x^*=\frac{B_3a}{B_4-B_3}, y^*=\frac{1}{B_2}(a+x^*)\left[B_1\left(1-\frac{x^*}{k}-pE\right)\right]$

$$v(x^*,y^*)=\begin{bmatrix} B_1\left(1-\frac{2x^*}{k}\right)-\frac{B_2y^*a}{(a+x^*)^2}-pE & \frac{-B_2x^*}{a+x^*} \\ \frac{B_4y^*a}{(a+x^*)^2} & -B_3+\frac{B_4x^*}{a+x^*} \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1\left(1-\frac{2x^*}{k}\right)-\frac{B_2y^*a}{(a+x^*)^2}-pE-\lambda & \frac{-B_2B_3}{B_4} \\ \frac{B_4y^*a}{(a+x^*)^2} & \frac{B_3(-B_4+B_3+B_4a)}{aB_4}-\lambda \end{vmatrix}=0$$

$$\begin{aligned} f(\lambda)=\lambda^2-\lambda&\left\{B_1\left(1-\frac{2x^*}{k}\right)-\frac{B_2y^*a}{(a+x^*)^2}-pE+\frac{B_3(-B_4+B_3+B_4a)}{aB_4}\right\} \\ &+\left(B_1\left(1-\frac{2x^*}{k}\right)-\frac{B_2y^*a}{(a+x^*)^2}-pE\right)\left(\frac{B_3(-B_4+B_3+B_4a)}{aB_4}-\lambda\right)+\left(\frac{B_2B_3}{B_4}\right)\left(\frac{B_4y^*a}{(a+x^*)^2}\right)=0 \end{aligned}$$

By using Descartes's rule there are two changes of a sign so we have two negative roots. Hence, we say the system is a stable node.

6.3. Variational matrix for Model – III:.

$$f(x,y)=B_1x\left(1-\frac{x}{k}\right)-\frac{B_2x^2y}{a^2+x^2}-pEx$$

$$g(x,y)=-B_3y+\frac{B_4x^2y}{a^2+x^2}$$

$$v(x,y)=\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

$$v(x,y) = \begin{bmatrix} B_1\left(1-\frac{2x}{k}\right) - \frac{2B_2a^2xy}{(a^2+x^2)^2} - pE & \frac{-B_2x^2}{a^2+x^2} \\ \frac{2a^2B_4xy}{(a^2+x^2)^2} & -B_3 + \frac{B_4x^2}{a^2+x^2} \end{bmatrix} \tag{III}$$

At (0,0) in III

$$v(0,0) = \begin{bmatrix} B_1 - pE & 0 \\ 0 & -B_3 \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1 - pE - \lambda & 0 \\ 0 & -B_3 - \lambda \end{vmatrix} = 0$$

$$\lambda = B_1 - pE \ ,$$
$$\lambda = -(pE - B_1) < 0 \ \ , \lambda = -B_3 < 0$$

The equilibrium point is a node since the roots are real distinct and negative.
At $(x^*, 0)$ in III where $x^* = k\left(1 - \frac{pE}{B_1}\right)$

$$v(x^*,0) = \begin{bmatrix} B_1\left(1-\frac{2x^*}{k}\right) - pE & \frac{-B_2{x^*}^2}{a^2+{x^*}^2} \\ 0 & -B_3 + \frac{B_4{x^*}^2}{a^2+{x^*}^2} \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$\begin{vmatrix} B_1\left(1-\frac{2x^*}{k}\right) - pE - \lambda & \frac{-B_2{x^*}^2}{a^2+{x^*}^2} \\ 0 & -B_3 + \frac{B_4{x^*}^2}{a^2+{x^*}^2} - \lambda \end{vmatrix} = 0$$

$$f(\lambda) = \lambda^2 + \lambda\left(B_3 + pE - B_1\left(1-\frac{2x^*}{k}\right) - \frac{B_4{x^*}^2}{a^2+{x^*}^2}\right) + B_1\left(1 - \frac{2x^*}{k} - pE\right)\left(\frac{B_4{x^*}^2}{a^2+{x^*}^2} - B_3\right) = 0$$

Now, $f(-\lambda) = \lambda^2 - \lambda\left(B_3 + pE - B_1\left(1-\frac{2x^*}{k}\right) - \frac{B_4{x^*}^2}{a^2+{x^*}^2}\right) + B_1\left(1 - \frac{2x^*}{k} - pE\right)\left(\frac{B_4{x^*}^2}{a^2+{x^*}^2} - B_3\right) = 0$

There are two changes of sign. So, by Descarte's rule, we conclude that the equilibrium point is a node since the roots are real distinct with a negative real part. Hence the system is asymptotically stable if

$$B_3 + pE > B_1\left(1 - \frac{2x^*}{k}\right) + \frac{B_4{x^*}^2}{a^2+{x^*}^2}$$

At (x^*, y^*) in III where $x^* = \left(\frac{B_3a^2}{B_4-B_3}\right)^{\frac{1}{2}}$, $y^* = \frac{1}{B_2x^*}\left(a^2+{x^*}^2\right)\left[B_1\left(1-\frac{x^*}{k}\right) - PE\right]$

$$v(x^*,y^*) = \begin{bmatrix} B_1\left(1-\frac{2x^*}{k}\right) - \frac{2B_2a^2x^*y^*}{(a^2+{x^*}^2)^2} - pE & \frac{-B_2{x^*}^2}{a^2+{x^*}^2} \\ \frac{2a^2B_4x^*y^*}{(a^2+{x^*}^2)^2} & -B_3 + \frac{B_4{x^*}^2}{a^2+{x^*}^2} \end{bmatrix}$$

The characteristic equation of the above matrix is given by:

$$f(\lambda) = \lambda^2 - \lambda\left\{\left(B_1\left(1-\frac{2x^*}{k}\right) - \frac{2B_2a^2x^*y^*}{(a^2+{x^*}^2)} - pE\right) + \left(-B_3 + \frac{B_4{x^*}^2}{a^2+{x^*}^2}\right)\right\}$$
$$+\left(B_1\left(1-\frac{2x^*}{k}\right) - \frac{2B_2a^2x^*y^*}{(a^2+{x^*}^2)} - pE\right)\left(-B_3+\frac{B_4{x^*}^2}{a^2+{x^*}^2}\right) + \left(\frac{B_2{x^*}^2}{a^2+{x^*}^2}\right)\left(\frac{2a^2B_4x^*y^*}{(a^2+{x^*}^2)^2}\right)$$

By using Descarte's rule, we have two negative roots since there are 2 changes of sign. Hence we say the system is a stable node.

7. Results and Discussion:

In this article, we investigate a two species prey-predator model based on fuzzy parameters. To overcome these two species model we use fuzzy concept (alpha - cut) which makes easier to analyze a dynamical behavior. The majority of researchers built their models on the assumption that the parameters in their models are well-known, however, the situation in real life is different. The prey-predator model is presented uniquely in this study, with biological parameters that are inherently inaccurate. Triangular Fuzzy Numbers have been used to represent the imprecise parameters. The signed distance approach method was used to defuzzify the fuzzy parameters by using alpha cut definition,and then the behaviors of the dynamical model in the fuzzy nature were noticed. We now present a prey-predator behavior model in both crisp and fuzzy forms, as follows:

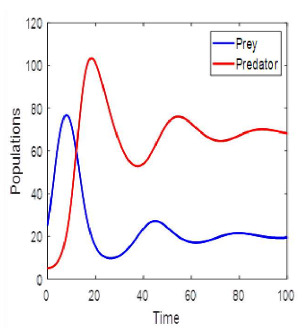
Model - I: For parameters: p = 1.5; E = 0.05; alpha = 0.5; k = 120; gamma = 0.1; beta = 0.005; delta = 0.005; The behaviors of the two species are shown in Figures (1a) and (1b) in both a crisp and a fuzzy environment, and they asymptotically converge to the equilibrium state value. Fig (1c) and Fig (1d) show a phase plane of the system shows that it is stable in both environments with the same initial condition.

Model - II: For parameters: a = 0.5; p = 1.5; E = 0.05; alpha = 0.5; k = 120; gamma = 0.1; beta = 0.005; delta = 0.005; A two species population asymptotically converges to the equilibrium state value, as shown in Figures (2a) and (2b). A phase picture of the system is shown in Figures (2c) and (2d), demonstrating that (x^*, y^*) is locally asymptotically stable in both fuzzy and crisp nature.

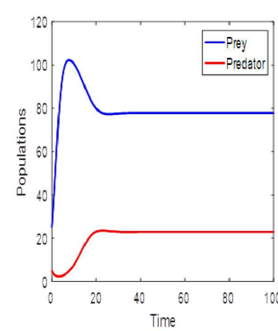
Model - III: For Parameters:a = 0.5; p = 0.5; E = 0.8; alpha = 0.5; k = 120; gamma = 0.1; beta = 0.005; delta = 0.005; Figures (3a) and (3b) indicate both prey and predator species vary a little at first, but as time goes on, all of the species spirals towards their particular steady-state levels of species are distinct in two distinct environments. Figures (3c) and (3d) depict a phase portrait of prey-predator biomass in two environments with identical initial conditions.

Comparison Result:

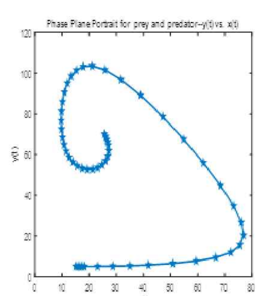
Model		Existing Method [9]	Proposed Method
Model I	(0,0)	Unstable	Unstable
	$(x^*,0)$	Asymptotically stable	Stable node
	(x^*,y^*)	Stable	Stable node
Model II	(0,0)	Unstable	Unstable
	$(x^*,0)$	Asymptotically stable	Asymptotically stable when $\frac{B_4 k}{a+k}\left(1-\frac{pE}{B_1}\right) > B_1 + pE + B_3$
	(x^*,y^*)	Asymptotically stable	Stable node
Model III	(0,0)	---	Node when $\lambda = -(pE - B_1) < 0$, $\lambda = -B_3 < 0$
	$(x^*,0)$	Unstable	Asymptotically stable when $B_3 + pE > B_1\left(1-\frac{2x^*}{k}\right) + \frac{B_4 x^{*2}}{a^2 + x^{*2}}$
	(x^*,y^*)	Asymptotically stable	Stable node



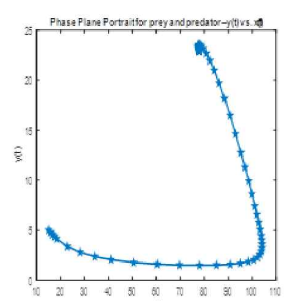
(**a**) Fig:1a



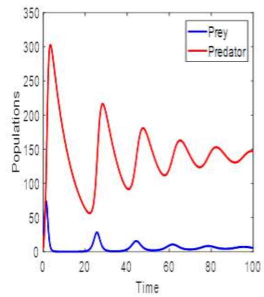
(**b**) Fig:1b



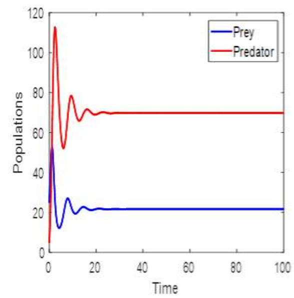
(**a**) Fig:1c



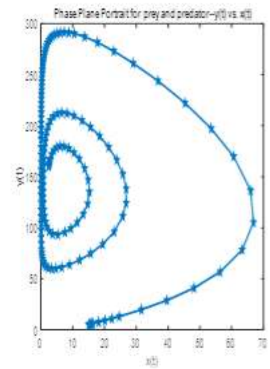
(**b**) Fig:1d



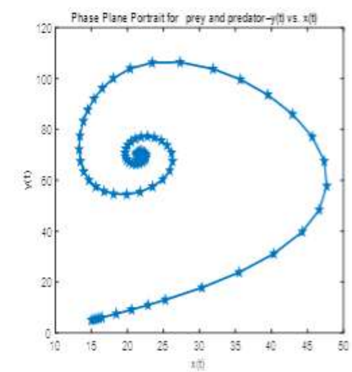
(**a**) Fig:2a



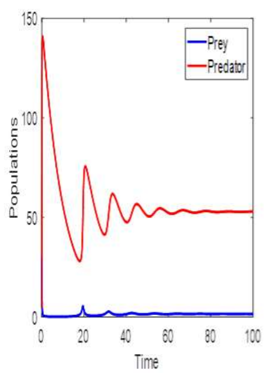
(**b**) Fig:2b



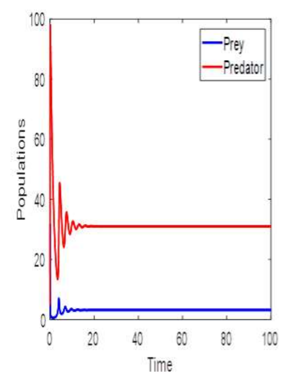
(**a**) Fig:2c



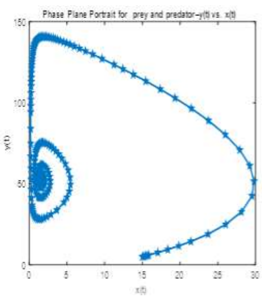
(**b**) Fig:2d



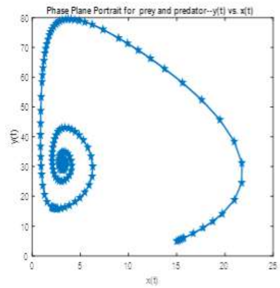
(**a**) Fig:3a



(**b**) Fig:3b



(**a**) Fig:3c



(**b**) Fig:3d

8. Conclusion

In this article, three mathematical Prey Predator population models are explored and analyzed. We explore the logistic equation of prey growth in these models. We examine Holling I type predator response function in model 1 and holling II type predator response function in model 2 and holling III type predator response function in model 3. The existence of equilibrium points is obtained based on three fuzzy models. The stability of equilibrium points is addressed using a variational matrix, and the criteria of asymptotical stability of equilibrium points are also derived. Numerical simulation is done by using MATLAB software. In both fuzzy and crisp nature, by using set of parameters value the figure depicts the dynamical behavior of a prey-predator model and a phase portrait of a system.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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