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WEIGHTED VALUE SHARING AND UNIQUENESS OF ENTIRE AND MEROMORPHIC FUNCTIONS OF A LINEAR DIFFERENTIAL POLYNOMIAL

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ABSTRACT. In this research article, we deal with the uniqueness of entire and meromorphic functions when two linear differential polynomial share a non-zero value and obtain some results .

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1. Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For some $a \in \mathbb{C} \cup \{\infty\}$, if the zero of $f-a$ and $g-a$ have the same locations as well as same multiplicities, we say that f and g share the value a CM (counting multiplicities). If we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities). Throughout the paper the elemental and standard notations of Nevanlinna's Value Distribution Theory of meromorphic functions which are discussed in ([2],[9]) have been adopted. A meromorphic function a is said to be a small with respect to f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$, outside a possible exceptional set of finite linear measure. Also, we use I to denote any set of infinite linear measure of $0 < r < \infty$. If $\alpha \equiv \alpha(z)$ is a small function, we define that f and g share α CM (IM) according as $f-\alpha$ and $g-\alpha$ share 0 CM (IM).

By using the definition of L(z) to denote an arbitrary polynomial of degree n, i.e.,

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 = a_n (z-c_1)^{l_1} + a_{n-1}(z-c_2)^{l_2} + \ldots + (z-c_s)^{l_s} \quad (1)$$

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where $a_i,\ i = 0,\ 1, \dots,\ n,\ a_n \neq 0$, and $c_j,\ j = 1,\ 2,\ ..., s$, are finite complex number constants; $c_1,\ c_2,\ ...,\ c_s$ are all distinct zeros of $L(z),\ l_1,\ l_2,\ ...,\ l_s.\ s,\ n$ are all positive integers satisfying the equality

$$l_1 + l_2 + \ldots + l_s = n \text{ and } l = max\,\{l_1,\ l_2 \ldots l_s\} \tag{2}$$

In 2016, Harina P. Waghamore and Rajeshwari S.[8] studied the existence of solutions for $[L(f)]^{(k)}$ and the corresponding uniqueness theorem and obtained the following results.

Theorem 1.1. [8] *Let f and g be two non - constant meromorphic functions and let $n,\ k,\ l$ be three positive integers. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share $(1, l)$, and one of the following conditions holds:*
(i) $l \geq 2$ and $(k+8)l > (k+7)n + 3k + 8$;
(ii) $l = 1$ and $(2k+10)l > (2k+9)n + 5k + 11$;
(iii) $l = 0$ and $(4k+14)l > (4k+13)n + 9k + 14$.
then either $f = b_1e^{bz} + c,\ g = b_2e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$ where $b_1,\ b_2$ and b are three constants such that $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.

Theorem 1.2. [8] *Let f and g be two non - constant entire functions, and let $n,\ k,\ l$ be three positive integers. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share $(1, l)$ and one of the following conditions holds:*
(i) $l \geq 2$ and $4l > 3n + 3k + 4$;
(ii) $l = 1$ and $11l > 9n + 8k + 9$;
(iii) $l = 0$ and $6l > 5n + 5k + 7$.
then either $f = b_1e^{bz} + c,\ g = b_2e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $b_1,\ b_2$ and b are three constants such that $(-1)^k(b_1b_2)^n(nb)^{2k} = 1$ and $R(\omega_1, \omega_2) = L(\omega_1) - L(\omega_2)$.

2. Definitions

In 2001, Lahiri [4] introduced a gradation of sharing of values or sets which is known as weighted sharing. Below we are recalling the notion.

Definition 2.1. ([3], [4]) Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that $f,\ g$ share the value a with weight k. We write $f,\ g$ share (a, k) to mean that $f,\ g$ share the value a with weight k. Also we note that $f,\ g$ share a value a IM or CM if and only if $f,\ g$ share $(a, 0)$ or (a, ∞) respectively.

Definition 2.2. (see [4]) For $S \subset \mathbb{C} \cup \{\infty\}$ we define $E_f(S; k)$ as $E_f(S; k) = \cup_{a \in S} E_k(a, f)$, where k is a non-negative integer or infinity.

If $E_f(S, k) = E_g(S, k)$ then we say that f and g share the set S with weight k and write f and g share (S, k).

In order to address our problem we require a linear differential polynomial of a special form.

Definition 2.3. ([6], [7]) Let f be a non - constant meromorphic function. Then we denote $L(f)$ a Linear Differential Polynomial of the form: $L(f) = f^{(q)}$ for $q = 1,\ 2,\ 3$ and $L(f) = \sum_{j=1}^{q-3} a_j f^{(j)} + f^{(q)}$ for $q \geq 4$, where $a_1, a_2, ...a_{q-3}$ are constants.

3. Lemmas

In this segment, we present a few lemmas which will be helpful to prove our main results.

Lemma 3.1. [12] *Let f be a non - constant meromorphic function, let k be a positive integer, and let c be a non - zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N(r,\frac{1}{f}) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}(r,\frac{1}{f}) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 3.2. *(* [8],[11]*) Let f be a non - constant meromorphic function, let k be a positive integer, then*

$$\begin{aligned} N_p\left(r, \frac{1}{f^{(k)}}\right) &\leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\ &\leq (p+k)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \end{aligned}$$

and clearly $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right)$.

Lemma 3.3. [5] *Let f and g be two non - constant entire functions, and let k be positive integer. If $f^{(k)}$ and $g^{(k)}$ share $(1,l)$ $(l = 0,1,2)$. Then*
(i)If $l = 0$,

$$\Theta(0,f) + \delta_k(0,f) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) > 5.$$

then either $f^{(k)}g^{(k)} = 1$ or $f \equiv g$;
(ii) If $l = 1$,

$$\frac{1}{2}[\Theta(0,f) + \delta_k(0,f) + \delta_{k+2}(0,f)] + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) + \Theta(0,g) + \delta_k(0,g) > \frac{9}{2}.$$

then either $f^{(k)}g^{(k)} = 1$ *or* $f \equiv g$;
(iii) If $l \geq 2$,

$$\Theta(0,f) + \delta_k(0,f) + \delta_{k+1}(0,f) + \delta_{k+2}(0,g) > 3.$$

then either $f^{(k)}g^{(k)} = 1$ *or* $f \equiv g$.

Lemma 3.4. [7] *Let* f *and* g *be two non - constant meromorphic functions,* $k(\geq 1)$ *and* $(l \geq 0)$ *be integers. If* $f^{(k)}$ *and* $g^{(k)}$ *share (1,l) (l=0,1,2).Then*
(i) If $l \geq 2$,

$$(k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > k+7.$$

then either $f^{(k)}g^{(k)} = 1$ *or* $f \equiv g$;
(ii) If $l = 1$,

$$(2k+3)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\delta_{k+2}(0,f) > 2k+9.$$

then either $f^{(k)}g^{(k)} = 1$ *or* $f \equiv g$;
(iii) If $l = 0$,

$$(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g) > 4k+13.$$

then either $f^{(k)}g^{(k)} = 1$ *or* $f \equiv g$.

Lemma 3.5. [1] *Let* $f(z)$ *be a non - constant entire function and let* $k(\geq 2)$ *be a positive integer. If* $f(f^{(k)} \neq 0)$, *then* $f = e^{az+b}$, *where* a *and* b *are constants.*

Lemma 3.6. [5] *Let* f *be a non - constant meromorphic function and* n, l *be a positive integers with* $n \geq l+2$, *if* $a \in \mathbb{C} \setminus \{0\}$ *then*

$$\overline{N}(r,0;L(f^n)) \leq (l+1)\overline{N}(r,0;f) + l\overline{N}(r,f) + S(r,f).$$
$$\overline{N}(r,L(f^n)) = \overline{N}(r,f).$$

Lemma 3.7. [5] *Let* f *and* g *be two non - constant meromorphic functions sharing* $(\infty,0)$ *such that* $L(f^n)L(g^n) = \alpha$, *where* α *is a non - zero constant and* $n \geq 1+l$. *Then* $f(z) = c_1\ exp(cz)$ *and* $g(z) = c_2\ exp(-cz)$, *where*

$$(c_1c_2)^n\left\{A\sum_{j=1}^{q-3}a_j(nc)^j + (nc)^q\right\}\left\{A\sum_{j=1}^{q-3}a_j(-nc)^j + (-nc)^q\right\} = \alpha$$

and $A = 0$ *if* $q = 1,\ 2,\ 3$ *and* $A = 1$ *if* $q \geq 4$.

4. Main results

In this paper, we study the existence of solutions for $[L(f^n)]^{(k)}$and the corresponding uniqueness theorems. Thus, we obtain the following results as a generalization of the theorems presented above.
We now state the main results of the paper.

Theorem 4.1. *Let f and g be two non - constant meromorphic functions, and let $n,\ k$ and l be three positive integers. If $[L(f^n)]^{(k)}$ and $[L(g^n)]^{(k)}$ share $(1,l)$, and one of the following conditions holds:*
(i) $l \geq 2$ and $n > 4l(k+2)+3k+8$;
(ii) $l = 1$ and $n > 6l(k+2)+5k+11$;
(iii) $l = 0$ and $n > 2l(5k+7)+9k+14$.
Then
(I) $L(f^n) = \omega L(g^n)$, where $\omega^d = 1$.
(II) $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ where

$$(c_1c_2)^n\left\{A\sum_{j=1}^{q-3}a_j(nc)^j+(nc)^q\right\}\left\{A\sum_{j=1}^{q-3}a_j(-nc)^j+(-nc)^q\right\}=\omega$$

and $\omega^d = 1$ and $A = 0$ if $q = 1,\ 2,\ 3$ and $A = 1$ if $q \geq 4$.

Proof. Let $L(f^n)$ is given by,

$$L(f^n) = f^{n-l}P$$

where P is a differential polynomial in f of degree atmost l and $n \geq l+1$ Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned}\Theta(0,L(f^n)) &= 1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{L(f^n)}\right)}{T(r,L(f^n))}\\ &\geq 1-\limsup_{r\to\infty}\frac{(l+1)\overline{N}(r,0;f)+l\overline{N}(r,f)+S(r,f)}{T(r,f^{n-l})+T(r,P)}\\ &\geq 1-\limsup_{r\to\infty}\frac{(l+1)T(r,f)+lT(r,f)}{(n-l+l)T(r,f)} \qquad (3)\\ &\geq 1-\limsup_{r\to\infty}\frac{(2l+1)T(r,f)}{nT(r,f)}\\ &\geq 1-\frac{2l+1}{n}.\end{aligned}$$

Similarly, we get

$$\Theta(0,L(g^n)) \geq 1-\frac{2l+1}{n}. \qquad (4)$$

Moreover, we have

$$\begin{aligned}\Theta(\infty, L(f^n)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, L(f^n)\right)}{T(r, L(f^n))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f^{n-l}) + T(r, P)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{(n-l+l)T(r, f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\ &\geq 1 - \frac{1}{n}.\end{aligned} \tag{5}$$

Similarly, we get

$$\Theta(\infty, L(g^n)) \geq 1 - \frac{1}{n}. \tag{6}$$

Also, we have,

$$\begin{aligned}\delta_k(0, L(f^n)) &= 1 - \limsup_{r\to\infty} \frac{N_k\left(r, L(f^n)\right)}{T(r, L(f^n))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{k\overline{N}\left(r, \frac{1}{L(f^n)}\right)}{T(r, f^{n-l}) + T(r, P)} \\ &\geq 1 - \frac{k(2l+1)}{n}.\end{aligned} \tag{7}$$

Similarly, we get

$$\delta_k(0, L(g^n)) \geq 1 - \frac{k(2l+1)}{n}. \tag{8}$$

$$\delta_{k+1}(0, L(f^n)) \geq 1 - \frac{(k+1)(2l+1)}{n}. \tag{9}$$

$$\delta_{k+1}(0, L(g^n)) \geq 1 - \frac{(k+1)(2l+1)}{n}. \tag{10}$$

$$\delta_{k+2}(0, L(f^n)) \geq 1 - \frac{(k+2)(2l+1)}{n}. \tag{11}$$

$$\delta_{k+2}(0, L(g^n)) \geq 1 - \frac{(k+2)(2l+1)}{n}. \tag{12}$$

Case I. If $l \geq 2$ and from (3) - (12) and also from Lemma 3.4, we get

$$(k+2)\Theta(\infty, f) + 2\Theta(\infty, f) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g)$$
$$> k + 7.$$

$$(k+2)\left(1-\frac{1}{n}\right)+2\left(1-\frac{1}{n}\right)+\left(1-\frac{2l+1}{n}\right)+\left(1-\frac{2l+1}{n}\right)$$
$$+1-\frac{(k+1)(2l+1)}{n}+1-\frac{(k+1)(2l+1)}{n}$$
$$>k+7.$$
$$n>4l(k+2)+3k+8.$$

Case II. If $l=1$ and from (3) - (12) and also from Lemma 3.4, we get
$(2k+3)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\delta_{k+2}(0,f)$
$>2k+9.$

$$(2k+3)\left(1-\frac{1}{n}\right)+2\left(1-\frac{1}{n}\right)+\left(1-\frac{2l+1}{n}\right)+\left(1-\frac{2l+1}{n}\right)$$
$$+1-\frac{(k+1)(2l+1)}{n}+1-\frac{(k+1)(2l+1)}{n}$$
$$+1-\frac{(k+2)(2l+1)}{n}>2k+9.$$
$$n>6l(k+2)+5k+11.$$

Case III. If $l=0$ and from (3) - (12) and also from Lemma 3.4, we get
$(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)>4k+13.$

$$(2k+3)\left(1-\frac{1}{n}\right)+(2k+4)\left(1-\frac{1}{n}\right)+\left(1-\frac{2l+1}{n}\right)+\left(1-\frac{2l+1}{n}\right)$$
$$+2\left(1-\frac{(k+1)(2l+1)}{n}\right)+3\left(1-\frac{(k+1)(2l+1)}{n}\right)$$
$$>4k+13$$
$$n>2l(5k+7)+9k+14.$$

Case IV. Let $F\equiv G$.Then $L(f^n)=\omega L(g^n)$, where $\omega^d=1$. This is possibility I of the theorem.

Case V. Let $FG\equiv 1$. Then $L(f^n)L(g^n)=\omega$ where $\omega^d=1$.Then by Lemma 3.7 we get the possibility II of the theorem. This proves the theorem.

□

Theorem 4.2. *Let f and g be two non - constant entire functions, and let n, k, l, m are positive integers. If $[L(f^n)]^{(k)}$ and $[L(g^n)]^{(k)}$ share $(1,l)$, and one of the following conditions holds:*
(i) $l\geq 2$ and $n>(6k+8)+3k+4$;
(ii) $l=1$ and $2n>(16k+18)l+8k+9$;
(iii) $l=0$ and $n>(10k+14)l+5k+7$.
Then

(I) $L(f^n) = \omega L(g^n)$ *where* $\omega^d = 1$.
(II) $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$ *where*

$$(c_1c_2)^n\left\{A\sum_{j=1}^{q-3} a_j(nc)^j + (nc)^q\right\}\left\{A\sum_{j=1}^{q-3} a_j(-nc)^j + (-nc)^q\right\} = \omega$$

and $\omega^d = 1$ *and* $A = 0$ *if* $q = 1,\ 2,\ 3$ *and* $A = 1$ *if* $q \geq 4$.

Proof. Let $L(f^n)$ is given by

$$L(f^n) = f^{n-l}P$$

where P is a differential polynomial in f of degree atmost l and $n \geq l+1$. Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned}
\Theta(0, L(f^n)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{L(f^n)}\right)}{T(r, L(f^n))} \\
&\geq 1 - \limsup_{r\to\infty} \frac{(l+1)\overline{N}(r, 0; f) + l\overline{N}(r, f) + S(r, f))}{T(r, f^{n-l}) + T(r, P)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{(l+1)T(r, f) + lT(r, f)}{(n-l+l)T(r, f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{(2l+1)T(r, f)}{nT(r, f)} \\
&\geq 1 - \frac{2l+1}{n}.
\end{aligned} \tag{13}$$

Similarly, we get

$$\Theta(0, L(g^n)) \geq 1 - \frac{2l+1}{n}. \tag{14}$$

Moreover, we have

$$\begin{aligned}
\Theta(\infty, L(f^n)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, L(f^n)\right)}{T(r, L(f^n))} \\
&\geq 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f^{n-l}) + T(r, P)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{(n-l+l)T(r, f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{T(r, f)}{nT(r, f)} \\
&\geq 1 - \frac{1}{n}.
\end{aligned} \tag{15}$$

Similarly, we get

$$\Theta(\infty, L(g^n)) \geq 1 - \frac{1}{n}. \tag{16}$$

Also, we have

$$\begin{aligned} \delta_k(0, L(f^n)) &= 1 - \limsup_{r\to\infty} \frac{N_k\left(r, L(f^n)\right)}{T(r, L(f^n))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{k\overline{N}\left(r, \frac{1}{L(f^n)}\right)}{T(r, f^{n-l}) + T(r, P)} \\ &\geq 1 - \frac{k(2l+1)}{n}. \end{aligned} \tag{17}$$

Similarly, we get

$$\delta_k(0, L(g^n)) \geq 1 - \frac{k(2l+1)}{n}. \tag{18}$$

$$\delta_{k+1}(0, L(f^n)) \geq 1 - \frac{(k+1)(2l+1)}{n}. \tag{19}$$

$$\delta_{k+1}(0, L(g^n)) \geq 1 - \frac{(k+1)(2l+1)}{n}. \tag{20}$$

$$\delta_{k+2}(0, L(f^n)) \geq 1 - \frac{(k+2)(2l+1)}{n}. \tag{21}$$

$$\delta_{k+2}(0, L(g^n)) \geq 1 - \frac{(k+2)(2l+1)}{n}. \tag{22}$$

Case I. If $l \geq 2$ and from (13) - (22) and also from Lemma 3.3, we get

$$\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+2}(0, g) > 3.$$

$$\begin{aligned} &\left(1 - \frac{2l+1}{n}\right) + \left(1 - \frac{k(2l+1)}{n}\right) + \left(1 - \frac{(k+1)(2l+1)}{n}\right) \\ &+ \left(1 - \frac{(k+2)(2l+1)}{n}\right) \\ &> 3. \end{aligned}$$

$$n > 2l(3k+4) + 3k + 4.$$

Case II. If $l = 1$ and from (13) - (22) and also from Lemma 3.3, we get

$$\frac{1}{2}[\Theta(0,f)+\delta_k(0,f)+\delta_{k+2}(0,f)]+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\Theta(0,g)+\delta_k(0,g) > \frac{9}{2},$$

$$\frac{1}{2}\left[1-\frac{2l+1}{n}+1-\frac{k(2l+1)}{n}+1-\frac{(k+2)(2l+1)}{n}\right]$$
$$+\left(1-\frac{(k+1)(2l+1)}{n}\right)+\left(1-\frac{(k+1)(2l+1)}{n}\right)$$
$$+\left(1-\frac{2l+1}{n}\right)+\left(1-\frac{k(2l+1)}{n}\right)>\frac{9}{2}.$$

$$2n > 2l(8k+9)+8k+9.$$

Case III. If $l=0$ and from (13) - ((22)) and also from Lemma 3.3, we get

$$\Theta(0,f)+\delta_k(0,f)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)>5.$$

$$\left(1-\frac{2l+1}{n}\right)+\left(1-\frac{k(2l+1)}{n}\right)+\left(1-\frac{(k+1)(2l+1)}{n}\right)+\left(1-\frac{(k+1)(2l+1)}{n}\right)$$
$$+\left(1-\frac{(k+2)(2l+1)}{n}\right)+\left(1-\frac{(k+2)(2l+1)}{n}\right)>5.$$

$$n > 2l(5k+7)+5k+7.$$

Case I. Let $F\equiv G$. Then $L(f^n)=\omega L(g^n)$, where $\omega^d=1$. This is possibility I of the theorem.

Case II. Let $FG\equiv 1$. Then $L(f^n)L(g^n)=\omega$ where $\omega^d=1$. Then by Lemma 3.7 we get the possibility II of the theorem.
This proves the theorem. □

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