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FULLY MODIFIED (p, q)-POLY-TANGENT POLYNOMIALS WITH TWO VARIABLES†

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Abstract. In this paper, we introduce a fully modified (p, q)-poly tangent polynomials and numbers of the first type. We investigate analytic properties that is related with (p, q)-Gaussian binomial coefficients. We also define (p, q)-Stirling numbers of the second kind and fully modified (p, q)-poly tangent polynomials and numbers of the first type with two variables. Moreover, we derive some identities are concerned with the modified tangent polynomials and the (p, q)-Stirling numbers.

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1. Introduction

These days, many researchers are interested in the applications of q-numbers and (p, q)-numbers. In areas of quantum mechanics, physics and mathematics, the applying theory is studied and extended actively. Especially, mathematicians in the fields of combinatorics, number theory and special functions, frequently explorer that(cf [4,5,6,8,9,10,11]). We also investigate the generalization of poly Bernoulli polynomials and poly tangent polynomials involving (p, q)-numbers.

Throughout this paper, we use the following notations. $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of complex numbers, respectively.

The (p, q)-numbers are known by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q},$$

where $0 < q < p \leq 1$. When $p = 1$, we have $[n]_{p,q} = [n]_q$ and $\lim_{q \to 1}[n]_q = n$.

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The (p,q)-factorial of n of order k is defined as

$$[n]_{p,q}^{(\underline{k})} = [n]_{p,q}[n-1]_{p,q}\cdots[n-k+1]_{p,q},$$

for $k = 1,2,3,\cdots$. If $k = n$, it is denoted $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}$ that is called (p,q)-factorial of n. The (p,q)-Gaussian binomial formula is defined by

$$(x+a)_{p,q}^{n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} a^{n-k} x^{k},$$

with the (p,q)-Gaussian binomial coefficient, $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$ $(n \geq k)$.

In [12], two type of the (p,q)-exponential functions are given as below

$$\begin{aligned} \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} &= e_{p,q}(x), \\ \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} &= E_{p,q}(x). \end{aligned} \tag{1.1}$$

The (p,q)-analogue of polylogarithm function $Li_{k,p,q}$ is known by

$$Li_{k,p,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^{k}}, (k \in \mathbb{Z})(cf[6,11]).$$

In [5], we defined fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type.

Definition 1.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, q \in \mathbb{R}$ and $0 < q < 1$, fully modified q-poly-Bernoulli polynomials $\widetilde{B}_{n,q}^{(k)}(x)$ of the first type and the fully modified q-poly-tangent polynomials $\widetilde{T}_{n,q}^{(k)}(x)$ of the first type are defined by

$$\begin{aligned} \frac{Li_{k,q}(1-e_q(-t))}{(e_q(t)-1)} e_q(xt) &= \sum_{n=0}^{\infty} \widetilde{B}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}, \\ \frac{[2]_q Li_{k,q}(1-e_q(-t))}{t(e_q(2t)+1)} e_q(xt) &= \sum_{n=0}^{\infty} \widetilde{T}_{n,q}^{(k)}(x) \frac{t^n}{[n]_q!}. \end{aligned} \tag{1.2}$$

When $x = 0$, $\widetilde{B}_{n,q}^{(k)} = \widetilde{B}_{n,q}^{(k)}(0)$, $\widetilde{T}_{n,q}^{(k)} = \widetilde{T}_{n,q}^{(k)}(0)$ are called fully modified q-poly-Bernoulli numbers of the first type and fully modified q-poly-tangent numbers of the first type. If $q \to 1$ in (1.2), we have the poly-Bernoulli polynomials $B_n^{(k)}(x)$ and the poly-tangent polynomials $T_n^{(k)}(x)$, respectively.

Substitute $k = 1, q \to 1$ in (1.2), we have the ordinary Bernoulli polynomials $B_n(x)$and the ordinary tangent polynomials $T_n(x)$, respectively.

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}, \qquad \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{xt}.$$

Now, we define a special extension of poly-Bernoulli polynomials from (p,q)-analogue of exponential functions and polylogarithmn function.

Definition 1.2. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$. We define fully modified (p,q)-poly-Bernoulli polynomials $\widetilde{B}^{(k)}_{n,q}(x)$ of the first type by

$$\sum_{n=0}^{\infty} \widetilde{B}^{(k)}_{n,p,q}(x)\frac{t^n}{n!} = \frac{Li_{k,p,q}(1 - e_{p,q}(-t))}{(e_{p,q}(t) - 1)} e_{p,q}(xt). \tag{1.3}$$

When $x = 0$, $\widetilde{B}^{(k)}_{n,p,q} = \widetilde{B}^{(k)}_{n,p,q}(0)$ are called fully modified (p,q)-poly-Bernoulli numbers of the first type. If $p = 1$ in (1.3), we get $\widetilde{B}^{(k)}_{n,p,q}(x) = \widetilde{B}^{(k)}_{n,q}(x)$.

Definition 1.3. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, 0 < q < p \leq 1$, the (p,q)-Bernoulli polynomials of order l are defined by

$$\sum_{n=0}^{\infty} B^{\langle l \rangle}_{n,p,q}(x)\frac{t^n}{n!} = \left(\frac{t}{e_{p,q}(t) - 1}\right)^l e_{p,q}(xt). \tag{1.4}$$

In this paper, we define fully modified (p,q)-poly-Bernoulli polynomials of the first type and fully modified (p,q)-poly-tangent polynomials of the first type. Furthermore, we derive a connection of the two polynomials and investigate some identities that are concerned with (p,q)-Gaussian binomials coefficients. We also construct fully modified (p,q)-poly-tangent polynomials of the first type with two variables using (p,q)-exponential function and observe recurrence formula that are related with (p,q)-Stirling numbers.

2. Some identities of fully modified (p,q)-poly-tangent polynomials of the first type

In this section, we introduce a fully modified (p,q)-poly-tangent numbers $T^{(k)}_{n,p,q}$ and polynomials $T^{(k)}_{n,p,q}(x)$ of the first type by the generating functions. We explore some identities of the polynomials and find a relation connected with (p,q)-analogue of the ordinary tangent polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+, p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, we define (p,q)-tangent polynomials $T_{n,p,q}(x)$ of the first type by

$$\sum_{n=0}^{\infty} T_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(xt).$$

Definition 2.2. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}, p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, we define a fully modified (p,q)-poly-tangent polynomials $\widetilde{T}^{(k)}_{n,p,q}(x)$ of the first type by

$$\sum_{n=0}^{\infty} \widetilde{T}^{(k)}_{n,p,q}(x)\frac{t^n}{n!} = \frac{[2]_{p,q} Li_{k,p,q}(1 - e_{p,q}(-t))}{t(e_{p,q}(2t) + 1)} e_{p,q}(xt). \tag{2.1}$$

When $x=0$, $\widetilde{T}^{(k)}_{n,p,q}=\widetilde{T}^{(k)}_{n,p,q}(0)$ are called fully modified (p,q)-poly-tangent numbers of the first type. Note that $p=1, [n]_{p,q}=[n]_q$, and $\widetilde{T}^{(k)}_{n,p,q}(x)=\widetilde{T}^{(k)}_{n,q}(x)$.

Theorem 2.3. *For $n\in\mathbb{Z}_+, k\in\mathbb{Z}$ and $p,q\in\mathbb{R}$ such that $0<q<p\leq 1$, the following result holds*

$$\widetilde{T}^{(k)}_{n,p,q}(x)=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}p^{\binom{n-l}{2}}\,\widetilde{T}^{(k)}_{l,p,q}\,x^{n-l}.$$

Using the definition of (p,q)-exponential function in (1.1), we derive the next theorem. The result shows that fully modified (p,q)-poly-tangent polynomials of the first type are connected with the fully modified (p,q)-poly-Bernoulli numbers and the (p,q)-analogue of the ordinary tangent polynomials.

Theorem 2.4. *For n be a nonnegative integer, $k\in\mathbb{Z}$ and $0<q<p\leq 1$, we get*

$$\widetilde{T}^{(k)}_{n,p,q}(x)=\sum_{l=0}^{n}\frac{\begin{bmatrix}n\\a\end{bmatrix}_{p,q}\begin{bmatrix}a\\l\end{bmatrix}_{p,q}}{[n-a+1]_{p,q}}\,p^{\binom{n-a+1}{2}}\,\widetilde{B}^{(k)}_{l,p,q}\,T_{a-l,p,q}(x).$$

Proof. Let $0<q<p\leq 1$. From the definitions of (p,q)-polylogarithm functon and (p,q)-exponential function, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{T}^{(k)}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}&=\frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(t)+1)}e_{p,q}(xt)\\&=\sum_{n=0}^{\infty}\widetilde{B}^{(k)}_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}T_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\frac{p^{\binom{n+1}{2}}}{[n+1]_{p,q}}\frac{t^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=0}^{a}\begin{bmatrix}n\\a\end{bmatrix}_{p,q}\begin{bmatrix}a\\l\end{bmatrix}_{p,q}\frac{p^{\binom{n-a+1}{2}}}{[n-a+1]_{p,q}}\widetilde{B}^{(k)}_{l,p,q}\,T_{a-l,p,q}(x)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

□

By definition of (p,q)-exponential function, (2.1) is expressed with (p,q)-Gaussian binomial coefficient and (p,q)-analogue of ordinary tangent polynomials.

Theorem 2.5. *Let $n\in\mathbb{N}, k\in\mathbb{Z}$ and $p,q\in\mathbb{R}$ such that $0<q<p\leq 1$. Then we have*

$$\begin{aligned}&[n]_{p,q}\widetilde{T}^{(k)}_{n-1,p,q}(x)\\&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{a=0}^{n}\sum_{\substack{n_1+\cdots+n_{i-1}=n\\n_1,\cdots,n_{i-1}\geq 0}}^{a}\begin{bmatrix}n\\a,n_1,\cdots,n_{i-1}\end{bmatrix}_{p,q}\frac{\binom{l+1}{i}}{[l+1]^k_{p,q}}(-1)^{a+i}\\&\times p^{\binom{a-n_1}{2}+\binom{n_1-n_2}{2}+\cdots+\binom{n_{i-2}-n_{i-1}}{2}}T_{n-a,p,q}(x).\end{aligned}$$

Proof. For $n \in \mathbb{N}, k \in \mathbb{Z}$ and $p, q \in \mathbb{R}$ such that $0 < q < p \leq 1$, using (1.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x)\frac{t^n}{[n]_{p,q}!} \\
&\quad= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt) \\
&\quad= \sum_{l=0}^{\infty}\frac{(1-e_{p,q}(-t))^{l+1}}{[l+1]_{p,q}^k}\frac{[2]_{p,q}}{t(e_{p,q}(2t)+1)}e_{p,q}(xt) \\
&\quad= \frac{1}{t}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\frac{(-1)^i\binom{l+1}{i}}{[l+1]_{p,q}^k}\sum_{n=0}^{\infty}T_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} \\
&\qquad\quad \times \sum_{n=0}^{\infty}\sum_{\substack{n_1+\cdots+n_{i-1}=n \\ n_1,\cdots,n_{i-1}\geq 0}}^{n}\begin{bmatrix} n \\ n_1,\cdots,n_{i-1}\end{bmatrix}_{p,q}(-1)^n \\
&\qquad\quad \times p^{\binom{n-n_1}{2}+\binom{n_1-n_2}{2}+\cdots+\binom{n_{i-2}-n_{i-1}}{2}}\frac{t^n}{[n]_{p,q}!} \\
&\quad= \frac{1}{t}\sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{\substack{n_1+\cdots+n_{i-1}=a \\ n_1,\cdots,n_{i-1}\geq 0}}^{a}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\begin{bmatrix} n \\ a,n_1,\cdots,n_{i-1}\end{bmatrix}_{p,q}\frac{\binom{l+1}{i}}{[l+1]_{p,q}^k} \\
&\qquad\quad \times (-1)^{a+i}p^{\binom{a-n_1}{2}+\binom{n_1-n_2}{2}+\cdots+\binom{n_{i-2}-n_{i-1}}{2}}T_{n-a,p,q}(x)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}[n]_{p,q}\widetilde{T}_{n-1,p,q}^{(k)}(x)\frac{t^n}{[n]_{p,q}!} \\
&\quad= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{\substack{n_1+\cdots+n_{i-1}=a \\ n_1,\cdots,n_{i-1}\geq 0}}^{a}\begin{bmatrix} n \\ a,n_1,\cdots,n_{i-1}\end{bmatrix}_{p,q}\frac{\binom{l+1}{i}}{[l+1]_{p,q}^k} \\
&\qquad \times (-1)^{a+i}p^{\binom{a-n_1}{2}+\binom{n_1-n_2}{2}+\cdots+\binom{n_{i-2}-n_{i-1}}{2}}T_{n-a,p,q}(x)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

□

The above result can be expressed with multivariate analogue of (p,q)-Rogers-Szegö polynomials. Moreover, the equality leads to a result that is represented with (p,q)-Hermite polynomials.

3. Properties of the fully modified (p, q)-poly-tangent polynomials of the first type with two variables

In this section, we introduce fully modified (p, q)-poly-tangent polynomials of the first type with two variables. We also consider special generating function of (p, q)-Stirling numbers and investigate some properties that is concerned with the generating function.

In [4], the authors introduced (q, r, w)-Stirling numbers. Naturally, the generating series of (p, q)-Stirling numbers of the second kind is defined by

$$\sum_{n=m}^{\infty} S_{p,q,r,w}(n+r+w, m+r+w)\frac{t^n}{[n]_{p,q}!} = \frac{(e_{p,q}(t)-1)^m}{[m]_{p,q}!}e_{p,q}(rt)E_{p,q}(wt),$$

where $n, m \in \mathbb{Z}_+$ with $0 \le m \le n$.
If $r = w = 0$, then we get the following generating function.

$$\frac{(e_{p,q}(t)-1)^m}{[m]_{p,q}!} = \sum_{n=m}^{\infty} S_{p,q}(n, m)\frac{t^n}{[n]_{p,q}!}. \tag{3.1}$$

For $0 \le n, m \le 5$, a few values of the (p, q)-Stirling numbers of the second kind are given as below.

TABLE 1. $S_{p,q}(n, m)$

n \ m	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	p	1	0	0
3	0	p^3	$\frac{2p(p^2+pq+q^2)}{p+q}$	1	0
4	0	p^6	$\frac{p^2(p^2+q^2)(3p^2+3pq+q^2)}{p+q}$	$3p(p^2+q^2)$	1

It makes that the (p, q)-polylogarithm function $Li_{k,p,q}$ is represented with the (p, q)-Stirling numbers of the second kind.

$$\frac{Li_{k,p,q}(1-e_{p,q}(-t))}{t} = \sum_{n=0}^{\infty}\sum_{l=1}^{n+1} \frac{[l]_{p,q}!}{[l]_{p,q}^k[n+1]_{p,q}} (-1)^{l+n+1}S_{p,q}(n+1, l)\frac{t^n}{[n]_{p,q}!} \tag{3.2}$$

Using the identity in (3.2), we derive the following result which is connected with (p, q)-Stirling numbers of the second kind and (p, q)-analogue of the ordinary tangent polynomials.

Theorem 3.1. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $p,q \in \mathbb{R}$ such that $0 < q < p \leq 1$, the following identity holds*

$$\widetilde{T}^{(k)}_{n,p,q}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[l]^k_{p,q}[a+1]_{p,q}}(-1)^{l+a+1}S_{p,q}(a+1,l)T_{n-a,p,q}(x).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$. By the recomposition of (p,q)-polylogarithm function in (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{T}^{(k)}_{n,p,q}(x)\frac{t^n}{n!} &= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt) \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+l}[l]_{p,q}!}{[l]^k_{p,q}[n+1]_{p,q}}S_{p,q}(n+1,l)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}T_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{l=1}^{a+1}\begin{bmatrix} n \\ a \end{bmatrix}_{p,q}\frac{(-1)^{l+a+1}[l]_{p,q}!}{[l]^k_{p,q}[a+1]_{p,q}}S_{p,q}(a+1,l)T_{n-a,p,q}(x)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Comparing the coefficient both sides, we get

$$\widetilde{T}^{(k)}_{n,p,q}(x) = \sum_{a=0}^{n}\sum_{l=1}^{a+1} \begin{bmatrix} n \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[l]^k_{p,q}[a+1]_{p,q}}(-1)^{l+a+1}S_{p,q}(a+1,l)T_{n-a,p,q}(x).$$

□

Furthermore, (3.2) leads the following identity that are constructed with (p,q)-Stirling numbers and (p,q)-Bernoulli polynomials of order l.

Theorem 3.2. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$. Then we get*

$$\widetilde{T}^{(k)}_{n,p,q}(x) = \sum_{a=0}^{n}\sum_{i=0}^{a}\frac{\begin{bmatrix} n \\ a \end{bmatrix}_{p,q}\begin{bmatrix} a \\ i \end{bmatrix}_{p,q}}{\begin{bmatrix} i+l \\ l \end{bmatrix}_{p,q}}S_{p,q}(i+l,l)B^{\langle l\rangle}_{i,p,q}(x)\widetilde{T}^{(k)}_{n-a,p,q}.$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$, fully modified (p,q)-poly-tangent polynomials can be indicated by formula that include the fully modified (p,q)-poly-tangent numbers, the (p,q)-Stirling numbers and the (p,q)-poly-Bernoulli polynomials of order l.

$$\begin{aligned}
&\frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt)\\
&\quad=\sum_{n=0}^{\infty}\widetilde{T}_{n,p,q}^{(k)}\frac{t^n}{[n]_{p,q}!}\sum_{n=l}^{\infty}[l]_{p,q}!S_{p,q}(n,l)\frac{t^{n-l}}{[n]_{p,q}!}\sum_{n=0}^{\infty}B_{n,p,q}^{\langle l\rangle}(x)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{n=0}^{\infty}\widetilde{T}_{n,p,q}^{(k)}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\frac{S_{p,q}(n+l,l)}{\begin{bmatrix}n+l\\ l\end{bmatrix}_{p,q}}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}B_{n,p,q}^{\langle l\rangle}(x)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{n=0}^{\infty}\sum_{a=0}^{n}\sum_{i=0}^{a}\begin{bmatrix}n\\ a\end{bmatrix}_{p,q}\begin{bmatrix}a\\ i\end{bmatrix}_{p,q}\frac{S_{p,q}(i+l,l)}{\begin{bmatrix}i+l\\ l\end{bmatrix}_{p,q}}\,B_{i,p,q}^{\langle l\rangle}(x)\,\widetilde{T}_{n-a,p,q}^{(k)}\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficient both sides, we get

$$\widetilde{T}_{n,p,q}^{(k)}(x)=\sum_{a=0}^{n}\sum_{i=0}^{a}\frac{\begin{bmatrix}n\\ a\end{bmatrix}_{p,q}\begin{bmatrix}a\\ i\end{bmatrix}_{p,q}}{\begin{bmatrix}i+l\\ l\end{bmatrix}_{p,q}}S_{p,q}(i+l,l)B_{i,p,q}^{\langle l\rangle}(x)\widetilde{T}_{n-a,p,q}^{(k)}.$$

□

Now, we introduce fully modified (p,q)-poly-tangent of the first type with two variables by using two generating functions in (1.1).

Definition 3.3. For $n\in\mathbb{Z}_+, k\in\mathbb{Z}$ and $0<q<p\leq 1$, the fully modified (p,q)-poly-tangent polynomials $\widetilde{T}_{n,p,q}^{(k)}(x,y)$ of the first type with two variables by

$$\sum_{n=0}^{\infty}\widetilde{T}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!}=\frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt)E_{p,q}(yt). \tag{3.2}$$

In next result, we show the relation between fully modified (p,q)-poly-tangent polynomials of the first kind and (3.2). Also in theorem 3.5, it is continued the recurrence formula from two polynomials.

Theorem 3.4. *For $n\in\mathbb{Z}_+, k\in\mathbb{Z}$, we get the addition theorem.*

$$\widetilde{T}_{n,p,q}^{(k)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_{p,q}\widetilde{T}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

Proof. Let $k \in \mathbb{Z}$ and n be a nonnegative integer. Then we get

$$\sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt)E_{p,q}(yt)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\widetilde{T}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}\right)\frac{t^n}{[n]_{p,q}!}.$$

Thus, we have

$$\widetilde{T}_{n,p,q}^{(k)}(x,y) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\widetilde{T}_{l,p,q}^{(k)}(x)q^{\binom{n-l}{2}}y^{n-l}.$$

□

Theorem 3.5. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $0 < q < p \leq 1$. We get*

$$\widetilde{T}_{n,p,q}^{(k)}(x,y) - \widetilde{T}_{n,p,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q} q^{\binom{n-l}{2}}\,\widetilde{T}_{l,p,q}^{(k)}(x)\,y^{n-1}. \tag{3.3}$$

Proof. For $n \in \mathbb{N}, k \in \mathbb{Z}, 0 < q < p \leq 1$, we have

$$\sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x,y)\frac{t^n}{[n]_{p,q}!} - \sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x)\frac{t^n}{[n]_{p,q}!}$$
$$= \frac{[2]_{p,q}Li_{k,p,q}(1-e_{p,q}(-t))}{t(e_{p,q}(2t)+1)}e_{p,q}(xt)(E_{p,q}(yt)-1)$$
$$= \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q} q^{\binom{n-l}{2}}\,\widetilde{T}_{l,p,q}^{(k)}(x)y^{n-1}\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}}$, the proof is complete. □

Substitute $y = 1$, we get

$$\widetilde{T}_{n,p,q}^{(k)}(x,1) - \widetilde{T}_{n,p,q}^{(k)}(x) = \sum_{l=0}^{n-1}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q} q^{\binom{n-l}{2}}\,\widetilde{T}_{l,p,q}^{(k)}(x).$$

Also, the above result is appeared with (p,q)-Stirling numbers and (p,q)-tangent polynomials as below.

Theorem 3.6. *Let $n \in \mathbb{N}, k \in \mathbb{Z}$ and $0 < q < p \leq 1$. We have*

$$\widetilde{T}_{n,p,q}^{(k)}(x,y) - \widetilde{T}_{n,p,q}^{(k)}(x) = \sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1}\begin{bmatrix} n \\ r+1 \end{bmatrix}_{p,q}\begin{bmatrix} r+1 \\ a \end{bmatrix}_{p,q}\frac{[l]_{p,q}!}{[a+1]_{p,q}[l]_{p,q}^k}$$
$$\times (-1)^{l+a+1}q^{\binom{a+1}{2}}y^{r-a+1}S_{2,p,q}(a+1,l)T_{n-r-1,p,q}(x).$$

Proof. Let $n \in \mathbb{N}, k \in \mathbb{Z}, 0 < q < p \leq 1$. Using the definition of the (p,q)-poly-Bernoulli polynomials, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} \widetilde{T}_{n,p,q}^{(k)}(x)\frac{t^n}{n!} \\
&\quad = \frac{[2]_{p,q} Li_{k,p,q}(1-e^{-t})}{t(e_{p,q}(2t)+1)} e_{p,q}(xt)\left(E_{p,q}(yt)-1\right) \\
&\quad = \sum_{n=0}^{\infty}\sum_{l=1}^{n+1} \frac{(-1)^{n+l+1}[l]_{p,q}!}{[n+1]_{p,q}[l]_{p,q}^k} S_{2,p,q}(n+1,l)\frac{t^n}{[n]_{p,q}!} \\
&\qquad\qquad \times \sum_{n=0}^{\infty} T_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} y^{n+1}\frac{t^{n+1}}{[n+1]_{p,q}!} \\
&\quad = \sum_{n=1}^{\infty}\sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}_{p,q} \begin{bmatrix} r+1 \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[a+1]_{p,q}[l]_{p,q}^k} \\
&\qquad\qquad \times (-1)^{l+a+1} q^{\binom{a+1}{2}} y^{r-a+1} S_{2,p,q}(a+1,l) T_{n-r-1,p,q}(x)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Hence, the recurrence formula is indicated by

$$\begin{aligned}
&\widetilde{T}_{n,p,q}^{(k)}(x,y) - \widetilde{T}_{n,p,q}^{(k)}(x) \\
&\quad = \sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}_{p,q} \begin{bmatrix} r+1 \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[a+1]_{p,q}[l]_{p,q}^k} \\
&\qquad\qquad \times (-1)^{l+a+1} q^{\binom{a+1}{2}} y^{r-a+1} S_{2,p,q}(a+1,l) T_{n-r-1,p,q}(x).
\end{aligned}$$

□

In the case $y = 1$, we have

$$\begin{aligned}
\widetilde{T}_{n,p,q}^{(k)}(x,1) - \widetilde{T}_{n,p,q}^{(k)}(x) &= \sum_{r=0}^{n-1}\sum_{a=0}^{r}\sum_{l=0}^{a+1} \begin{bmatrix} n \\ r+1 \end{bmatrix}_{p,q} \begin{bmatrix} r+1 \\ a \end{bmatrix}_{p,q} \frac{[l]_{p,q}!}{[a+1]_{p,q}[l]_{p,q}^k} \\
&\qquad \times (-1)^{l+a+1} q^{\binom{a+1}{2}} S_{2,p,q}(a+1,l) T_{n-r-1,p,q}(x).
\end{aligned}$$

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