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CAPUTO-FABRIZIO FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS VIA NEW DHAGE ITERATION METHOD

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Abstract. In this paper, we study the following hybrid Caputo-Fabrizio fractional differential equation:

$$ {}^{\mathcal{CF}}_{a}\mathfrak{D}^{\theta}_{\vartheta}\left[\omega(\vartheta) - \mathfrak{F}(\vartheta, \omega(\vartheta))\right] = \mathfrak{G}(\vartheta, \omega(\vartheta)), \quad \vartheta \in \mathfrak{J} := [a, b], $$
$$ \omega(a) = \varphi_a \in \mathbb{R}, $$

The result is based on a Dhage fixed point theorem in Banach algebra. Further, an example is provided for the justification of our main result.

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1. Introduction

Fractional differential equations appear in many applications of real world problem, for example, in electromagnetic, in economy, in biology, etc ([3, 4, 19]). In [7], Akhadkulov *et al.* discussed via a new version of Kransoselskii-type fixed-point theorem under a nonlinear $\mathcal{D}$-contraction condition (see, Dhage version of Kransoselskii-type fixed-point theorem [9]) the following fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators of orders $0 < \lambda < 1$ and $\gamma > 0$:

$$ \begin{cases} \mathbb{D}^{\lambda}[\varpi(\varepsilon) - \Phi(\varepsilon, \varpi(\varepsilon))] = \Psi\left(\varepsilon, \varpi(\varepsilon), \mathbb{I}^{\gamma}(\varpi(\varepsilon))\right), \text{ a.e. } \varepsilon \in \mathbf{J}, \gamma > 0, \\ \varpi\left(\varepsilon_0\right) = \varpi_0, \end{cases} $$

where $\mathbf{J} = [\varepsilon_0, \varepsilon_0 + \ell]$, for some fixed $\varepsilon_0 \in \mathbb{R}$ and $\ell > 0$ and $\Phi \in C(\mathbf{J} \times \mathbb{R}, \mathbb{R}), \Psi \in \mathcal{C}(\mathbf{J} \times \mathbb{R}^2, \mathbb{R})$. In [15], the authors considered the functional integro-differential

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equations of fractional order

$$\frac{d^{\varrho}}{d\varepsilon^{\varrho}}[\varpi(\varepsilon)-\Phi(\varepsilon,\varpi(\varepsilon))]=\Psi\left(\varepsilon,\int_0^{\varepsilon}\Upsilon\left(s,\varpi_s\right)\right)ds,\qquad \varepsilon\in\mathbb{R}_+,$$

where $0<\varrho<1$, $\varpi_{\varepsilon}:\mathbb{R}_+\to\mathbb{R}, \Phi(\varepsilon,\varpi)=\Phi:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}, \Psi(\varepsilon,\varpi)=\Psi:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$. Dhage and Jadhav [14] studied the existence of solution for hybrid differential equation:

$$\begin{cases} \dfrac{d}{d}[\varpi(\varepsilon)-\Phi(\varepsilon,\varpi(\varepsilon))]=\Psi(\varepsilon,\varpi(\varepsilon)), \varepsilon\in\mathbf{J},\\ \varpi\left(\varepsilon_0\right)=\varpi_0\in\mathbb{R},\end{cases}$$

where $\Phi,\Psi\in C(\mathbf{J}\times\mathbb{R},\mathbb{R}\backslash\{0\})$. In [18], Lu et *al.* established, under the φ -Lipschitz contraction condition, the existence result for the following fractional hybrid differential problems via the Riemann-Liouville derivative of order $0<\xi<1$:

$$\begin{cases} \mathbb{D}^{\xi}[\varpi(\varepsilon)-\Phi(\varepsilon,\varpi(\varepsilon))]=\Psi(\varepsilon,\varpi(\varepsilon)),\ \text{a.e. } \varepsilon\in\mathbf{J},\\ \varpi\left(\varepsilon_0\right)=\varpi_0\in\mathbb{R},\end{cases}$$

where $\Phi,\Psi\in\mathcal{C}(\mathbf{J}\times\mathbb{R},\mathbb{R})$.

On the other hand, fractional-order differential equations with the Caputo-Fabrizio (in short $\mathfrak{C}.\mathfrak{F}$) derivative have been paid more and more attentions from 2015. (see [1, 2, 5, 6, 8, 16, 17]). In this article, we investigate the following hybrid fractional differential equations (in short HFDE):

$$\begin{gathered} {}^{\mathcal{CF}}_{a}\mathfrak{D}^{\theta}_{\vartheta}\left[\omega(\vartheta)-\mathfrak{F}(\vartheta,\omega(\vartheta))\right]=\mathfrak{G}(\vartheta,\omega(\vartheta)),\quad \vartheta\in\mathfrak{J}:=[a,b],\\ \omega(a)=\varphi_a,\end{gathered}\tag{1}$$

where $b>a>0$, $\mathfrak{F},\mathfrak{G}:\mathfrak{J}\times\mathbb{R}\to\mathbb{R}$ are given functions and ${}^{\mathcal{CF}}_{a}\mathfrak{D}^{\theta}_{\vartheta}$ is the $\mathfrak{C}.\mathfrak{F}$ derivative of order $\theta\in(0,1)$.

We prove the existence of a solution for HFDE (1) using a Dhage fixed point theorem in Banach algebra (see [11]). This article is orderly as follows: In Section 2, we present some definitions and results are presented. In Section 3, we develop the monotone iterative technique and prove the existence of solution for the problem (1) by using Dhage fixed point theorem (Theorem 2.6). In the fourth section, an example is constructed to illustrate the applicability of the proved results.

2. Preliminaries

Let us introduce the Banach spaces $\mathcal{C}:=\{f:\mathfrak{J}\to\mathbb{R}, f \text{ is continuous}\}$, with the norm

$$\|\phi\|_{\infty}=\sup_{\vartheta\in\mathfrak{J}}|\phi(\vartheta)|,$$

and

$$L^1(\mathfrak{J},\mathbb{R}):=\{f:\mathfrak{J}\to\mathbb{R}, f \text{ is measurable and Lebesgue integrable}\},$$

with the norm

$$\|\phi\|_{L^1} = \int_a^b |\phi(\vartheta)|d\vartheta.$$

Let us denote $\mathbb{Y}(\theta) := \dfrac{2}{(2-\theta)\mathfrak{R}(\theta)}$.

Definition 2.1 ([8, 2, 1, 17])**.** Let $0 < \theta < 1$ and $\mathfrak{R}(\theta)$ be a normalization function such that $\mathfrak{R}(0) = \mathfrak{R}(1) = 1$. We define the $\mathfrak{C}.\mathfrak{F}$ fractional integral of order θ for a function $\chi \in L^1(\mathbf{I})$ by

$${}^{\mathcal{CF}}I^{\theta}\chi(\vartheta) = (1-\theta)\mathbb{Y}(\theta)\chi(\vartheta) + \theta\mathbb{Y}(\theta)\int_0^{\vartheta}\chi(s)ds, \quad \vartheta \geq 0.$$

Definition 2.2 ([8, 2, 1, 17])**.** Let $0 < \theta < 1$ and $\chi \in C^1(\mathbf{I})$. We define the $\mathfrak{C}.\mathfrak{F}$ fractional derivative for a function χ of order θ by

$${}^{\mathcal{CF}}\mathfrak{D}^{\theta}\chi(\vartheta) = \frac{1}{(1-\theta)\mathbb{Y}(\theta)}\int_0^{\vartheta}\exp(-\frac{\theta}{1-\theta}(\vartheta - s))\chi'(s)ds; \ \vartheta \in \mathbf{I}.$$

Remark 2.1. $({}^{\mathcal{CF}}\mathfrak{D}^{\theta})(\chi) = 0 \iff \chi = C^{st}$.

Definition 2.3. A mapping $\mathcal{K} : \Xi \to \Xi$ is called:

(i) **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$,
i.e. if $\omega \preceq \varpi \Rightarrow \mathcal{K}\omega \preceq \mathcal{K}\varpi$ for all $\omega, \varpi \in \Xi$;

(ii) **monotone nonincreasing** if $\omega \preceq \varpi \Rightarrow \mathcal{K}\omega \succeq \mathcal{K}\varpi$ for all $\omega, \varpi \in \Xi$;

(iii) **monotone** if it is either monotone nondecreasing or monotone nonincreasing on Ξ.

Definition 2.4 ([12])**.** Let Ξ on a normed linear space. An operator $\mathcal{K} : \Xi \to \Xi$ is called:

(i) **compact** if $\mathcal{K}(\Xi)$ is a relatively compact subset of Ξ.

(ii) **totally bounded** if for any bounded subset S of Ξ, $\mathcal{K}(S)$ is a relatively compact subset of Ξ.

(iii) **completely continuous** on Ξ if $\mathcal{K}$ is continuous and totally bounded.

Definition 2.5 (Partially nonlinear $\mathcal{D}$-Lipschitz mapping [12])**.** Let $(\Xi, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{K} : \Xi \to \Xi$ is called:

(i) **partially nonlinear $\mathcal{D}$-Lipschitz** if there exists an upper semi-continuous nondecreasing function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Upsilon(0) = 0$ and

$$\|\mathcal{K}\xi - \mathcal{K}\zeta\| \leq \Upsilon(\|\xi - \zeta\|) \tag{2}$$

for all comparable elements $\xi, \zeta \in \Xi$;

(ii) **partially Lipschitz** if $\Upsilon(r) = \ell r$ where $\ell > 0$, with a Lipschitz constant ℓ;

(iii) **partially contraction** with contraction constant ℓ if $\ell < 1$;

(iv) **nonlinear $\mathcal{D}$-contraction** if it is a nonlinear $\mathcal{D}$-Lipschitz with $\Upsilon(r) < r$ for $r > 0$.

Theorem 2.6 ([11]). *Let $\big(\Xi, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in Ξ. Let $\Psi, \Phi : \Xi \to \Xi$ be two nondecreasing operators such that*

(i) *Ψ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(ii) *Φ is partially continuous and partially compact, and*
(iii) *there exists an element $\omega_0 \in \Xi$ such that $\omega_0 \preceq \Psi\omega_0 + \Phi\omega_0$ or $\omega_0 \succeq \Psi\omega_0 + \Phi\omega_0$.*

Then, the operator equation $\Psi\omega + \Phi\omega = \omega$ has a solution ω^* in Ξ and the sequence $\{\omega_n\}$ of successive iterations defined by $\omega_{n+1} = \Psi\omega_n + \Phi\omega_n$, $n = 0, 1, \ldots$, converges monotonically to ω^*.

3. Existence of Solutions

Definition 3.1. A function $u \in C(\mathfrak{J}, \mathbb{R})$ is said to be a lower solution of the HFDE (1) if it satisfies

$$\begin{cases} {}^{\mathcal{CF}}{}_{a}\mathcal{D}^{\theta}_{\tau}\left[\omega(\vartheta) - \mathfrak{F}(\vartheta, \omega(\vartheta))\right] \leq \mathfrak{G}(\vartheta, \omega(\theta)), \\ \omega(a) \leq \varphi_a, \end{cases} \tag{3}$$

for all $\tau \in \mathfrak{J}$. Similarly, an upper solution $v \in C(\mathfrak{J}, \mathbb{R})$ for the HFDE (1) is defined on $\mathfrak{J}$ by reversing the order.

Definition 3.2. A function $\omega \in C(\mathfrak{J}, \mathbb{R})$ is said to be a solution of HFDE (1) if it satisfies equation

$${}^{\mathcal{CF}}_{a}\mathfrak{D}^{\theta}_{\tau}\left[\omega(\vartheta) - \mathfrak{F}(\vartheta, \omega(\vartheta))\right] = \mathfrak{G}(\vartheta, \omega(\vartheta)), \tag{4}$$

on $\mathfrak{J}$, and the condition $\omega(a) = \varphi_a$.

Let us introduce the following assumptions:

($\mathbf{C_I}$) The functions $\mathfrak{F}, \mathfrak{G} : \mathfrak{J} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.
($\mathbf{C_{II}}$) The functions $\mathfrak{F}, \mathfrak{G}$ are nondecreasing in ω, for all $\vartheta \in \mathfrak{J}$.
($\mathbf{C_{III}}$) There exist constants $\Delta_{\mathfrak{F}}, \nabla_{\mathfrak{G}} > 0$ such that

$$\begin{aligned} |\mathfrak{F}(\vartheta, \omega)| &\leq \Delta_{\mathfrak{F}}, \\ |\mathfrak{G}(\vartheta, \omega)| &\leq \nabla_{\mathfrak{G}}, \end{aligned} \tag{5}$$

for all $\vartheta \in \mathfrak{J}$ and $\omega \in \mathbb{R}$.

($\mathbf{C_{IV}}$) There exists a $\mathfrak{D}-$contraction Ω such that

$$0 \leq \mathfrak{F}(\vartheta, \omega) - \mathfrak{F}(\vartheta, \varpi) \leq \Omega(\omega - \varpi),$$

for all $\vartheta \in \mathfrak{J}$ and $\omega, \varpi \in \mathbb{R}$, with $\omega \geq \varpi$.

($\mathbf{C_V}$) There exists a lower solution $\upsilon \in C(\mathfrak{J}, \mathbb{R})$ of problem (1), that is

$$\begin{cases} {}^{\mathcal{CF}}{}_{a}\mathcal{D}^{\theta}_{\tau}\left[\upsilon(\vartheta) - \mathfrak{F}(\vartheta, \upsilon(\vartheta))\right] \leq \mathfrak{G}(\vartheta, \upsilon(\vartheta)), \\ \upsilon(a) \leq \varphi_a. \end{cases} \tag{6}$$

Lemma 3.3 ([1]). *Let $\alpha \in (0,1]$. For a given continuous function $\mathfrak{H} : \mathfrak{G} \to \mathbb{R}$, a function $\omega \in C(\mathfrak{J}, \mathbb{R})$ is a solution of the Cauchy problem*

$$\begin{cases} {}^{\mathcal{CF}}{}_{a}\mathcal{D}^{\theta}_{\vartheta}\omega(\vartheta) = \mathfrak{H}(\vartheta), \quad \vartheta \in \mathfrak{J}, \\ \omega(a) = \varphi_a \in \mathbb{R}, \end{cases} \tag{7}$$

if and only if it is a solution of the nonlinear integral equation

$$\omega(\vartheta) = \varphi_a + \mathfrak{A}(\theta)\left(\mathfrak{H}(\vartheta) - \mathfrak{H}(a)\right) + \mathfrak{B}(\theta)\int_a^{\vartheta} \mathfrak{H}(s)ds, \tag{8}$$

where

$$\mathfrak{A}(\theta) = (1-\theta)\mathbb{Y}(\theta),\ \mathfrak{B}(\theta) = \theta\mathbb{Y}(\theta).$$

Theorem 3.4. *Suppose that $(\boldsymbol{C_I}) - (\boldsymbol{C_V})$ are satisfied. Then, the fractional Cauchy problem (1) has a solution ω^* defined on $\mathfrak{J}$ and the sequence $\{\omega_n\}_{n=0}^{\infty}$ of successive approximations given by*

$$\begin{cases} \omega_0 = u(\vartheta), \\ \omega_{n+1}(\vartheta) = \mathfrak{F}(\vartheta, \omega_n(\vartheta)) + \varphi_a - \mathfrak{F}(a, \varphi_a) \\ \qquad +\mathfrak{A}(\theta)\left[\mathfrak{G}(\vartheta, \omega_n(\vartheta)) - \mathfrak{G}(a, \varphi_a)\right] + \mathfrak{B}(\theta)\int_a^{\vartheta} \mathfrak{G}(\sigma, \omega_n(\sigma))d\sigma, \end{cases} \tag{9}$$

i.e.

$$\begin{cases} \omega_0(\vartheta) = u(\vartheta), \\ \omega_{n+1}(\vartheta) = \mathfrak{F}(\vartheta, \omega_n(\vartheta)) + \varphi_a - \mathfrak{F}(a, \varphi_a) \\ \qquad +(1-\theta)\mathbb{Y}(\theta)\left[\mathfrak{G}(\vartheta, \omega_n(\vartheta)) - \mathfrak{G}(a, \varphi_a)\right] + \theta\mathbb{Y}(\theta)\int_a^{\vartheta} \mathfrak{G}(\sigma, \omega_n(\sigma))d\sigma \end{cases} \tag{10}$$

converges monotonically to ω^*.

Proof. $\Xi = C(\mathfrak{J}, \mathbb{R})$ is a partially ordered Banach space. Then, consider the equivalent operator equation

$$\Psi\omega(\vartheta) + \Phi\omega(\vartheta) = \omega(\vartheta),$$

where

$$\Psi\omega(\vartheta) = \mathfrak{F}(\vartheta, \omega(\vartheta)), \tag{11}$$

and

$$\begin{aligned} \Phi\omega(\vartheta) = \varphi_a - \mathfrak{F}(a, \varphi_a) + (1-\theta)\mathbb{Y}(\theta)\left[\mathfrak{G}(\vartheta, \omega(\vartheta)) - \mathfrak{G}(a, \varphi_a)\right] \\ + \theta\mathbb{Y}(\theta)\int_a^{\vartheta} \mathfrak{G}(\sigma, \omega(\sigma))d\sigma, \end{aligned} \tag{12}$$

for $\vartheta \in \mathfrak{J}$.

Step I: Ψ *and* Φ *are nondecreasing operators on* Ξ.
Let $\omega, \varpi \in \Xi$ where $\omega \geq \varpi$. Then, by hypothesis $(\mathbf{C_{II}})$ and for $\vartheta \in \mathfrak{J}$, we get

$$\begin{aligned}\omega(\vartheta) \geq \varpi(\vartheta) &\Longrightarrow \mathfrak{F}(\vartheta, \omega(\vartheta)) \geq \mathfrak{F}(\vartheta, \varpi(\vartheta)) \\ &\Longrightarrow \Psi\omega(\vartheta) \geq \Psi\varpi(\vartheta).\end{aligned}$$

Then, Ψ is a nondecreasing operator on Ξ into Ξ. For $\vartheta \in \mathfrak{J}$ and by $(\mathbf{C_{II}})$, we get

$$\begin{aligned}\Phi\omega(\vartheta) - \Phi\varpi(\vartheta) &= (1-\theta)\mathbb{Y}(\theta)\left[\mathfrak{G}(\vartheta, \omega(\vartheta)) - \mathfrak{G}(\vartheta, \varpi(\vartheta))\right] \\ &\quad + \theta\mathbb{Y}(\theta)\int_a^\vartheta \left[\mathfrak{G}(\sigma, \omega(\sigma)) - \mathfrak{G}(\sigma, \varpi(\sigma))\right] d\sigma \geq 0.\end{aligned}$$

Then, Φ is nondecreasing operator on Ξ into Ξ.

Step II: Ψ *is a partially bounded and partially nonlinear* $\mathfrak{D}$*-contraction operator on* Ξ. Let $\omega \in \Xi$, then for $\vartheta \in \mathfrak{J}$ and by $(\mathbf{C_{III}})$, we get

$$|\Psi\omega(\vartheta)| = |\mathfrak{F}(\vartheta, \omega(\vartheta))| \leq \Delta_{\mathfrak{F}}.$$

Thus,

$$\| \Psi\omega \| \leq \Delta_{\mathfrak{F}}.$$

Then, Ψ is bounded on Ξ and so partially bounded.
On the other hand, let $\omega, \varpi \in \Xi$ where $\omega \geq \varpi$. Then, for $\vartheta \in \mathfrak{J}$, by hypothesis $(\mathbf{C_{IV}})$, we get

$$\begin{aligned}|\Psi\omega(\vartheta) - \Psi\varpi(\vartheta)| &= |\mathfrak{F}(\vartheta, \omega(\vartheta)) - \mathfrak{F}(\vartheta, \varpi(\vartheta))| \\ &\leq \Omega(|\omega(\vartheta) - \varpi(\vartheta)|) \\ &\leq \Omega(\|\omega - \varpi\|).\end{aligned}$$

Then, for each $\omega, \varpi \in \Xi$ where $\omega \geq \varpi$, we get

$$\|\Psi\omega - \Psi\varpi\| \leq \Omega(\|\omega - \varpi\|).$$

Thus, Ψ is a partially nonlinear $\mathfrak{D}-$contraction on Ξ and, thus partially continuous.

Step III: Φ *is a partially continuous on* Ξ.
Let $\{\omega_n\}$ be a sequence of elements of a chain $\mathcal{C}$ in Ξ such that $\omega_n \to \omega^*$ for

each $n \in \mathbb{N}$. Then, by $(\mathbf{C_I})$ and the dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty}(\Phi\omega_n)(\vartheta) &= \lim_{n\to\infty}\Big[\varphi_a - \mathfrak{F}(a,\varphi_a) + (1-\theta)\mathbb{Y}(\theta)\left[\mathfrak{G}(\vartheta,\omega_n(\vartheta)) - \mathfrak{G}(a,\varphi_a)\right] \\
&\quad + \theta\mathbb{Y}(\theta)\int_a^{\vartheta}\mathfrak{G}(\sigma,\omega_n(\sigma))d\sigma\Big] \\
&= \varphi_a - \mathfrak{F}(a,\varphi_a) + (1-\theta)\mathbb{Y}(\theta)\Big[\lim_{n\to\infty}\mathfrak{G}(\vartheta,\omega_n(\vartheta)) - \mathfrak{G}(a,\varphi_a)\Big] \\
&\quad + \theta\mathbb{Y}(\theta)\int_a^{\vartheta}\Big[\lim_{n\to\infty}\mathfrak{G}(\vartheta,\omega_n(\tau))\Big]d\sigma \\
&= \varphi_a - \mathfrak{F}(a,\varphi_a) + (1-\theta)\mathbb{Y}(\theta)\left[\mathfrak{G}(\vartheta,\omega^*(\vartheta)) - \mathfrak{G}(a,\varphi_a)\right] \\
&\quad + \theta\mathbb{Y}(\theta)\int_a^{\vartheta}\mathfrak{G}(\sigma,\omega^*(\sigma))d\sigma \\
&= (\Phi\omega^*)(\vartheta),
\end{aligned}$$

for all $\vartheta \in \mathfrak{J}$. This shows that $\{\Phi\omega_n\}$ converges to $\Phi\omega^*$ pointwise on $\mathfrak{J}$ and and the convergence is monotonic by the property of $\mathfrak{G}$.
Next, we will show that $\{\Phi\omega_n\}$ is an equicontinuous sequence of functions in Ξ. Let $\vartheta_1, \vartheta_2 \in \mathfrak{J}$ be arbitrary with $\vartheta_1 < \vartheta_2$. Then, by $(\mathbf{C_{III}})$ we have

$$\begin{aligned}
\left|\Phi\omega_n(\vartheta_2) - \Phi\omega_n(\vartheta_1)\right| &\le (1-\theta)\mathbb{Y}(\theta)\left|\mathfrak{G}(\vartheta_2,\omega_n(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega_n(\vartheta_1))\right| \\
&\quad + \theta\mathbb{Y}(\theta)\left|\int_a^{\vartheta_2}\mathfrak{G}(\sigma,\omega_n(\sigma))\,d\sigma - \int_a^{\vartheta_1}\mathfrak{G}(\sigma,\omega_n(\sigma))\,d\sigma\right| \\
&\le (1-\theta)\mathbb{Y}(\theta)\left|\mathfrak{G}(\vartheta_2,\omega_n(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega_n(\vartheta_1))\right| \\
&\quad + \theta\mathbb{Y}(\theta)\int_{\vartheta_1}^{\vartheta_2}\left|\mathfrak{G}(\sigma,\omega_n(\sigma))\right|d\sigma \\
&\le (1-\theta)\mathbb{Y}(\theta)\left|\mathfrak{G}(\vartheta_2,\omega_n(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega_n(\vartheta_1))\right| \\
&\quad + \nabla_{\mathfrak{G}}\theta\mathbb{Y}(\theta)(\vartheta_2 - \vartheta_1).
\end{aligned}$$

Then, we have

$$\left|\Phi\omega_n(\vartheta_2) - \Phi\omega_n(\vartheta_1)\right| \to 0 \text{ as } \vartheta_2 \to \vartheta_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\Phi\omega_n \to \Phi\omega^*$ is uniformly and hence Φ is a partially continuous on Ξ.

Step IV: *Φ is a partially compact on Ξ.*
Let $\mathcal{C}$ be a chain in Ξ. We shall show that $\Phi(\mathcal{C})$ is uniformly bounded and equicontinuous in Ξ. Let $\widetilde{\omega} \in \Phi(\mathcal{C})$ be arbitrary. We have $\widetilde{\omega} = \Phi(\omega)$ for some $\omega \in \mathcal{C}$, and by $(\mathbf{C_{III}})$, we get

$$
\begin{aligned}
\widetilde{\omega}(\vartheta) &= |\Phi\omega(\vartheta)| \\
&\leq |\varphi_a - \mathfrak{F}(a,\varphi_a)| + (1-\theta)\mathbb{Y}(\theta)\,|\mathfrak{G}(\vartheta,\omega(\vartheta)) - \mathfrak{F}(a,\varphi_a)| \\
&\quad + \theta\mathbb{Y}(\theta)\int_a^{\vartheta} |\mathfrak{G}(\sigma,\omega(\sigma))|\,d\sigma \\
&\leq |\varphi_a - \mathfrak{F}(a,\varphi_a)| + 2(1-\theta)\mathbb{Y}(\theta)\nabla_{\mathfrak{G}} + \theta\mathbb{Y}(\theta)\nabla_{\mathfrak{G}}(b-a) := \mathbf{M},
\end{aligned}
$$

for all $\vartheta \in \mathfrak{J}$. Taking the supremum over ϑ, we obtain $\|\widetilde{\omega}\| = \|\Phi\omega\| \leq \mathbf{M}$ for each $\widetilde{\omega} \in \Phi(\mathcal{C})$. Hence, $\Phi(\mathcal{C})$ is a uniformly bounded subset of Ξ.
Next, we will show that $\Phi(\mathcal{C})$ is an equicontinuous set in Ξ. Let $\vartheta_1, \vartheta_2 \in \mathfrak{J}$ where $\vartheta_1 < \vartheta_2$. Then, by $(\mathbf{C_{III}})$ we get

$$
\begin{aligned}
|\Phi\omega(\vartheta_2) - \Phi\omega(\vartheta_1)| &\leq (1-\theta)\mathbb{Y}(\theta)\,|\mathfrak{G}(\vartheta_2,\omega(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega(\vartheta_1))| \\
&\quad + \theta\mathbb{Y}(\theta)\left|\int_a^{\vartheta_2} \mathfrak{G}(\sigma,\omega(\sigma))\,d\sigma - \int_a^{\vartheta_1} \mathfrak{G}(\sigma,\omega(\sigma))\,d\sigma\right| \\
&\leq (1-\theta)\mathbb{Y}(\theta)\,|\mathfrak{G}(\vartheta_2,\omega(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega(\vartheta_1))| \\
&\quad + \theta\mathbb{Y}(\theta)\int_{\vartheta_1}^{\vartheta_2} |\mathfrak{G}(\sigma,\omega(\sigma))|\,d\sigma \\
&\leq (1-\theta)\mathbb{Y}(\theta)\,|\mathfrak{G}(\vartheta_2,\omega_n(\vartheta_2)) - \mathfrak{G}(\vartheta_1,\omega_n(\vartheta_1))| \\
&\quad + \theta\mathbb{Y}(\theta)\nabla_{\mathfrak{G}}(\vartheta_2 - \vartheta_1).
\end{aligned}
$$

Then, we have

$$
|\Phi\omega(\vartheta_2) - \Phi\omega(\vartheta_1)| \to 0 \quad \text{as} \quad \vartheta_1 \to \vartheta_2
$$

uniformly for all $\omega \in C$. This shows that $\Phi(C)$ is an equicontinuous set in Ξ. Hence $\Phi(C)$ is compact subset of Ξ and consequently Φ is a partially compact operator on Ξ into itself.

Step V: υ satisfies the operator inequality $\upsilon \leq \Phi\upsilon$. Since the hypothesis $(\mathbf{C_V})$ holds, ω is a lower solution of (1) defined on $\mathfrak{J}$, i.e

$$
\begin{cases}
{}^{\mathcal{CF}}{}_a\mathcal{D}_\tau^\theta\,[\upsilon(\vartheta) - \mathfrak{F}(\vartheta,\upsilon(\vartheta))] \leq \mathfrak{G}(\vartheta,\upsilon(\vartheta)) \\
\upsilon(a) \leq \varphi_a,
\end{cases}
$$

for all $\vartheta \in \mathfrak{J}$. By integrating of inequality

$$
{}^{\mathcal{CF}}{}_a\mathcal{D}_\tau^\theta\,[\upsilon(\vartheta) - \mathfrak{F}(\vartheta,\upsilon(\vartheta))] \leq \mathfrak{G}(\vartheta,\upsilon(\vartheta)), \tag{13}
$$

from a to ϑ, we get

$$
\begin{aligned}
\upsilon(\vartheta) \leq \mathfrak{F}(\vartheta,\upsilon(\vartheta)) + \varphi_a - \mathfrak{F}(a,\varphi_a) + (1-\theta)\mathbb{Y}(\theta)\,[\mathfrak{G}(\vartheta,\omega(\vartheta)) - \mathfrak{G}(a,\varphi_a)] \\
+\theta\mathbb{Y}(\theta)\int_a^{\vartheta} \mathfrak{G}(\sigma,\omega(\sigma))d\sigma,
\end{aligned}
$$

for all $\vartheta \in \mathfrak{J}$. This show that υ is a lower solution of the operator inequality $\upsilon \leq \Psi\upsilon + \Phi\upsilon$. Thus Ψ and Φ satisfies all conditions in Theorem 2.6. We conclude that the operator equation $\Psi\omega + \Phi\omega = \omega$ has a solution. Furthermore, the sequence

$\{\omega_n\}$ of successive approximations defined by (10) converges monotonically to ω^*. $\square$

4. An Example

Consider the following hybrid fractional Cauchy problem:

$$ {}^{\mathcal{CF}}{}_0\mathcal{D}_\vartheta^{\frac{1}{2}} \left[\omega(\vartheta) - \frac{1}{25}\left(\frac{\omega(\vartheta)}{5+\omega(\vartheta)} + 2 \right) \right] = \pi + \frac{1}{4} e^{-\vartheta} \arctan \omega(\vartheta), \ \ \vartheta \in \mathfrak{J} = [0,1], \tag{14} $$

$$ \omega(0) = \varphi_0 \in \mathbb{R}. \tag{15} $$

Set

$$ \mathfrak{F}(\vartheta,\omega) = \frac{1}{25}\left(\frac{\omega}{5+\omega} + 2 \right), $$

and

$$ \mathfrak{G}(\vartheta,\omega) = \pi + \frac{1}{4} e^{-\vartheta} \arctan \omega, $$

for all $\vartheta \in \mathfrak{J}, \ \omega \in \mathbb{R}$. Clearly, the functions $\mathfrak{F}$ and $\mathfrak{G}$ are jointly continuous and nondecreasing in ω for all $\vartheta \in \mathfrak{J}$. Then conditions $(\mathbf{C_I})$ and $(\mathbf{C_{II}})$ are satisfies. Furthermore, the functions $\mathfrak{F}$ and $\mathfrak{G}$ satisfy the condition $(\mathbf{C_{III}})$ with $\Delta_{\mathfrak{F}} = \dfrac{13}{150}$ and $\nabla_{\mathfrak{G}} = \dfrac{9\pi}{8}$. On the other hand, let $\omega, \varpi \in \mathbb{R}$ where $\omega \geq \varpi$, and $\vartheta \in \mathfrak{J}$, then,

$$ \begin{aligned} 0 \leq \mathfrak{F}(\vartheta,\omega) - \mathfrak{F}(\vartheta,\varpi) &= \frac{1}{25}\left[\frac{\omega}{5+\omega} - \frac{\varpi}{5+\varpi} \right] \\ &= \frac{1}{5}\left[\frac{\omega - \varpi}{(5+\omega)(5+\varpi)} \right] \\ &\leq \frac{1}{5}(\omega - \varpi) \\ &= \Omega(\omega - \varpi), \end{aligned} $$

for all $\vartheta \in \mathfrak{J}$, where $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Omega(\vartheta) = \dfrac{1}{5}\vartheta < \vartheta, \ \vartheta > 0$, is a $\mathfrak{D}-$contraction. This shows that the function $\mathfrak{F}$ satisfies the condition $(\mathbf{C_{IV}})$.

Finally, $\upsilon(\vartheta) = 0, \ \text{if } \vartheta \in [0,1]$, is a lower solution of the HFDE (14) defined on $\mathfrak{J}$. Indeed,

$$ \begin{aligned} 0 = \upsilon(\vartheta) &\leq \mathfrak{F}(\vartheta,0) + \varphi_0 - \mathfrak{F}(0,\varphi_0) + \left(1 - \frac{1}{2}\right) \mathbb{Y}\left(\frac{1}{2}\right) [\mathfrak{G}(\vartheta,0) - \mathfrak{G}(0,\varphi_0)] \\ &\quad + \frac{1}{2}\mathbb{Y}\left(\frac{1}{2}\right) \int_0^\vartheta \mathfrak{G}(\sigma,0) d\sigma \\ &\leq \frac{2}{25} - \frac{1}{25}\left[\frac{\varphi_0}{5+\varphi_0} + 2 \right] + \varphi_0 + \mathbb{Y}\left(\frac{1}{2}\right) \left[\frac{\pi}{2} - \frac{1}{8}\arctan \varphi_0 \right] \\ &\leq \varphi_0 - \frac{\varphi_0}{25(5+\varphi_0)} + \mathbb{Y}\left(\frac{1}{2}\right) \left[\frac{\pi}{2} - \frac{1}{8}\arctan \varphi_0 \right], \end{aligned} $$

for some $\varphi_0 \in \mathbb{R}$. Then, the condition $(\mathbf{C_V})$ is true. Thus, all conditions, $(\mathbf{C_I})-(\mathbf{C_V})$, are satisfied. It follows from Theorem 3.4 that the problem (14)-(15) as a solution solution ω^* on $\mathfrak{J} = [0, 1]$, which is a limit of the monotone sequence $(\omega_n),\ n = 0, 1, \ldots,$ defined by

$$\omega_0(\vartheta) = \varphi_0, \text{ for } \vartheta \in [0, 1],$$

where $\varphi_0 - \dfrac{\varphi_0}{25(5+\varphi_0)} + \mathbb{Y}\left(\dfrac{1}{2}\right)\left[\dfrac{\pi}{2} - \dfrac{1}{8}\arctan\varphi_0\right] \geq 0$ and

$$\begin{cases} \omega_{n+1}(\vartheta) = \mathfrak{F}(\vartheta, \omega_n(\vartheta)) + \varphi_0 - \mathfrak{F}(0, \varphi_0) \\ \qquad\qquad + \frac{1}{2}\mathbb{Y}\left(\frac{1}{2}\right)\left[\mathfrak{G}(\vartheta, \omega_n(\vartheta)) - \mathfrak{G}(0, \varphi_0)\right] + \frac{1}{2}\mathbb{Y}\left(\frac{1}{2}\right)\int_0^{\vartheta}\mathfrak{G}(\sigma, \omega_n(\sigma))d\sigma. \end{cases}$$

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