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SECURE DOMINATION PARAMETERS OF HALIN GRAPH WITH PERFECT K-ARY TREE

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ABSTRACT. Let G be a simple undirected graph. A planar graph known as a Halin graph(HG) is characterised by having three connected and pendent vertices of a tree that are connected by an outer cycle. A subset S of V is said to be a dominating set of the graph G if each vertex u that is part of V is dominated by at least one element v that is a part of S. The domination number of a graph is denoted by the $\gamma(G)$, and it corresponds to the minimum size of a dominating set. A dominating set S is called a secure dominating set if for each $v \in V \backslash S$ there exists $u \in S$ such that v is adjacent to u and $S_1 = (S\backslash\{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of a secure dominating set of G is equal to the secure domination number $\gamma_s(G)$. In this article we found the secure domination number of Halin graph(HG) with perfet k-ary tree and also we determined secure domination of rooted product of special trees.

AMS Mathematics Subject Classification : 05C05, 05C17, 05C38, 05C69, 05C76.
Key words and phrases : Tree, Halin graph, perfect k-ary tree, dominating set, secure dominating set.

1. Introduction

Let $G = (V, E)$ be an undirected simple graph, where V be a vertex set and which are connected by edge set E. A graph is said to be planar if its edges may only intersect at the vertices of other graphs. The number of the edges that are incident in the vertices of a graph is referred to as the vertex's degree. A regular graph having every degree of vertices are same. The connected graph and acyclic graph is called tree. In a graph G, the subgraph of G that is induced by all vertices adjacent to vertex V is referred to as the neighbourhood of vertex V. If tree is called Homeomorphically irreducible vertex of degree two. The halin graph $HG = T \cup C$ is planar graph contains homeomorphically irreducible

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tree and pendent vertices are joined by an outer cycle. The Halin graph having atleast four vertices. For the purpose of researching graphs with a minimum of three connections, Rudolf Halin developed the Halin graph. In 1975, Bondy proved that Halin graph [3] is Hamiltonian. Barefoot [1987] showed that Halin graphs are Hamiltonian connected [4]. It has been shown that many problems that are difficult in NP for generalised graphs may be resolved in polynomial period for Halin graphs. Pully blank Naddef and Cornuejols established that a Hamiltonian cycle may be discovered in polynomial time by using a Halin graph. In this study, we just take into consideration simple graphs. A subset S of V is said to be a dominating set of the graph G if each vertex u that is part of V is dominated by at least one element v that is a part of S. The domination number of a graph is denoted by the $\gamma(G)$, and it corresponds to the minimum size of a dominating set. In the literature, there are many different domintion numbers. For a more detailed analysis of domination theory, the focus on the importance should refer to [5]. In 2004, Cockayne et al. established the notion of a secure dominance number [11]. By their definition, a dominating set S is called a secure dominating set if for each $v \in V \backslash S$ there exists $u \in S$ such that v is adjacent to u and $S_1 = (S \backslash \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of a secure dominating set of G is equal to the secure domination number $\gamma_s(G)$. Secure domination number has been extensively researched in the literature. By extending a simple constructive characterization of γ-excellent trees, Mynhardt et al. in 2005 obtained a constructive characterization of trees with equal domination and secure domination numbers, where a graph G is γ-excellent if and only if every vertex of G is contained in some minimum dominating set of G [8]. Even when limited to bipartite graphs and split graphs, Merouane et al. demonstrated in [15] that the issue of determining the secure domination number is in the NP-complete class. A linear-time technique was put out by Araki et al. in 2018 [2] for determining the secure domination number of appropriate interval graphs. The zero-divisor graph's secure domination number was discovered by Mohamed Ali et al. [14]. We can find more relevant information on secure domination number in [6], [7], [12], [13], [15], [17]. Consider if G is a graph with n vertices and that H is a different graph with v as the root vertex. A k-ary tree with vertices on all internal vertices of the same degree and all branch vertices at the same height [5]. The rooted product of G and H is defined as the graph with one copy of G and n copies of H identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for each $1 \leq i \leq n$ [16]. In this article, we analyse the secure domination number of the Halin graph(HG) with the perfect k-ary tree, and we also found the secure domination number of the rooted product of special trees.

1.1. Formation of Halin Graph (HG). Let T_1 is rooted perfect K-ary tree and P is a path with $V_{(p+2)}$ vertices. The rooted, perfect K-ary trees operate as a branch that extends from the inner vertex of P. The branches expand till stage l. An example of$T_{(l,m)}$ is specified from the below Figure (2). Now, all the pendent vertices of $T_{(l,m)}$ is connected with an exterior cycle, it connect as required of HG[10].

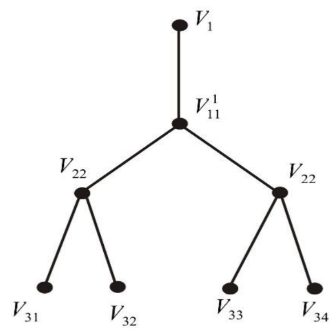

FIGURE 1. Perfect binary tree.

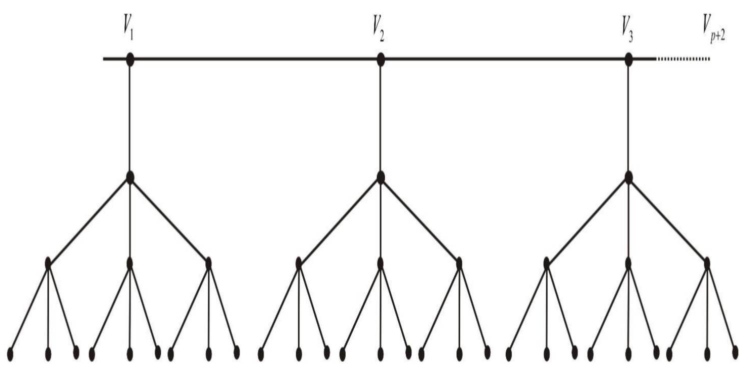

FIGURE 2. Hoeomorphically irreducible tree.

2. Main Results

During this section, we established the secure domination of the halin graph and the Cubic halin graph with the perfect binary as well as k-ary tree.

Proposition 2.1. [11] *Let G be any graph, then*

$$\gamma(G)) \leq \gamma_s(G))$$

Theorem 2.2. [11] *Let P_n be the graph with order n. Then*

$$\gamma_s(P_n) = \lceil \frac{3n}{7} \rceil.$$

Theorem 2.3. *If T_l is perfect K-ary tree, then the secure domination number is*

$$\gamma_s(T_l) = \sum_{m=1}^{\lceil \frac{l-1}{3} \rceil} k^{l-m}$$

Proof. Let we start with $(l-1)$ stage vertices. It is maximum number of sequences of vertices adjacent to private neighbors any two vertices i, j stages where $i \neq j$ will have private neighbors in $i \pm 1$ and $j \pm 1$ stage. Now proceed as follows:

$$\gamma_s(T_l) = k^{(l-1)} + k^{(l-4)} + k^{(l-7)} + k^{(l-10)} + ... + k^{(l-m).}$$

Where l is number of stages and m is any integer. Then we have the secure domination number is

$$\gamma_s(T_l) = \sum_{m=1}^{\lceil \frac{l-1}{3} \rceil} k^{l-m}.$$

□

Theorem 2.4. *If $H_1(l,m)$ is Halin graph with has perfect K-ary tree, then the secure domination number of the Halin graph is*

$$\gamma_s(H_1(l,m)) = \begin{cases} \lceil \frac{P}{2} \rceil + P\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}, & if\ l \equiv\ 0(\text{mod } 2) \\ 1 + P\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}, & if\ l \equiv\ 1(\text{mod } 2) \end{cases}$$

Proof. Let us assume that u is one of the internal vertices of the middle path. Take into consideration the vertices in the l, $l-1$, $l-2$ levels of the tree. Let T_l is adjoining to u. Therefore, there are two components that make up the proof of the theorem.

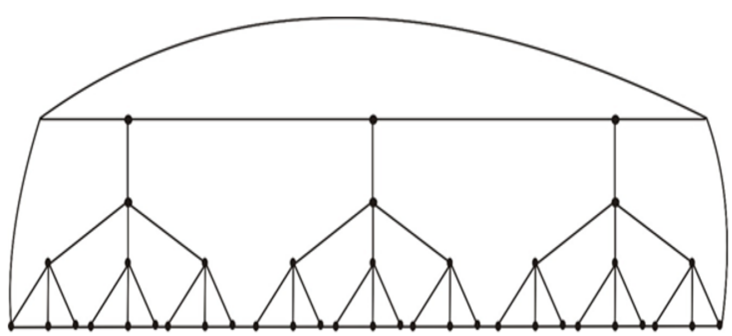

FIGURE 3. Halin graph with perfect 3-ary tree.

Case (i): When $l \equiv\ 0 \pmod 2$ then this type of set of vertices are middle path vertices dominated by themselves.

$$\begin{aligned}|s| &= \left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right) \\ &\quad + \left(k^{(l-1)}k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right) + ... \\ &\quad + \left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right)(p - times). \\ |S| &= P\left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right) \\ \gamma_s(H_1(l,m)) &= \left\lceil \frac{p}{2} \right\rceil + P\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} k^{l-m}, if\ l \equiv\ 0 (\text{mod } 2).\end{aligned}$$

Case (ii): When $l \equiv\ 1 \pmod 2$ there is no need for the vertex on the middle path to be included in the secure dominating set for this form of vertex set.

$$\begin{aligned}|s| &= 1 + \left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right) \\ &\quad + \left(k^{(l-1)}k^{(l-3)} + k^{(l-5)} + 2^{(l-7)} + ... + k^{(l-m)}\right) + ... \\ &\quad + \left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right)(p - times). \\ |S| &= 1 + P\left(k^{(l-1)} + k^{(l-3)} + k^{(l-5)} + k^{(l-7)} + ... + k^{(l-m)}\right) \\ \gamma_s(H_1(l,m)) &= 1 + P\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} k^{l-m}, if\ l \equiv\ 0 (\text{mod } 2).\end{aligned}$$

□

Corollary 2.5. *If $H_1(1,2)$ is cubic Halin graph then secure domination number is*

$$\gamma_s(H_1(1,2)) = \begin{cases} \left\lceil \frac{P}{2} \right\rceil + P\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}, & if\ l \equiv\ 0 (\text{mod } 2) \\ 1 + P\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}, & if\ l \equiv\ 1 (\text{mod } 2) \end{cases}$$

Proof. Let $H_1(1,2)$ is the Halin graph with contains $v_1, v_2, v_3, \ldots\ldots, v_{(p+2)}$ central path vertices. Then the perfect binary tree is contained inside each vertex, operating as a branch. An outer cycle is often what connects the end vertices to each other (figure 4). It generates a graph of cubic Halin. By using Theorem 2.4, in the cubic Halin graph, we are able to calculate the secure dominance number.

□

Theorem 2.6. *Let T_l be a perfect binary tree with n vertices. If $(T_l \bigodot T_l)$ is a rooted product of two graphs with n vertices, then the secure domination number is*

$$\gamma_s(T_l \bigodot T_l) = \begin{cases} n\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}, & if\ l \equiv\ 0 (\text{mod } 2) \\ 1 + n\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}. & \end{cases}$$

Proof. This proof is finding a secure dominating set and whose cardinality are minimum. Consider $\{v_{11}, v_{12}......v_{1n}\}$, $\{v_{21}, v_{22}......v_{2n}\}$,...... $\{v_{n1}, v_{n2}......v_{ln}\}$ be the set vertices of perfect binary tree, by attaching a perfect binary tree at each vertex with l stages. Let T_l be a perfect binary tree whose vertices are defined as follows: the vertex set of T_l is defined for l stages. Stage 0 consists of a single vertex v_0. Stage 1 consists of two vertices namely $v_{11}, andv_{12}$ Stage 2 consists of four vertices namely $v_{21}, v_{22}, v_{23}, v_{24}$. Similarly, the stage consists of

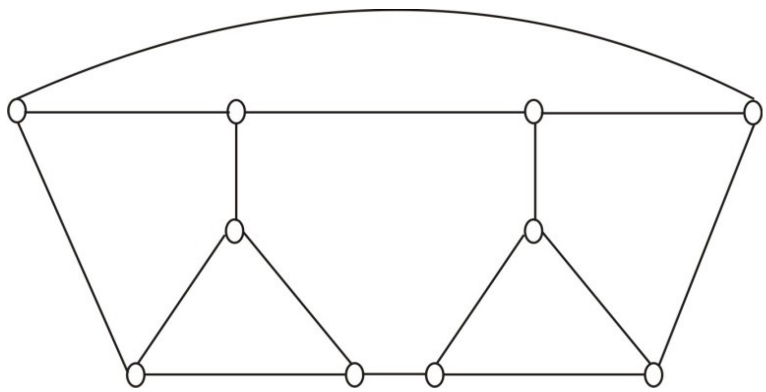

Figure 4. Cubic Halin graph with perfect binary tree.

2^l vertices namely $v_{l1}, v_{l2}, ..., v_{ll}$
At each stage j^th there are 2^j vertices with the number of vertices in T_l is n. Where $|n| = 2^0 + 2^1 + ... + 2^l$.

Now the rooted product $T_l \bigodot T_l$ is obtained by attaching a perfect binary tree of order n to each vertex of a perfect binary tree T_l.
The vertices of j^{th} stage of a perfect binary tree T_l dominates the vertices of $(j+1)^{th}$ and $(j-1)^{th}$ stage where j≠l and 0.

Case (i): When $l \equiv 0 \pmod 2$, the vertices of alternate stages of original graph T_l starting from stage 0 and all the vertices of alternate stages of perfect binary tree are attached to the original graph T_l starting from $l-1$ gives the minimum dominating set. These rooted vertices are dominated by themselves, then

$$\begin{aligned} \gamma_s(T_l \bigodot T_l) &= \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) \\ &\quad + \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) + ... \\ &\quad + \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) (n - times) \\ &= n\left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) \end{aligned}$$

Hence, the secure domination number of rooted product of perfect binary tree is

$$\gamma_s(T_l \bigodot T_l) = n\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}, if\ l \equiv 0 (\text{mod } 2).$$

Case (ii): Suppose $l \equiv 1 \pmod 2$, the vertices of alternate stage of original graph T_l starting from the stage 1 and the vertices of all the alternate stages of perfect binary tree to the original graph T_l starting from stage $l-1$, gives the

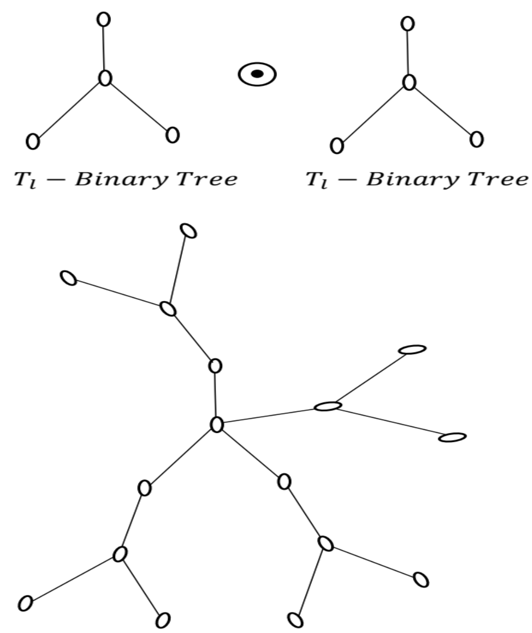

FIGURE 5. Rooted product of perfect binary tree.

secure dominating set.

$$\begin{aligned}\gamma_s(T_l \bigodot T_l) &= \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) \\ &\quad + \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) + ... \\ &\quad + \left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right) (n - times) \\ &= 1 + n\left(2^{(l)} + 2^{(l-2)} + 2^{(l-4)} + 2^{(l-6)} + ... + 2^{(l-m)}\right)\end{aligned}$$

Hence, the secure domination number of rooted product of perfect binary tree is

$$\gamma_s(T_l \bigodot T_l) = 1 + n\sum_{m=1}^{\left\lceil \frac{l-1}{2} \right\rceil} 2^{l-m}, if\ l \equiv 1(\text{mod } 2).$$

□

Theorem 2.7. *Let T_l be a perfect k-ary tree with n vertices. If $(T_l \bigodot T_l)$ is a rooted product of two graphs with n vertices, then the secure domination number* *is*

$$\gamma_s(T_l \bigodot T_l) = \begin{cases} n\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}, & if\ l \equiv\ 0(\text{mod } 2) \\ 1 + n\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}. \end{cases}$$

Proof. This proof is finding a secure dominating set and whose cardinality are minimum. Consider $\{v_{11}, v_{12}......v_{1n}\}$, $\{v_{21}, v_{22}......v_{2n}\}$,...... $\{v_{n1}, v_{n2}......v_{ln}\}$ be the set vertices of perfect k-ary tree, by attaching a perfec t k-ary tree at each vertex with l stages[10]. Let T_l be a perfect k-ary tree whose vertices are defined as follows: the vertex set of T_l is defined for l stages. Stage 0 consists of a single vertex v_0. Stage 1 consists of two vertices namely $v_{11}, and v_{12}$ Stage 2 consists of four vertices namely $v_{21}, v_{22}, v_{23}, v_{24}$. Similarly, the stage consists of k^l vertices namely $v_{l1}, v_{l2}, ..., v_{ll}$
At each stage j^th there are k^j vertices with the number of vertices in T_l is n. Where $|n| = k^0 + k^1 + ... + k^l$.

Now the rooted product $T_l \bigodot T_l$ is obtained by attaching a perfect binary tree of order n to each vertex of a perfect k-ary tree T_l.
The vertices of j^{th} stage of a perfect k-ary tree T_l dominates the vertices of $(j+1)^{th}$ and $(j-1)^{th}$ stage where j≠l and 0.
Case (i): When $l \equiv\ 0$ (mod 2) , the vertices of alternate stages of original graph T_l starting from stage 0 and all the vertices of alternate stages of perfect k-ary tree are attached to the original graph T_l starting from $l-1$ gives the minimum dominating set. "These rooted vertices are dominated by themselves, then

$$\begin{aligned} \gamma_s(T_l \bigodot T_l) &= \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) \\ &\quad + \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) + ... \\ &\quad + \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right)(n - times) \\ &= n\left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) \end{aligned}$$

Hence, the secure domination number of rooted product of the perfect k-ary tree is

$$\gamma_s(T_l \bigodot T_l) = n\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}, if\ l \equiv\ 0(\text{mod } 2).$$

Case (ii): Suppose $l \equiv\ 1(\text{mod } 2)$, the vertices of alternate stage of original graph T_l starting from the stage 1 and the vertices of all the alternate stages of perfect k-ary tree to the original graph T_l starting from stage $l-1$, gives the secure dominating set.

$$\begin{aligned} \gamma_s(T_l \bigodot T_l) &= \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) \\ &\quad + \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) + ... \\ &\quad + \left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right)(n - times) \\ &= 1 + n\left(k^{(l)} + k^{(l-2)} + k^{(l-4)} + k^{(l-6)} + ... + k^{(l-m)}\right) \end{aligned}$$

Hence, the secure domination number of rooted product of the perfect k-ary tree is

$$\gamma_s(T_l \bigodot T_l) = 1 + n\sum_{m=1}^{\lceil \frac{l-1}{2} \rceil} k^{l-m}, if\ l \equiv\ 1(\text{mod } 2).$$

□

3. Conclusion

In this article, we build the different families of Halin graphs with complete k-ary trees, determine its secure domination number of rooted products of perfect binary trees, and investigate some graph parameters related to those graphs. In addition, we build the new relations of Halin graphs by using entire k-ary trees as our building blocks.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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