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DIFFERENTIAL EQUATIONS AND ZEROS FOR NEW MIXED-TYPE HERMITE POLYNOMIALS

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ABSTRACT. In this paper, we find induced differential equations to give explicit identities of these polynomials from the generating functions of 2-variable mixed-type Hermite polynomials. Moreover, we observe the structure and symmetry of the zeros of the 2-variable mixed-type Hermite equations.

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1. Introduction

The ordinary Hermite numbers H_n and Hermite polynomials $H_n(v)$ are usually defined by the generating functions

$$e^{t(2v-t)} = \sum_{n=0}^{\infty} H_n(v)\frac{t^n}{n!}, \quad e^{-t^2} = \sum_{n=0}^{\infty} H_n\frac{t^n}{n!},$$

respectively. Clearly, $H_n = H_n(0)$.

It can be seen that these numbers and polynomials play an important role in various fields of mathematics, applied mathematics, and physics, including number theory, combinations, special functions, and differential equations. Various interesting properties about them are obtained, see [1]-[3]. The ordinary Hermite polynomials $H_n(u)$ satisfy the Hermite differential equation

$$\frac{d^2H(v)}{dv^2} - 2v\frac{dH(v)}{dv} + 2nH(v) = 0, n = 0, 1, 2, \ldots.$$

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We recall that the 2-variable Hermite polynomials $H_n(v,w)$ defined by the generating function(see [2])

$$\sum_{n=0}^{\infty} H_n(v,w)\frac{t^n}{n!} = e^{t(v+wt)} \tag{1}$$

are the solution of heat equation

$$\frac{\partial}{\partial w}H_n(v,w) = \frac{\partial^2}{\partial v^2}H_n(v,w), \quad H_n(v,0) = v^n. \tag{2}$$

Observe that

$$H_n(2v,-1) = H_n(v).$$

Motivated by their importance and potential for applications in certain problems in probability, combinatorics, number theory, differential equations, numerical analysis and other areas of mathematics and physics, several kinds of some special numbers and polynomials were recently studied by many authors, see [1]-[4]. Many mathematicians have studied the area of the degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Genocchi polynomials, and degenerate tangent polynomials, see [4]-[8]. Mixed-type Hermit polynomials, which differ from the results of studies of [10], will be introduced. In this paper, we can see the properties related to the mixed-type Hermit polynomials and mixed-type Hermit polynomials is a solution of differential equation of certain initial value problems.

In [11], Hwang and Ryoo proposed the 2-variable degenerate Hermite polynomials $\mathcal{H}_n(v,w,\lambda)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathcal{H}_n(v,w,\lambda)\frac{t^n}{n!} = (1+\lambda)^{\dfrac{vt+wt^2}{\lambda}}. \tag{3}$$

Since $(1+\lambda)^{\frac{t}{\lambda}} \to e^t$ as $\lambda \to 0$, it is evident that (3) reduces to (1). The 2-variable degenerate Hermite polynomials $\mathcal{H}_n(v,w,\lambda)$ in generating function (3) are the solution of equation

$$\begin{aligned} &\left(\frac{\log(1+\lambda)}{\lambda}\right)\frac{\partial}{\partial v}\mathcal{H}_n(v,w,\lambda) = \frac{\partial^2}{\partial v^2}\mathcal{H}_n(v,w,\lambda), \\ &\mathcal{H}_n(v,0,\lambda) = \left(\frac{\lambda}{\log(1+\lambda)}\right)^n v^n. \end{aligned} \tag{4}$$

Since $\dfrac{\lambda}{\log(1+\lambda)} \to 1$ as λ approaches to 0, it is apparent that (4) descends to (2).

The differential equations arising from the generating functions of special numbers and polynomials were also studied, see [8]-[14]. Hermit polynomials are well known as they play a very important role in mathematical and physical engineering fields. When $\lambda \to 1$, the mixed-type Hermit polynomials become Hermit polynomials. Therefore, the study of mixed-type Hermit polynomials is one of the important studies to understand the characteristics of Hermit polynomials.

This paper is organized as follows. In Section 2, the zeros of the 2-variable mixed-type Hermite equations by using the computer are introduced. Furthermore, we observe the pattern of scattering phenomenon for the zeros of 2-variable mixed-type Hermite equations. Our paper ends with Section 3, where the conclusions and future directions of this work are presented.

2. Differential equations associated with 2-variable mixed-type Hermite polynomials

In this section, we construct the differential equations with coefficients $a_i(N,u,v,\lambda)$ arising from the generating functions of the 2-variable mixed-type Hermite polynomials:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\right)^N \mathfrak{G}(t,u,v|\lambda) - a_0(N,u,v,\lambda)(1+\lambda t^2)^{-N}\mathfrak{G}(t,u,v|\lambda)\\
&- a_1(N,u,v,\lambda)(1+\lambda t^2)^{-N}t^2\mathfrak{G}(t,u,v|\lambda) - \cdots\\
&- a_{2N}(N,u,v,\lambda)(1+\lambda t^2)^{-N}t^{2N}\mathfrak{G}(t,u,v|\lambda) = 0.
\end{aligned}$$

By using the coefficients of this differential equation, we derive explicit identities for the 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u,v,\lambda)$. Recall that

$$\mathfrak{G} = \mathfrak{G}(t,u,v|\lambda) = e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}} = \sum_{n=0}^{\infty}\mathbf{H}_n(u,v,\lambda)\frac{t^n}{n!}, \quad \lambda,u,v,t\in\mathbb{C}. \tag{5}$$

Then, by (5), we have

$$\begin{aligned}
\mathfrak{G}(t,u,v|\lambda)^{(1)} &= \frac{\partial}{\partial t}\mathfrak{G}(t,u,v|\lambda) = \frac{\partial}{\partial t}\left(e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}}\right)\\
&= \left(u+\frac{2vt}{1+\lambda t^2}\right)e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}} = \left(\frac{u+2vt+u\lambda t^2}{1+\lambda t^2}\right)\mathfrak{G}(t,u,v|\lambda),
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathfrak{G}^{(2)} &= \frac{\partial}{\partial t}\mathfrak{G}^{(1)}(t,u,v|\lambda)\\
&= \left(\frac{u^2+2v}{(1+\lambda t^2)^2}\right)\mathfrak{G}(t,u,v|\lambda) + \left(\frac{4uv}{(1+\lambda t^2)^2}\right)t\mathfrak{G}(t,u,v|\lambda)\\
&\quad + \left(\frac{2u^2\lambda-2v\lambda+4v^2}{(1+\lambda t^2)^2}\right)t^2\mathfrak{G}(t,u,v|\lambda) + \left(\frac{4uv\lambda}{(1+\lambda t^2)^2}\right)t^3\mathfrak{G}(t,u,v|\lambda)\\
&\quad + \left(\frac{u^2\lambda^2}{(1+\lambda t^2)^2}\right)t^4\mathfrak{G}(t,u,v|\lambda).
\end{aligned} \tag{7}$$

Continuing this process which is already shown in (7), we can guess that

$$\begin{aligned}\mathfrak{G}^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N \mathfrak{G}(t,u,v|\lambda) \\ &= \sum_{i=0}^{2N} a_i(N,u,v,\lambda)(1+\lambda t^2)^{-N} t^i \mathfrak{G}(t,u,v|\lambda), (N=0,1,2,\ldots).\end{aligned} \tag{8}$$

Differentiating (8) with respect to t, we have

$$\begin{aligned}\mathfrak{G}^{(N+1)} &= \frac{\partial \mathfrak{G}(t,u,v|\lambda)}{\partial t} \\ &= \sum_{i=0}^{2N} a_i(N,u,v,\lambda)(i)t^{i-1}(1+\lambda t^2)^{-N}\mathfrak{G}(t,u,v|\lambda) \\ &\quad + \sum_{i=0}^{2N} a_i(N,u,v,\lambda)t^i(-N)(2\lambda t)(1+\lambda t^2)^{-N-1}\mathfrak{G}(t,u,v|\lambda) \\ &\quad + \sum_{i=0}^{2N} a_i(N,u,v,\lambda)t^i(1+\lambda t^2)^{-N}\mathfrak{G}^{(1)}(t,u,v,|\lambda) \\ &= \sum_{i=0}^{2N-1}(i+1)a_{i+1}(N,u,v,\lambda)t^i(1+\lambda t^2)^{-(N+1)}\mathfrak{G}(t,u,v|\lambda) \\ &\quad + \sum_{i=0}^{2N}(u)a_i(N,u,v,\lambda)t^i(1+\lambda t^2)^{-(N+1)}\mathfrak{G}(t,u,v|\lambda) \\ &\quad + \sum_{i=1}^{2N+1}(2v-2N\lambda+(i-1)\lambda)a_{i-1}(N,u,v,\lambda)t^i(1+\lambda t^2)^{-(N+1)}\mathfrak{G}(t,u,v|\lambda) \\ &\quad + \sum_{i=2}^{2N+2}(u\lambda)a_{i-2}(N,u,v,\lambda)t^i(1+\lambda t^2)^{-(N+1)}\mathfrak{G}(t,u,v|\lambda).\end{aligned} \tag{9}$$

Now replacing N by $N+1$ in (8), we find

$$\mathfrak{G}^{(N+1)} = \sum_{i=0}^{2N+2} a_i(N+1,u,v,\lambda)t^i(1+\lambda t^2)^{-(N+1)}\mathfrak{G}(t,u,v|\lambda). \tag{10}$$

Comparing the coefficients on both sides of (9) and (10), we obtain

$$a_0(N+1,u,v,\lambda) = a_1(N,u,v,\lambda) + ua_0(N,u,v,\lambda). \tag{11}$$

$$a_1(N+1,u,v,\lambda) = 2a_2(N,u,v,\lambda) + ua_1(N,u,v,\lambda) + (2v-2N\lambda)a_0(N,u,v,\lambda). \tag{12}$$

For $2 \leq i \leq 2N-1$, we obtain

$$\begin{aligned} a_i(N+1,u,v,\lambda) = {} & (i+1)a_{i+1}(N,u,v,\lambda) + ua_i(N,u,v,\lambda) \\ & + (2v - 2N\lambda + (i-1)\lambda)a_{i-1}(N,u,v,\lambda) \\ & + (u\lambda)a_{i-2}(N,u,v,\lambda). \end{aligned} \tag{13}$$

For $i = 2N$, we obtain

$$\begin{aligned} a_{2N}(N+1,u,v,\lambda) = {} & ua_{2N}(N,u,v,\lambda) + (2v-\lambda)a_{2N-1}(N,u,v,\lambda) \\ & + (u\lambda)a_{2N-2}(N,u,v,\lambda). \end{aligned} \tag{14}$$

For $i = 2N+1$, we obtain

$$a_{2N+1}(N+1,u,v,\lambda) = (2v)a_{2N}(N,u,v,\lambda) + (u\lambda)a_{2N-1}(N,u,v,\lambda). \tag{15}$$

For $i = 2N+2$, we obtain

$$a_{2N+2}(N+1,u,v,\lambda) = (u\lambda)a_{2N}(N,u,v,\lambda). \tag{16}$$

In addition, by (8), we have

$$\mathfrak{G}(t,u,v|\lambda) = \mathfrak{G}^{(0)}(t,u,v|\lambda) = a_0(0,u,v,\lambda)\mathfrak{G}(t,u,v|\lambda). \tag{17}$$

By (17), we get

$$a_0(0,u,v,\lambda) = 1. \tag{18}$$

It is not difficult to show that

$$\begin{aligned} & u(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda) + 2vt(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda) + u\lambda t^2(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda) \\ & = a_0(1,u,v,\lambda)(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda) + a_1(1,u,v,\lambda)t(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda) \\ & \quad + a_2(1,u,v,\lambda)t^2(1+\lambda t^2)^{-1}\mathfrak{G}(t,u,v|\lambda). \end{aligned} \tag{19}$$

Thus, by (6) and (19), we also get

$$a_0(1,u,v,\lambda) = u, \quad a_1(1,u,v,\lambda) = 2v, \quad a_2(1,u,v,\lambda) = u\lambda. \tag{20}$$

From (11), we note that

$$\begin{aligned} & a_0(N+1,u,v,\lambda) = a_1(N,u,v,\lambda) + ua_0(N,u,v,\lambda), \\ & a_0(N,u,v,\lambda) = a_1(N-1,u,v,\lambda) + ua_0(N-1,u,v,\lambda), \\ & \cdots, \\ & a_0(N+1,u,v,\lambda) = \sum_{i=0}^{N} u^i a_1(N-i,u,v,\lambda) + u^{N+1}. \end{aligned} \tag{21}$$

From (12), we note that

$$
\begin{aligned}
&a_1(N+1,u,v,\lambda) = 2a_2(N,u,v,\lambda) + ua_1(N,u,v,\lambda) + (2v-2N\lambda)a_0(N,u,v,\lambda),\\
&a_1(N,u,v,\lambda) = 2a_2(N-1,u,v,\lambda) + ua_1(N-1,u,v,\lambda)\\
&\qquad\qquad\qquad + (2v-2(N-1)\lambda)a_0(N-1,u,v,\lambda),\\
&\dots,\\
&a_1(N+1,u,v,\lambda)\\
&= 2\sum_{i=0}^{N} u^i a_2(N-i,u,v,\lambda) + \sum_{i=0}^{N} u^i (2v-2(N-i)\lambda)a_0(N-i,u,v,\lambda).
\end{aligned}
\tag{22}
$$

From (14), we have

$$
\begin{aligned}
&a_{2N}(N+1,u,v,\lambda)\\
&= ua_{2N}(N,u,v,\lambda) + (2v-\lambda)a_{2N-1}(N,u,v,\lambda) + (u\lambda)a_{2N-2}(N,u,v,\lambda),\\
&a_{2N-2}(N,u,v,\lambda) = ua_{2N-2}(N-1,u,v,\lambda) + (2v-\lambda)a_{2N-3}(N-1,u,v,\lambda)\\
&\qquad\qquad + (u\lambda)a_{2N-4}(N-1,u,v,\lambda), \dots,\\
&a_{2N}(N+1,u,v,\lambda) = u\sum_{i=0}^{N} (u\lambda)^i a_{2N-2i}(N-i,u,v,\lambda)\\
&\qquad\qquad + (2v-\lambda)\sum_{i=0}^{N-1} (u\lambda)^i a_{2N-(2i+1)}(N-i,u,v,\lambda).
\end{aligned}
\tag{23}
$$

Again, by (15), we have

$$
\begin{aligned}
&a_{2N+1}(N+1,u,v,\lambda) = (2v)a_{2N}(N,u,v,\lambda) + (u\lambda)a_{2N-1}(N,u,v,\lambda),\\
&a_{2N-1}(N,u,v,\lambda) = (2v)a_{2N-2}(N-1,u,v,\lambda) + (u\lambda)a_{2N-3}(N-1,u,v,\lambda), \dots,\\
&a_{2N+1}(N+1,u,v,\lambda) = (2v)\sum_{i=0}^{N-1} (u\lambda)^i a_{2N-2i}(N-i,u,v,\lambda) + (2v)(u\lambda)^N.
\end{aligned}
\tag{24}
$$

From (16), we have

$$
\begin{aligned}
&a_{2N+2}(N+1,u,v,\lambda) = (u\lambda)a_{2N}(N,u,v,\lambda),\\
&a_{2N}(N,u,v,\lambda) = (u\lambda)a_{2N-2}(N-1,u,v,\lambda), \dots,\\
&a_{2N+2}(N+1,u,v,\lambda) = (u\lambda)^{N+1}.
\end{aligned}
\tag{25}
$$

Continuing this process, we can deduce that, for $2 \leq i \leq 2N-1$,

$$\begin{aligned} a_i(N+1,u,v,\lambda) &= (i+1)\sum_{k=0}^{N} u^i a_{i+1}(N-k,u,v,\lambda) \\ &\quad + (2v+\lambda(i-2N-1))\sum_{k=0}^{N} u^i a_{i-1}(N-k,u,v,\lambda) \\ &\quad + (u\lambda)\sum_{k=0}^{N} u^i a_{i-2}(N-k,u,v,\lambda). \end{aligned} \tag{26}$$

Note that, here the matrix $a_i(N,u,v,\lambda)_{0\leq i\leq 2N+2, 0\leq j\leq N+1}$ is given by

$$\begin{pmatrix} 1 & u & 2v+u^2 & 6uv+u^3 & \cdots & \cdot \\ 0 & 2v & 4uv & \cdot & \cdots & \cdot \\ 0 & u\lambda & 2u^2\lambda v+4v^2-2v\lambda & \cdot & \cdots & \cdot \\ 0 & 0 & 4uv\lambda & \cdot & \cdots & \cdot \\ 0 & 0 & u^2\lambda^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & u^{N+1}\lambda^{N+1} \end{pmatrix}$$

Therefore, by (18)-(26), we obtain the following theorem.

Theorem 2.1. *For $N = 0, 1, 2, \ldots,$ the differential equation*

$$\left(\frac{\partial}{\partial t}\right)^N \mathfrak{G}(t,u,v|\lambda) - \sum_{i=0}^{N} a_i(N,u,v,\lambda)(1+\lambda t^2)^{-N} t^i \mathfrak{G}(t,u,v|\lambda) = 0$$

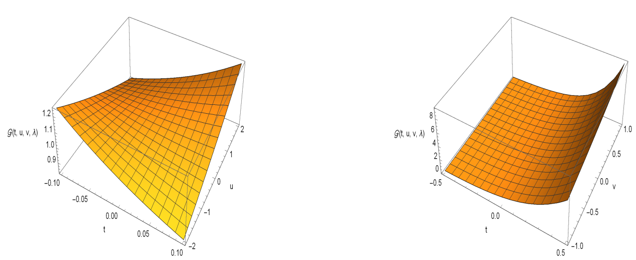
has a solution

$$\mathfrak{G} = \mathfrak{G}(t,u,v|\lambda) = e^{ut}(1+\lambda t^2)^{\frac{v}{\lambda}},$$

where

$$
\begin{aligned}
a_0(N+1,u,v,\lambda) &= \sum_{i=0}^{N} u^i a_1(N-i,u,v,\lambda) + u^{N+1},\\
a_1(N+1,u,v,\lambda) &= 2\sum_{i=0}^{N} u^i a_2(N-i,u,v,\lambda)\\
&\qquad + \sum_{i=0}^{N} u^i (2v-2(N-i)\lambda) a_0(N-i,u,v,\lambda),\\
a_{2N}(N+1,u,v,\lambda) &= u\sum_{i=0}^{N} (u\lambda)^i a_{2N-2i}(N-i,u,v,\lambda)\\
&\qquad + (2v-\lambda)\sum_{i=0}^{N-1} (u\lambda)^i a_{2N-(2i+1)}(N-i,u,v,\lambda),\\
a_{2N+1}(N+1,u,v,\lambda) &= (2v)\sum_{i=0}^{N-1} (u\lambda)^i a_{2N-2i}(N-i,u,v,\lambda) + (2v)(u\lambda)^N,\\
a_{2N+2}(N+1,u,v,\lambda) &= (u\lambda)^{N+1},\\
a_i(N+1,u,v,\lambda) &= (i+1)\sum_{k=0}^{N} u^i a_{i+1}(N-k,u,v,\lambda)\\
&\qquad + (2v+\lambda(i-2N-1))\sum_{k=0}^{N} u^i a_{i-1}(N-k,u,v,\lambda)\\
&\qquad + (u\lambda)\sum_{k=0}^{N} u^i a_{i-2}(N-k,u,v,\lambda), (2 \le i \le 2N-1).
\end{aligned}
$$

Here is a plot of the surface for this solution. In Figure 1(A), we choose



(A) $-\lambda = 1/3$ and $v = 1$. (B) $\lambda = 1/2$ and $u = 4$.

FIGURE 1. The surface for the solution $\mathcal{G}(t,u,v,\lambda)$

$-2 \leq u \leq 2, -\dfrac{1}{10} \leq t \leq \dfrac{1}{10}, \lambda = 1/3$, and $v = 1$. In Figure 1(B), we choose $-1 \leq v \leq 1, -\dfrac{1}{2} \leq t \leq \dfrac{1}{2}, \lambda = 1/2$, and $u = 4$.

Making N-times derivative for (5) with respect to t, we have

$$\left(\frac{\partial}{\partial t}\right)^N \mathfrak{G}(t,u,v|\lambda) = \sum_{m=0}^{\infty} \mathbf{H}_{m+N}(u,v,\lambda)\frac{t^m}{m!}. \tag{27}$$

By (27) and Theorem 7, we have

$$\begin{aligned}
&a_0(N,u,v,\lambda)(1+\lambda t^2)^{-N}\mathcal{G}(t,u,v,\lambda) + a_1(N,u,v,\lambda)(1+\lambda t^2)^{-N} t\mathcal{G}(t,u,v,\lambda)\\
&+\cdots\\
&+ a_{2N-1}(N,u,v,\lambda)(1+\lambda t^2)^{-N} t^{2N-1}\mathcal{G}(t,u,v,\lambda) + a_{2N}(N,u,v,\lambda)(1+\lambda t^2)^{-N} t^{2N}\mathcal{G}(t,u,v,\lambda)\\
&= \sum_{m=0}^{\infty} \mathbf{H}_{m+N}(u,v,\lambda)\frac{t^m}{m!}.
\end{aligned}$$

Hence we have the following theorem.

Theorem 2.2. *For $N = 0,1,2,\ldots,$ and $0 \leq m \leq 2N$, we get*

$$\sum_{k=0}^{[\frac{m}{2}]} (N)_k \lambda^k \mathbf{H}_{N+m-2k}(u,v,\lambda)\frac{m!}{(m-2k)!k!} = \sum_{i=0}^{m} \frac{\mathbf{H}_{m-i}(u,v,\lambda)a_i(N,u,v,\lambda)m!}{(m-i)!}. \tag{28}$$

If we take $m = 0$ in (28), then we have the below corollary.

Corollary 2.3. *For $N = 0,1,2,\ldots,$ we have*

$$\mathbf{H}_N(u,v,\lambda) = a_0(N,u,v,\lambda)\mathbf{H}_0(u,v,\lambda) = a_0(N,u,v,\lambda),$$

where

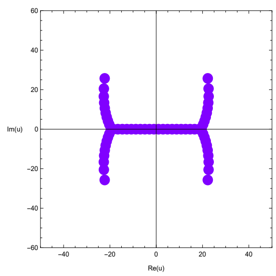
$$\begin{aligned}
&a_0(0,u,v,\lambda) = 1,\\
&a_0(N+1,u,v,\lambda) = \sum_{i=0}^{N} u^i a_1(N-i,u,v,\lambda) + u^{N+1}.
\end{aligned}$$

The first few of them are

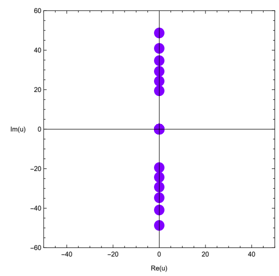
$$\begin{aligned}
&\mathbf{H}_0(u,v,\lambda) = 1,\\
&\mathbf{H}_1(u,v,\lambda) = u,\\
&\mathbf{H}_2(u,v,\lambda) = u^2 + 2v,\\
&\mathbf{H}_3(u,v,\lambda) = u^3 + 6uv,\\
&\mathbf{H}_4(u,v,\lambda) = u^4 + 12u^2v + 12v^2 - 12v\lambda,\\
&\mathbf{H}_5(u,v,\lambda) = u^5 + 20u^3v + 60uv^2 - 60uv\lambda,\\
&\mathbf{H}_6(u,v,\lambda) = u^6 + 30u^4v + 180u^2v^2 + 120v^3 - 180u^2v\lambda - 360v^2\lambda + 240v\lambda^2.
\end{aligned}$$

3. Zeros of the 2-variable mixed-type Hermite polynomials

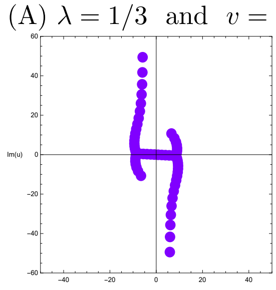
This section shows the benefits of supporting the theoretical prediction through numerical experiments and finding new interesting pattern of the zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$. We use Mathematica in order to obtain numerical results and visualize the roots of $\mathbf{H}_n(u, v, \lambda)$. The 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u, v, \lambda)$ can be determined explicitly. We investigate the zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$. The zeros of the $\mathbf{H}_n(u, v, \lambda) = 0$ for $n = 60, v = -2, -2 - i, 2 + i, 2 - i, \lambda = 1/3$, and $u \in \mathbb{C}$ are displayed in Figure 2. In



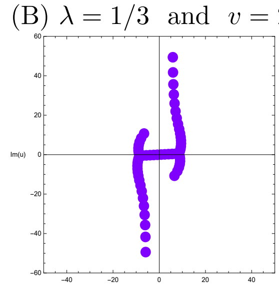
(A) $\lambda = 1/3$ and $v = -2$.



(B) $\lambda = 1/3$ and $v = 2$.



(C) $\lambda = 1/3$ and $v = 2 + i$.

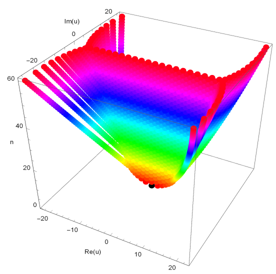


(D) $\lambda = 1/3$ and $v = -2 - i$.

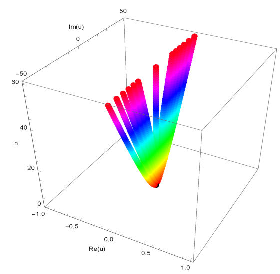
FIGURE 2. Zeros of $\mathbf{H}_n(u, v, \lambda) = 0$

Figure 2A, we choose $n = 60, \lambda = 1/3$ and $v = -2$. In Figure 2B, we choose $n = 60, \lambda = 1/3$ and $v = -2 - i$. In Figure 2C, we choose $n = 60, \lambda = 1/3$ and $v = 2 + i$. In Figure 2D, we choose $n = 60, \lambda = 1/3$ and $v = 2 - i$.

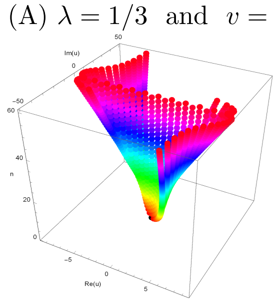
Stacks of zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$ for $1 \leq n \leq 60, \lambda = 1/3$ from a 3-D structure are presented in Figure 3.



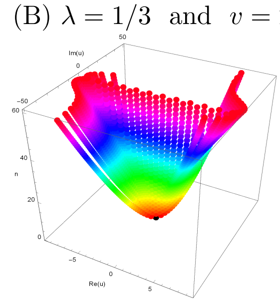
(A) $\lambda = 1/3$ and $v = -2$.



(B) $\lambda = 1/3$ and $v = 2$.



(C) $\lambda = 1/3$ and $v = -2 + i$.



(D) $\lambda = 1/3$ and $v = 2 - i$.

FIGURE 3. Stacks of zeros of $\mathbf{H}_n(u, v, \lambda) = 0, 1 \leq n \leq 60$

In Figure 3A, we choose $v = -2$. In Figure 3B, we choose $v = 2$. In Figure 3C, we choose $v = 2 + i$. In Figure 3D, we choose $v = 2 - i$.

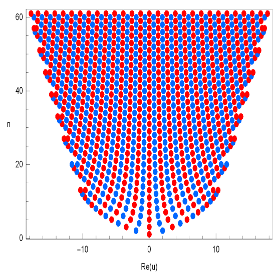
Our numerical results for approximate solutions of real zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$ are displayed(Tables 1, 2).

Table 1. Numbers of real and complex zeros of $\mathbf{H}_n(u, v, \lambda) = 0$

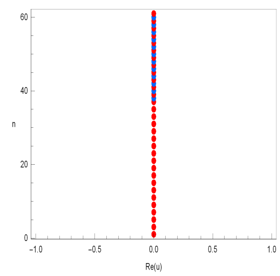
degree n	$v = -2, \lambda = 1/3$		$v = 2, \lambda = 1/3$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	0	2
3	3	0	1	2
4	4	0	0	4
5	5	0	1	4
6	6	0	0	6
7	7	0	1	6
8	4	4	0	8
9	5	4	1	8
10	6	4	0	10
11	7	4	1	11

We observed a remarkable regular structure of the complex roots of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$. We are also hoping to verify the same kind of regular structure of the complex roots of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u,v,\lambda)=0$(Table 1).

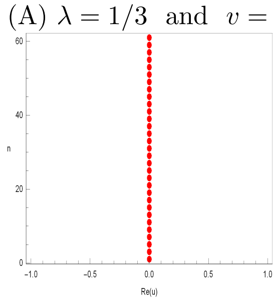
Plot of real zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u,v,\lambda)=0$ for $1\le n\le 60, \lambda=1/3$ structure are presented in Figure 4. In Figure 4A,



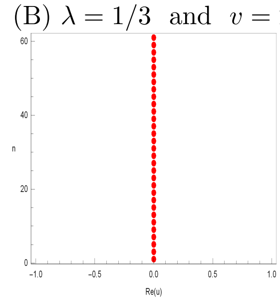
(A) $\lambda=1/3$ and $v=-2$.



(B) $\lambda=1/3$ and $v=2$.



(C) $\lambda=1/3$ and $v=-2+i$.



(D) $\lambda=1/3$ and $v=-2-i$.

FIGURE 4. Real zeros of $\mathbf{H}_n(u,v,\lambda)=0$ for $1\le n\le 60$

we choose $v=-2$. In Figure 4B, we choose $v=2$. In Figure 4C, we choose $v=-2+i$. In Figure 4D, we choose $v=-2-i$.

Next, the approximate solutions satisfying $\mathbf{H}_n(u,v,\lambda)=0, u\in\mathbb{C}$ were calculated. The results are given in Table 2. In Table 2, we choose $v=-2$ and $\lambda=1/3$.

Table 2. Approximate solutions of $\mathbf{H}_n(u,v,\lambda)=0, u\in\mathbb{R}$

degree n	u
1	0
2	−2.0000, 2.0000
3	−3.4641, 0, 3.4641
4	−4.6239, −1.6184, 1.6184, 4.6239
5	−5.5637, −3.0076, 0, 3.0076, 5.5637
6	−6.3144, −4.2490, −1.4403, 1.4403, 4.2490, 6.3144, 4.2490, 6.3144
7	−6.8626, −5.4166, −2.7505, 0, 2.7505, 5.4166, 6.8626
8	−3.9639, −1.3353, 1.3353, 3.9639

4. Observations and future directions

We constructed differential equations arising from the generating function of the 2-variable mixed-type Hermite polynomials $\mathbf{H}_n(u, v, \lambda)$. We also investigated the symmetry of the zeros of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$ for various variables u and v. As a result, the distribution of the zeros of 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$ has a very regular pattern. So, we organize the following series of conjectures with numerical experiments:

Let us use the following notations. $R_{\mathbf{H}_n(u,v,\lambda)}$ denotes the number of real zeros of $\mathbf{H}_n(u, v, \lambda) = 0$ lying on the real plane $Im(u) = 0$ and $C_{\mathbf{H}_n(u,v,\lambda)}$ denotes the number of complex zeros of $\mathbf{H}_n(u, v, \lambda) = 0$. Since n is the degree of the polynomial $\mathbf{H}_n(u, v, \lambda)$, we have $R_{\mathbf{H}_n(u,v,\lambda)} = n - C_{\mathbf{H}_n(u,v,\lambda)}$.

We can observe a stable regular pattern of the complex roots of the 2-variable mixed-type Hermite equations $\mathbf{H}_n(u, v, \lambda) = 0$ for v and λ. Therefore, the following conjecture is possible.

Conjecture 4.1. Let m be odd positive integer. For $a > 0$, prove or disprove that

$$R_{\mathbf{H}_m(u,a,\lambda)} = 1, \quad C_{\mathbf{H}_m(u,a,\lambda)} = 2\left[\frac{m}{2}\right],$$

where $\mathbb{C}$ is the set of complex numbers.

Conjecture 4.2. Let m be odd positive integer and $a \in \mathbb{C}$. Prove or disprove that

$$\mathbf{H}_m(0, a, \lambda) = 0.$$

As a result of investigating more v and λ variables, it is still unknown whether the conjecture 1 and conjecture 2 is true or false for all variables v and λ.

We observe that solutions of the 2-variable mixed-type Hermite equations $\mathbf{H}_m(u, b, \lambda) = 0$ has the $Re(u) = 0$ reflection symmetry for $b \in \mathbb{R}$. It is expected that solutions of the 2-variable mixed-type Hermite equations $\mathbf{H}_m(u, b, \lambda) = 0$ does not obtain a $Re(u) = b$ reflection symmetry for $b \in \mathbb{C} \setminus \mathbb{R}$ (see Figure 2, Figure 3, Figure 4).

Conjecture 4.3. Prove that the zeros of $\mathbf{H}_m(u, a, \lambda) = 0, a \in \mathbb{R}$, has $Im(u) = 0$ reflection symmetry analytic complex functions. Prove that the zeros of $\mathbf{H}_m(u, a, \lambda) = 0, a \in \mathbb{C} \setminus \mathbb{R}$, has not $Im(u) = 0$ reflection symmetry analytic complex functions.

Finally, we consider a more general problem. How many zeros does $\mathbf{H}_m(u, v, \lambda)$ have? We are not able to decide if $\mathbf{H}_n(u, v, \lambda) = 0$ has n distinct solutions. We would like to know the number of complex zeros $C_{\mathbf{H}_m(u,v,\lambda)}$ of $\mathbf{H}_m(u, v, \lambda) = 0$.

Conjecture 4.4. For $a \in \mathbb{C}$, prove or disprove that $\mathbf{H}_m(u, a, \lambda) = 0$ has n distinct solutions.

As a result of investigating more m variables, it is still unknown whether the conjecture is true or false for all variables m(see Tables 1 and 2).

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

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