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SEMISIMPLE INJECTIVE COMODULE

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ABSTRACT. In this paper, we define the concept of a semisimple injective comodule. We show that it is a generalization of some known comodules, and we give a useful characterization of this notion. Finally, we obtain a new characterization of simple semiartinian coring using semisimple injective comodules.

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1. Introduction

This paper's objectives are to introduce and investigate the notion of semisimple injective comodule.

All the modules in this study are unitary S-modules, and S is a commutative associative ring with identity. For a right S-module M, we designate the socle of M by $soc(M)$. I. Amin, M. Yousif, and N. Zeyada [1] created *soc*-injective and strongly *soc*-injective modules; is *soc*-N-injective for any two modules M and N if each S-homomorphism $\mathrm{f : soc(N) \longrightarrow M}$ extends to N. If the right S-module S_S is *soc*-injective, a ring S is said to be right (self-) *soc*-injective. If M is *soc*-N-injective for every module N, then M is strongly *soc*-injective.

A *coalgebra* over a ring S is an S-module T with linear maps

$$\Delta : T \to T \otimes_S T, \quad \varepsilon : T \to S,$$

called *coproduct* and em counit, respectively, with the commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\Delta} & T \to T \otimes_S T \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ T \to T \otimes_S T & \xrightarrow{\Delta} & T \to T \otimes_S T \to T \otimes_S T, \end{array}$$

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$$\begin{array}{ccc} T & \xrightarrow{\Delta} & T \to T \otimes_S T \\ {\scriptstyle \Delta}\downarrow & & \downarrow \\ T \to T \otimes_S T & \longrightarrow & T. \end{array}$$

An S-module M is a *right* T-*comodule* if there exists an S-linear map inducing commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_S C \\ {\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle I\Delta} \\ M \otimes_S C & \xrightarrow{\rho^M I} & M \otimes_S C \otimes_S C, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M \otimes_S C\ . \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle I} \\ & & M \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{\rho^M} & M \to T \otimes_S T \\ {\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle I\ \Delta} \\ M \to T \otimes_S T & \xrightarrow{\rho^M I} & M \to T \otimes_S T \to T \otimes_S T \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho^M} & M \to T \otimes_S T\ . \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle I} \\ & & M \end{array}$$

Given right T-comodules M and N, a T-*comodule morphism* is an S-linear map f : M → N with a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle \rho^M} \\ M \to T \otimes_S T & \xrightarrow{fI} & N \to T \otimes_S T \end{array}$$

.

The category of right T-comodules, which we designate by M^T, is made up of right T-comodules and comodule homomorphisms.

In the additive category of M^T, coproducts and cokernels are present. However, kernels are not always guaranteed to exist. Monomorphisms don't have to be injective maps, though. If every T-monomorphism U → N is a contraction, a right T-comodule N is referred to as semisimple in the M^T, and if all of these monomorphisms are isomorphisms, N is referred to as simple. Give D its own category. A morphism $\mathrm{f} : \mathrm{A} \longrightarrow \mathrm{B}$ in D is referred to as a coretraction if $\mathrm{g} \in \mathrm{Mor}(\mathrm{B}, \mathrm{A})$ exists and $\mathrm{fg} = \mathrm{id_A}$.

Direct sums and cokernels are features of the category $\mathbf{M}^T$, while T is a subgenerator. If T is a flat S-module, $\mathbf{M}^T$ is a Grothendieck category. The forgetful functor $(\hat{a}')R : \mathbf{M}^T \to \mathbf{M}^S$ is directly adjacent to the functor $\hat{\mathrm{a}}' \otimes_{\mathrm{S}} \mathrm{T} : \mathbf{M}_S \to \mathbf{M}^T$. Any monomorphism $f : K \to L$ of S-modules is a monomorphism in $\mathbf{M}^T$ as is $f \otimes I_T : K \otimes_S T \to L \otimes_S T$. The product of the $M_\lambda \otimes_S T$ in $\mathbf{M^T}$ is $(\Pi_\Lambda M_\lambda) \otimes_S T$ for any family of S-modules $\{M_\lambda\}_\Lambda$. Since $\mathbf{M^T}$ is a Grothendieck category if ${}_S T$ is flat, we can define exact sequences in $\mathbf{M^T}$ and describe them. In $\mathbf{M^T}$, hom-tensor relationships. If X is any S-module, then the following

S-linear map exists for any $M \in \mathbf{M}^T$,

$$\varphi : Hom^T(M, X \otimes_S T) \to Hom_S(M, X), f \longmapsto (I_X \otimes \varepsilon) \circ f,$$

is bijective, with inverse map $h \longmapsto (h \otimes I_T) \circ \rho^M$. Also for any $M, N \in \mathbf{M}^T$, the S-linear map

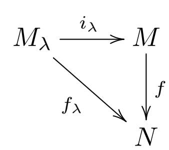
$$\psi : Hom^T(X \otimes_R M, N) \to Hom_S(X, Hom^T(M, N)), g \longmapsto [x \longmapsto g(x \otimes \hat{a}')],$$

is bijective, with inverse map $h \longmapsto [x \otimes m \longmapsto h(x)(m)]$.

Coproducts in $\mathbf{M}^T$. Let $\{M_\lambda, \varrho_\lambda^M\}_\Lambda$ be a family ofT-comodules. Put $M = \bigoplus_\Lambda M_\lambda$, the coproduct in M_S, $i_\lambda : M_\lambda \longrightarrow M$ the canonical inclusions, and consider the linear maps M_λ

$\varrho_\lambda^M : M_\lambda \longrightarrow M_\lambda \otimes_S T \subseteq M \otimes_S T$

Note that $M_\lambda \otimes_S T \subseteq M \otimes_S T$ is a pure submodule because the inclusions i_λ are S-splitting. The properties of coproducts of S-modules lead to a special coaction, $\varrho^M : M \longrightarrow M \otimes_S T$,, which is coassociative and counital since all the M are, and which transforms M into a T-comodule for which the $i_\lambda : M_\lambda \longrightarrow M$ are T-morphisms with the following universal property:
Let $\{f_\lambda : M_\lambda \longrightarrow N\}_\Lambda$ be a family of morphisms in M^T. For each of the distinct T-morphisms $f : M \longrightarrow N$ that exist, the following diagram of C-morphisms commutes:



The direct limit of direct families of T-comodules is generated from the direct limit in MR, just like the direct limit of the coproduct. Both constructions are particular examples of a general remark made in 38.25 regarding the colimits of F-coalgebras.

For all of the undefined terms used in this study, we mention [2], [3], and [8].

2. Main results

For any right S-module L define the map

$$\alpha_{L,K} : N \otimes K \longrightarrow Hom_{\mathbb{Z}}(K^*, L), n \otimes k \longmapsto [f \longmapsto nf(k)]$$

K satisfies the α-condition provided $\alpha_{T,K}$ is injective for all right S-modules N. M satisfies the α-condition if and only if M is locally projective in $\mathbf{M}^S$. If a coring T satisfies the left α- condition, then $\mathbf{M}^T$ is an abelian category. In the following we may consider T satisfies the left α- condition.

Definition 2.1. (1) A morphism $f : A \to B$ in $\mathbf{M}^T$ is called a monomorphism if for $g, h \in Mor_T(D, A)$, $fg = fh$ implies $g = h$.

(2) A morphism $f : A \to B$ in $\mathbf{M}^T$ is called a coretraction if there exists $g \in Mor_T(B, A)$ with $gf = I_A$.

(3) An T-comodule M is called semisimple if any monomorphism $f : N \to M$ is a coretraction.

Proposition 2.2. [3, 4.13(2)] *Let C be a left S-module. Then for $L \in \mathbf{M}^T$ the following are equivalent:*

(1) *L is semisimple;*
(2) *every subcomodule of L is a direct summand;*
(3) *L is a sum of simple subcomodules;*
(4) *L is a direct sum of simple subcomodules.*

Lemma 2.3. *Let L be a T-comodule. If L is simple R-module, then L is simple T-comodule.*

Proof. Suppose that L has a subcomodule K, so K is a submodule of L and K=0 or $K = L$. Therefore L has no subcomodules and L is simple comodule. □

Let M_i be a T-comodule for all $i \in I$. If M_i is simple S-module for every $i \in I$, then $M = \bigoplus_{i\in I} M_i$ is semisimple S-module. By above Lemma M is semisimple T-comodule.

Lemma 2.4. *If F_S is a free module and $_SN$ is a semsimple module then F_SN is semsimple F-comodule.*

Proof. For a ring S with unit we have $RN = N$ and $\mu_S : R_SN \mapsto N$ is an S-isomorphism. Since the tensor product commutes with direct sums (see 12.4), we obtain, for every free right S-module $F_R \cong R_R^{(\wedge)}$, $\wedge$ an index set, a Z-isomorphism $F_SN \cong N^{(\wedge)}$.

□

Definition 2.5. An object $M \in \mathbf{M}^T$ is relative semisimple injective in $\mathbf{M}^T$ if, for any T-comodule morphism $i : N \to L$ (with N is a semisimple in $\mathbf{M}^T$) that is an S-module coretraction, and for every morphism $f : N \to M$ in $\mathbf{M}^T$, there exists a right T-comodule morphism $g : L \to M$ such that $g \circ i = f$.

Proposition 2.6. *Let Q be a semisimple right T-comodule. Then the following are equivalent,*

(1) *Q is relative semisimple injective;*
(2) *every T-comodule map $i : Q \to L$ that is a coretraction in $\mathbf{M}_S$ is a coretraction in $\mathbf{M}^T$.*
(3) *the coaction $\varrho : Q \to Q \otimes_S T$ is a coretraction in $\mathbf{M}^T$*

Proof. (1)⇒(2) If Q is relative semisimple injective, take $N = Q$ and $f = I_Q$ to obtain the assertion.
(2)⇒ (3) Since $\varrho^Q : Q \to Q \otimes_S T$ is a right T-comodule map that is a coretraction in $\mathbf{M}_S$, so its a coretraction $\mathbf{M}^T$.
(3)→(1) See [3, 3.18]. □

Definition 2.7. An object $Q \in \mathbf{M}^T$ is semisimple injective in $\mathbf{M}^T$ if, for any monomorphism $M \to N$ in $\mathbf{M}^T$ where M is a semisimple comodule, the canonical map $Hom^T(N,Q) \to Hom^T(M,Q)$ is surjective.

Example 2.8. Any injective comodule is a semisimple injective comodule.

Proposition 2.9. *Let N be a T-comodule and $\{L_i : i \in I\}$ a family of T-comodules. Then the product $\Pi_{i\in I} L_i$ is semisimple N-injective if and only if L_i is semisimple N-injective $\forall i \in I$.*

Proof. Routine.

□

Proposition 2.10. *Let L be a comodule.*

1. *Let $\{L_i : i \in I\}$ a family of comodules in $\mathbf{M}^T$. Then the direct product $\prod_{i\in I} L_i$ is semisimple injective if and only if L_i is semisimple injective for all $i \in I$.*
2. *Let T and N be two comodules in $\mathbf{M}^T$ with $T \cong N$. If T is semisimple injective, then N is semisimple injective.*
3. *Let T and N be two comodules in $\mathbf{M}^T$ with N is a direct summand of T. If T is semisimple injective, then N is semisimple injective.*

Theorem 2.11. *For a left S-module T, the following statements are true:*

1. *If every semisimple T-comodule is a semisimple module and $X \in \mathbf{M}_S$ is strongly soc-injective in $\mathbf{M}_S$, then $X \otimes_S T$ is semisimple injective in $\mathbf{M}^T$.*
2. *If ${}_SF$ is free left S-module and N is semisimple injective in $\mathbf{M}^T$, then $Hom^T(F,N)$ is soc-injective in $\mathbf{M}_S$.*

Proof. (1) From hom-tensor relations in M^T we have $\varphi1$, and $\varphi2$ isomorphisms . There exist α surjective homomorphism from X is strongly *soc*-injective in $\mathbf{M}_S$ and we can define f as a surjective homomorphism by using $\varphi1$, $\varphi2$, and α.

$$\begin{array}{ccc} Hom^T(N, X\otimes_S T) & \xrightarrow[surjective]{f} & Hom^T(M, X\otimes_S T) \\ \varphi1 \downarrow \cong & & \cong \downarrow \varphi2 \\ Hom_S(N,X) & \xrightarrow[surjective]{\alpha} & Hom_S(M,X) \end{array}$$

(2) From hom-tensor relations in M^T we have $\varphi1$, and $\varphi2$ isomorphisms . There exist α surjective homomorphism from N is semisimple injective in $\mathbf{M}^T$ and we can defined f is surjective homomorphism by using $\varphi1$, $\varphi2$, and α.

$$\begin{array}{ccc} Hom_S(L, hom^T(L,N)) & \xrightarrow[surjective]{f} & Hom_S(K, hom^T(L,N)) \\ \cong \uparrow \varphi1 & & \varphi2 \uparrow \cong \\ Hom^T(L\otimes_S L, N) & \xrightarrow[surjective]{\alpha} & Hom^T(K\otimes_S L, N) \end{array}$$

□

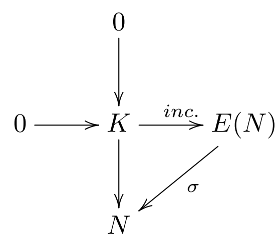
Note that T satisfies the left α- condition, so the category M^T is a Grothendieck category and every comodule in M^C has an injective hull.

Proposition 2.12. *Let N be a comodule in $\mathbf{M}^T$. The following are equivalent:*

(1) *N is semisimple injective comodule.*
(2) *N is semisimple $E(N)$- injective comodule.*
(3) *$N = E \oplus T$, where E is an injective comodule and T has no semisimple subcomodules.*

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (3): If N has no semisimple subcomodule, we are done. Assume that N has a nonzero semisimple subcomodule and K is the sum of all simple subcomodule of $E(N)$. Consider the following diagram:



where K is the sum of all simple subcomodule in $E(N)$ and i is the inclusion map. Since N is semisimple injective. So, there exists an T-morphism $\sigma : E(K) \to N$; which extends i. Since $E(K)$ is the injective envelope of K, σ is an embedding of $E(K)$ in N. If we write $E = \sigma(E(K))$, then $N = E \oplus T$ for some submodule T of M. Clearly, E is injective and T has no nonzero semisimple subcomodule.

(3) $\Rightarrow$ (1): This is clear since comodules have no semisimple subcomodule are semisimple injective and finite direct sums of semisimple injective comodules are semisimple injective. □

Definition 2.13. A coring is said to be simple if it does not contain non-trivial sub-bicomodules.

Definition 2.14. The coring T is said to be left semiartinian if it is semiartinian as an object in ${}^T M$, namely, every factor comodule of T contains a (nonzero) simple sub Comodule.

Theorem 2.15. *Let T be an S-coring where ${}_S T$ and T_R are projective modules. The statements that follow are equivalent:*

(1) *T is a simple left semi-artinian coring;*
(2) *T is a simple coring and has a simple left subcomodule;*
(3) *T is simple and each semisimple injective comodule in T is injective.*

Proof. (1) $\Rightarrow$ (2). Clear.

(2) $\Rightarrow$ (3). Let Q be an injective comodule that is semisimple. Q is then equal to $E \bigoplus T$ according to Proposition 2.12, where E is an injective comodule and T lacks a simple subcomodule. Therefore, assuming Q is an injective comodule and $T = 0$.

(3) $\Rightarrow$ (1). Assume that M has no simple subcomdules and N is a submodule of M, so $N \bigoplus M$ is injective comodule. Then N is injective comodule and its a summand of M. Thus M is a semisimple which is a contradiction.

□

Conflicts of interest The authors declare no conflict of interest.

Data availability Not applicable

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