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FRACTIONAL FIELD WITH STANDARD FRACTIONAL VECTOR CROSS PRODUCT

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ABSTRACT. In this research we have used the definition of standard fractional vector cross product to obtain fractional curl and fractional field of a standing wave, a travelling wave, a transverse wave, a vector field in x-y plane, a complex vector field and an electric field. Fractional curl and fractional field for a complex order are also discussed. We have supported the study with calculation of impedance at $\gamma = 0, 0 < \gamma < 1, \gamma = 1$. The formula discussed in this paper are useful for study of polarization, reflection, impedance, boundary conditions where fractional solutions have applications.

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1. Introduction

In 1967, Crowe [2] laid the foundation of vector calculus which revolutionised the study of applied mathematics. In the year 1967, Caputo [1] gave a definition of fractional derivative, which has practical relevance, as this definition requires integer order initial states to solve fractional differential equations; which otherwise requires fractional order initial-states. Das [3], [4] gave details of fractional calculus including fractional cross product and fractional curl. Mishra and Patnaik [9] applied fractional cross product in radiation characteristic. Tripathi and Kim [8] defined some properties of fractional vector cross product in euclidean 3 space. Later in 2022, Kankarej and Singh, [6], [7] gave an alternative definition of fractional cross product called standard fractional vector cross product and derived its properties. They also studied the properties for a vector pair.

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This research gives insight to physics of various aspects of electromagnetic wave propagation. In this paper we see, how fractional curl and field are calculated for different forms of electromagnetic wave propagation. We gave the proposed definition of standard fractional vector cross product and then in later sections calculated the fractional curl and fractional field for a standing wave, a complex vector field, an electric field, a travelling wave, a transverse wave. Complex order curl is also a part of this research.

2. Standard Fractional Vector Cross Product

Further to the study of standard fractional cross vector product in [6],[7] where we defined :

Definition 2.1 Let R^3 be the Euclidean 3-space equipped with standard inner product $\langle ., . \rangle$. Let (e_1, e_2, e_3) be standard orthonormal basis of R^3 and $\gamma \in [0, 1]$ a real number. Then, for vectors $a = a_1e_1 + a_2e_2 + a_3e_3,\ b = b_1e_1 + b_2e_2 + b_3e_3$ in R^3, the *Standard Fractional Vector Cross Product* is defined by

$$\begin{aligned} a \times^{\gamma} b = & \big\{(a_2b_3 - a_3b_2)\sin\big(\frac{\gamma\pi}{2}\big) + (a_2 + a_3)b_1\cos\big(\frac{\gamma\pi}{2}\big) - (b_2 + b_3)a_1\cos\big(\frac{\gamma\pi}{2}\big)\big\}e_1 \\ & +\big\{(a_3b_1 - a_1b_3)\sin\big(\frac{\gamma\pi}{2}\big) + (a_3 + a_1)b_2\cos\big(\frac{\gamma\pi}{2}\big) - (b_3 + b_1)a_2\cos\big(\frac{\{\gamma\pi}{2}\big)\big\}e_2 \\ & +\big\{(a_1b_2 - a_2b_1)\sin\big(\frac{\gamma\pi}{2}\big) + (a_1 + a_2)b_3\cos\big(\frac{\gamma\pi}{2}\big) - (b_1 + b_2)a_3\cos\big(\frac{\gamma\pi}{2}\big)\big\}e_3. \end{aligned} \tag{1}$$

From eqn (1) we have,

$$e_i \times^{\gamma} e_j = \cos\big(\frac{\gamma\pi}{2}\big)e_j + \sin\big(\frac{\gamma\pi}{2}\big)e_k - \cos\big(\frac{\gamma\pi}{2}\big)e_i, \tag{2}$$

$$e_j \times^{\gamma} e_i = \cos\big(\frac{\gamma\pi}{2}\big)e_i - \sin\big(\frac{\gamma\pi}{2}\big)e_k - \cos\big(\frac{\gamma\pi}{2}\big)e_j, \tag{3}$$

$$e_l \times^{\gamma} e_l = 0\ for\ l = \{1, 2, 3\}. \tag{4}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). The equations (2), (3) and (4) are similar to that in [6], [7].

3. Fractional Curl For A Standing Wave With Standard Vector Cross Product

From above section we have understood that fractional vector cross product is the fractional rotation of angle $\frac{\gamma\pi}{2}$ of the vector on which the cross product operation is carried on and the rotation about the axis of the vector which is doing the cross product operation. The operation being linear we can use superposition to get the expression for the standard fractional vector cross product.

Let us take a stationary wave given as $\bar{F}(z) = \hat{x}(\cos z) + \hat{y}(0) + \hat{z}(0)$. It is a vector field directed towards x axis with variation in z - direction. Thus it has a gradient in z - direction. This string is vibrating between $z = -\frac{\pi}{2}$ and $z = \frac{\pi}{2}$ in x - y plane but directed at x - axis. We see that the normal curl of this " standing stationary wave' of vibrating string using eq (27) of [7] takes the matrix form by [5]:

$$\begin{aligned}\nabla_z \times \bar{F}(z) &= \begin{bmatrix} 0 & -\hat{x}({}_{-\infty}D_z^1) & 0 \\ \hat{y}({}_{-\infty}D_z^1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos z \\ 0 \\ 0 \end{bmatrix} = \hat{y}({}_{-\infty}D_z^1)\cos z \\ &= \hat{y}(-\sin z) = \hat{y}\{\cos(\frac{\pi}{2} + z)\}.\end{aligned} \tag{5}$$

The fractional curl operation will be a fractional rotation of this vector field in the R^3 plane by an angle $\gamma(90°)$ about z axis in anticlockwise direction and the vector field will be advanced by an angle $\gamma(90°)$. Thus we get tilted fractional curl field of this vector field which will have reflections on x -z and y - z planes. The expression for this tilted curl can be written as from eqn(41) of [7] in the matrix form by [5]. Thus we have

$$\nabla \times^\gamma \bar{F} = \cos\left(\frac{\gamma\pi}{2}\right) \begin{bmatrix} \nabla_{yz}^\gamma & -\partial_x^\gamma & -\partial_x^\gamma \\ -\partial_y^\gamma & \nabla_{zx}^\gamma & -\partial_y^\gamma \\ -\partial_z^\gamma & -\partial_z^\gamma & \nabla_{xy}^\gamma \end{bmatrix} \begin{bmatrix} \hat{x}\cos z \\ 0 \\ 0 \end{bmatrix} + \sin\left(\frac{\gamma\pi}{2}\right) \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x^\gamma & \partial_y^\gamma & \partial_z^\gamma \\ \cos z & 0 & 0 \end{vmatrix}, \tag{6}$$

where $\nabla_{yz}^\gamma = \frac{\partial^\gamma}{\partial y^\gamma} + \frac{\partial^\gamma}{\partial z^\gamma}, \nabla_{zx}^\gamma = \frac{\partial^\gamma}{\partial z^\gamma} + \frac{\partial^\gamma}{\partial x^\gamma}, \nabla_{xy}^\gamma = \frac{\partial^\gamma}{\partial x^\gamma} + \frac{\partial^\gamma}{\partial y^\gamma}$,

$\frac{\partial^\gamma}{\partial x^\gamma} = \partial_x^\gamma, \frac{\partial^\gamma}{\partial y^\gamma} = \partial_y^\gamma, \frac{\partial^\gamma}{\partial z^\gamma} = \partial_z^\gamma$

which gives

$\nabla \times^\gamma \bar{F} = \sin\left(\frac{\gamma\pi}{2}\right)(\hat{y} - \hat{z})\frac{\partial^\gamma}{\partial z^\gamma}\cos z,$

$\nabla \times^\gamma \bar{F} = \sin\left(\frac{\gamma\pi}{2}\right)(\hat{y} - \hat{z})(-\sin z)$

$= \sin\left(\frac{\gamma\pi}{2}\right)(\hat{y} - \hat{z})\cos(\frac{\pi}{2} + z)$

4. Fractional Curl For A Complex vector Field With Standard Fractional Vector Cross Product

Let us take an example of a complex vector field $\bar{F} = \hat{x}(Ae^{ik_0 z})$ which is a x directed field varying in z direction. Thus $f_x = Ae^{ik_0 z}, f_y = 0$ and $f_z = 0$ which represents a wave travelling in x direction and harmonically in z direction. For the given complex vector field $\frac{\partial^\gamma F_x}{\partial x^\gamma} = \frac{\partial^\gamma F_x}{\partial y^\gamma} = 0$ which means $\nabla_{xy}^\gamma \bar{F} = 0$. Using this $\nabla^\gamma_{yz} = \frac{\partial^\gamma}{\partial z^\gamma}$ and $\nabla^\gamma_{zx} = \frac{\partial^\gamma}{\partial z^\gamma}$. Using this eqn (6) becomes

$$\nabla \times^\gamma \bar{F} = \cos\big(\frac{\gamma\pi}{2}\big) \begin{bmatrix} \partial^\gamma_z & 0 & 0 \\ 0 & \partial^\gamma_z & 0 \\ -\partial^\gamma_z & -\partial^\gamma_z & 0 \end{bmatrix} \begin{bmatrix} \hat{x} f_x \\ 0 \\ 0 \end{bmatrix} + \sin\big(\frac{\gamma\pi}{2}\big) \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial^\gamma_z \\ f_x & 0 & 0 \end{vmatrix} \\ = \sin\big(\frac{\gamma\pi}{2}\big)\hat{y}(\partial^\gamma_z f_x) \tag{7}$$

Using ${}_{-\infty}D^\gamma_t e^{\lambda t} = \frac{\partial^\gamma e^{\lambda t}}{\partial t^\gamma} = \lambda^\gamma e^{\lambda t}$, we write the fractional curl as

$$curl\bar{F} = \nabla \times^\gamma \{\hat{x}(Ae^{ik_0 z})\} = (ik_0)^\gamma A\hat{y} \sin\big(\frac{\gamma\pi}{2}\big)e^{ik_0 z}, \tag{8}$$

Thus the fractional field is given by using $\bar{F}_f = (ik_0)^{-\gamma}[\nabla \times^\gamma \bar{F}]$ as given in the following equation

$$\bar{F}_f = (ik_0)^{-\gamma}[\nabla \times^\gamma \{\hat{x}(Ae^{ik_0 z})\}] = A\hat{y} \sin\big(\frac{\gamma\pi}{2}\big)e^{ik_0 z}. \tag{9}$$

Let us take a vector field directed in z direction having variation in x - y plane as $\bar{F} = \hat{z}F(x, y) = \hat{z}(e^{i\lambda x + i\mu y})$. The given wave is z-directed travelling in x-y plane making an angle ϕ such that $tan\phi = \frac{\mu}{\lambda}$. Using eqn (6) the fractional curl of the vector is given below:

$$\begin{aligned} \nabla \times^\gamma \{\hat{z}F(x, y)\} &= \cos\big(\frac{\gamma\pi}{2}\big) \begin{bmatrix} \nabla^\gamma_{yz} & -\partial^\gamma_x & -\partial^\gamma_x \\ -\partial^\gamma_y & \nabla^\gamma_{zx} & -\partial^\gamma_y \\ -\partial^\gamma_z & -\partial^\gamma_z & \nabla^\gamma_{xy} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \hat{z}\bar{F}(x, y) \end{bmatrix} \\ &\quad + \sin\big(\frac{\gamma\pi}{2}\big) \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial^\gamma_x & \partial^\gamma_y & \partial^\gamma_z \\ 0 & 0 & F(x, y) \end{vmatrix} \\ &= \cos\big(\frac{\gamma\pi}{2}\big)\{-\hat{z}\partial^\gamma_x \bar{F}(x, y) - \hat{z}\partial^\gamma_y \bar{F}(x, y) + \hat{z}\nabla^\gamma_{xy}\bar{F}(x, y)\} \\ &\quad + \sin\big(\frac{\gamma\pi}{2}\big)\{\hat{x}\partial^\gamma_y F(x, y) - \hat{y}\partial^\gamma_x F(x, y)\} \\ &= \sin\big(\frac{\gamma\pi}{2}\big)\{\hat{x}\partial^\gamma_y F(x, y) - \hat{y}\partial^\gamma_x F(x, y)\}. \end{aligned} \tag{10}$$

We can write $F(x, y) = e^{i\lambda\hat{x} + i\mu\hat{y}} = F(\hat{r}) = e^{i|r|\hat{r}}$ where $|r| = \sqrt{\lambda^2 + \mu^2}$ and $\hat{r}$ is the vector in radial direction, that is $\hat{r} = \hat{x} + \hat{y}$ where $\hat{x}, \hat{y}$ are unit vectors of cartesian coordinates.

As we know ${}_{-\infty}D^\gamma_t e^{\lambda t} = \frac{\partial^\gamma e^{\lambda t}}{\partial^\gamma t}$. So, we have

$\frac{\partial^\gamma F(\hat{r})}{\partial \hat{r}^\gamma} = \frac{\partial^\gamma (e^{(i|r|\hat{r})})}{\partial \hat{r}^\gamma} = (i|r|)^\gamma e^{i|r|\hat{r}}$. This is a fractional derivative in $\hat{r}$ direction.

It's component in x-axis given by $\frac{\partial^\gamma F(x,y)}{\partial x^\gamma}$ is obtained by the multiplication of fractional derivative in the direction of $\hat{r}$ with projection on x-axis i.e. $\lambda(|r|)$. Similarly it's component in y-axis given by $\frac{\partial^\gamma F(x,y)}{\partial y^\gamma}$ is obtained by the multiplication of fractional derivative in the direction of $\hat{r}$ with projection on y-axis i.e. $\mu(|r|)$.

Thus we have

$$\begin{aligned}\partial_x^\gamma F(x,y) = \frac{\partial^\gamma (e^{(i|r|\hat{r})})}{\partial \hat{x}^\gamma} = \frac{\partial^\gamma (e^{(i\lambda\hat{x}+i\mu\hat{y})})}{\partial \hat{x}^\gamma} &= i^\gamma |r|^\gamma \frac{\lambda}{|r|} \\ &= i^\gamma \lambda \big(\sqrt{\lambda^2+\mu^2}\big)^{\gamma-1} e^{i(\lambda\hat{x}+\mu\hat{y})},\end{aligned} \tag{11}$$

$$\begin{aligned}\partial_y^\gamma F(x,y) = \frac{\partial^\gamma (e^{(i|r|\hat{r})})}{\partial \hat{y}^\gamma} = \frac{\partial^\gamma (e^{(i\lambda\hat{x}+i\mu\hat{y})})}{\partial \hat{y}^\gamma} &= i^\gamma |r|^\gamma \frac{\mu}{|r|} \\ &= i^\gamma \mu \big(\sqrt{\lambda^2+\mu^2}\big)^{\gamma-1} e^{i(\lambda\hat{x}+\mu\hat{y})}.\end{aligned} \tag{12}$$

Similarly the component in z direction is $i^\gamma |r|^\gamma$ or $i^\gamma \big(\sqrt{\lambda^2+\mu^2}\big)$.

So the fractional curl for this vector field is:

$$\begin{aligned}\nabla \times^\gamma \big(\hat{z} e^{i(\lambda\hat{x}+\mu\hat{y})}\big) &= \hat{x} i^\gamma \mu (\sqrt{\lambda^2+\mu^2}\big)^{\gamma-1} \sin(\frac{\gamma\pi}{2}) e^{i(\lambda\hat{x}+\mu\hat{y})} \\ &\quad -\hat{y} i^\gamma \lambda (\sqrt{\lambda^2+\mu^2}\big)^{\gamma-1} \sin(\frac{\gamma\pi}{2}) e^{i(\lambda\hat{x}+\mu\hat{y})}.\end{aligned} \tag{13}$$

Putting $F = \hat{z} e^{i(\lambda\hat{x}+\mu\hat{y})}$, $k_x = \lambda$ and $k_y = \mu$ and $k = \sqrt{k_x^2 + k_y^2} = \sqrt{\lambda^2+\mu^2}$
Thus the fractional field can be written as

$$\begin{aligned}F_f = (ik)^{-\gamma} \nabla \times^\gamma \big(\hat{z} e^{i(\lambda\hat{x}+\mu\hat{y})} &= (ik)^{-\gamma} \hat{x} i^\gamma k_y (k)^{\gamma-1} \sin(\frac{\gamma\pi}{2}) e^{i(k_x\hat{x}+k_y\hat{y})} \\ &\quad + (ik)^{-\gamma} (-\hat{y} i^\gamma) k_x (k)^{\gamma-1} \sin(\frac{\gamma\pi}{2}) e^{i(k_x\hat{x}+k_y\hat{y})},\end{aligned} \tag{14}$$

$$F_f = [\hat{x} \frac{k_y}{k} \sin(\frac{\gamma\pi}{2}) + \hat{y} \frac{k_x}{k} (-\sin(\frac{\gamma\pi}{2})) \tag{15}$$

For $\gamma = 0, F_f = 0$

For $\gamma = 1, F_f = [\hat{x} \frac{k_y}{k} - \hat{y} \frac{k_x}{k}] e^{i(k_x\hat{x}+k_y\hat{y})}$

5. Fractional Field For An Electric Field With Standard Fractional Vector Cross Product

Let us consider a field of the form $\bar{E} = z e^{ik(x\cos\phi + y sin\phi)}$.

$$\begin{aligned}\nabla \times^{\gamma} \bar{E} &= \nabla \times^{\gamma} \left\{ z e^{ik(x\cos\phi + ysin\phi)} \right\} \\ &= [\hat{x}z\sin\phi - \hat{y}z\cos\phi]\sin(\frac{\gamma\pi}{2})e^{ik(x\cos\phi+ysin\phi)}\end{aligned} \tag{16}$$

The fractional field of this electric field can be written as:

$$\bar{E_f} = (ik)^{-i\gamma}\nabla \times^{\gamma} \bar{E} = (ik)^{-\gamma}[\hat{x}z\sin\phi - \hat{y}z\cos\phi]\sin(\frac{\gamma\pi}{2})e^{ik(x\cos\phi+ysin\phi)}. \tag{17}$$

6. Fractional Field For A Travelling Wave With Standard Fractional Vector Cross Product

Let us consider a travelling wave given by $\bar{F} = E_0(\hat{x} + i\hat{y})e^{ik\hat{z}}$. This is a polarized vector field with E_0 radius in x - y plane but travelling in z direction. We have $\bar{F}_x = E_0 e^{ik\hat{z}}, \bar{F}_y = iE_0e^{ik\hat{z}}, \bar{F}_z = 0$. The fractional curl for $\gamma \neq 0$ gives

$$\begin{aligned}\nabla \times^{\gamma} \bar{F} &= \cos\left(\frac{\gamma\pi}{2}\right)\begin{bmatrix}\nabla^{\gamma}_{yz} & -\partial^{\gamma}_{x} & -\partial^{\gamma}_{x} \\ -\partial^{\gamma}_{y} & \nabla^{\gamma}_{zx} & -\partial^{\gamma}_{y} \\ -\partial^{\gamma}_{z} & -\partial^{\gamma}_{z} & \nabla^{\gamma}_{xy}\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ E_0(\hat{x}+i\hat{y})e^{ik\hat{z}}\end{bmatrix} \\ &\quad + \sin\left(\frac{\gamma\pi}{2}\right)\begin{vmatrix}\hat{x} & \hat{y} & \hat{z} \\ \partial^{\gamma}_{x} & \partial^{\gamma}_{y} & \partial^{\gamma}_{z} \\ 0 & 0 & E_0(\hat{x}+i\hat{y})e^{ik\hat{z}}\end{vmatrix} \\ &= \sin\left(\frac{\gamma\pi}{2}\right)\{\hat{x}\partial^{\gamma}_{y}E_0(\hat{x}+i\hat{y})e^{ik\hat{z}} - \hat{y}\partial^{\gamma}_{x}E_0(\hat{x}+i\hat{y})e^{ik\hat{z}}\} \\ &= \sin\left(\frac{\gamma\pi}{2}\right)\{\hat{x}iE_0e^{ik\hat{z}} - \hat{y}E_0e^{ik\hat{z}}\}(ik)^{\gamma}\end{aligned} \tag{18}$$

Using ${}_{-\infty}D^{\gamma}_{t}e^{\lambda t} = \frac{\partial^{\gamma}e^{\lambda t}}{\partial t^{\gamma}} = \lambda^{\gamma}e^{\lambda t}, \pm i = e^{\pm\frac{i\pi}{2}}$ and $\cos\chi \pm i\sin\chi = e^{\pm i\chi}$, if the vector field is right circularly polarized we write the fractional curl as

$$\begin{aligned}\nabla \times^{\gamma} \bar{F} &= \{-\hat{x}E_0e^{ik\hat{z}}\cos\left(\frac{\gamma\pi}{2}\right) + \hat{x}iE_0e^{ik\hat{z}}\sin\left(\frac{\gamma\pi}{2}\right)\}(ik)^{\gamma} \\ &\quad + \{i\hat{y}E_0e^{ik\hat{z}}\cos\left(\frac{\gamma\pi}{2}\right) - \hat{y}E_0e^{ik\hat{z}}\sin\left(\frac{\gamma\pi}{2}\right)\}(ik)^{\gamma} \\ &= \{-\hat{x}E_0e^{ik\hat{z}}\{e^{\frac{-i\pi\gamma}{2}}\} + i\hat{y}E_0e^{ik\hat{z}}\{e^{\frac{-i\pi\gamma}{2}}\}\}(ik)^{\gamma} = (-\hat{x}+i\hat{y})E_0e^{ik\hat{z}}\{e^{\frac{-i\pi}{2}}\}^{\gamma}(ik)^{\gamma}.\end{aligned} \tag{19}$$

If the vector field is left circularly poralized the fractional curl is given by

$$\nabla \times^{\gamma} \bar{F} = (i)^{\gamma}\{-\hat{x} - i\hat{y}\}E_0e^{ik\hat{z}}(ik)^{\gamma}. \tag{20}$$

Thus the fractional field for circularly polarized traveling wave is

$$\bar{F_f} = (\pm i)^{\gamma}(-\hat{x} \pm i\hat{y})E_0e^{ik\hat{z}} = (-\hat{x} \pm i\hat{y})E_0e^{ik\hat{z}\pm\frac{i\pi\gamma}{2}}. \tag{21}$$

7. Complex Order Curl

Let us consider a travelling wave given by $\bar{F} = E_0(\hat{x}+i\hat{y})e^{ik\hat{z}}$. Putting $\gamma = i\delta$ in eqns (19) - (21), we have fractional curl for right circularly polarized wave as

$$\begin{aligned} \nabla \times^{i\delta} \bar{F} &= (-i)^{i\delta}(-\hat{x}+i\hat{y})E_0 e^{ik\hat{z}}(ik)^{i\delta} \\ &= (e^{\frac{i\pi}{2}})^{i\delta}\{-\hat{x}+i\hat{y}\}E_0 e^{ik\hat{z}}(ik)^{i\delta} = \{-\hat{x}+i\hat{y}\}e^{\frac{-\pi\delta}{2}}E_0 e^{ik\hat{z}}(ik)^{i\delta} \end{aligned} \tag{22}$$

The fractional curl for a left circularly polarized wave is

$$\nabla \times^{i\delta} \bar{F} = \{-\hat{x}-i\hat{y}\}e^{\frac{\pi\delta}{2}}E_0 e^{ik\hat{z}}(ik)^{i\delta}. \tag{23}$$

Thus the fractional field for circularly polarized traveling wave is

$$\bar{F}_f = (-\hat{x} \pm i\hat{y})e^{\frac{\pm\pi\delta}{2}+ik\hat{z}}E_0. \tag{24}$$

We have fractional curl of complex order for right circularly polarized wave as

$$\nabla \times^{\gamma+i\delta} \bar{F} = (-i)^{\gamma}(-\hat{x}+i\hat{y})E_0 e^{\frac{-\pi\delta}{2}+ik\hat{z}}(ik)^{\gamma+i\delta}. \tag{25}$$

The fractional curl for a left circularly polarized wave is

$$\nabla \times^{\gamma+i\delta} \bar{F} = (i)^{\gamma}\{-\hat{x}-i\hat{y}\}E_0 e^{\frac{\pi\delta}{2}+ik\hat{z}}(ik)^{\gamma+i\delta}. \tag{26}$$

Thus the fractional field for circularly polarized traveling wave is

$$\bar{F}_f = (-\hat{x} \pm i\hat{y})E_0 e^{\frac{\pm\pi\delta}{2}}e^{ik\hat{z}\pm\frac{i\pi\gamma}{2}}. \tag{27}$$

Interpretation of the curl with complex order:
The real part of the fractional order $\gamma > 0$ gives the lead and lag of the spatial phase of vector field for right and left circularly polarized vector field. The imaginary part $\delta > 0$ increases and decreases the radius of the amplitude of vector field for right and left circularly polarized vector field.

8. Polarized Transverse Wave with Standard Fractional Vector Cross Product

Let us consider a right and left circularly polarized transverse electromagnetic(TEM) uniform plane wave propagating along +z direction has electric and magnetic fields given as:

$$\begin{aligned} \bar{E}_{\pm} &= (-\hat{x} \pm \hat{y})E_0 e^{ikz} \\ \eta\bar{H}_{\pm} &= (\mp i)(-\hat{x} \pm \hat{y})E_0 e^{ikz} \end{aligned} \tag{28}$$

Using eqns (19) - (21) we have the fractional fields given as,

$$\begin{aligned} \bar{E}_{f(\pm)} &= (\mp)^{\gamma}(-\hat{x} \pm \hat{y})E_0 e^{ikz} \\ \eta\bar{H}_{f(\pm)} &= (\mp)^{\gamma+1}(-\hat{x} \pm \hat{y})E_0 e^{ikz} for\ 0 < \gamma < 1 \end{aligned} \tag{29}$$

We understand that $(\pm i)^{\gamma}$ is the eigen value for the operator $(ik)^{-\gamma}\times^{\gamma}$ and the original circularly polarized electric and magnetic fields are eigen vectors for the same operator.

Now let us take a standing wave formed from two TEM uniform plane wave propagating in opposite direction having electric and magnetic fields as

$$\begin{aligned} \bar{E} &= \hat{x}\big[E_0 e^{-ikz} - E_0 e^{+ikz}\big] = -\hat{x}2iE_0\sin(kz), \\ \eta\bar{H} &= \hat{y}\big[E_0 e^{-ikz} + E_0 e^{+ikz}\big] = \hat{y}2E_0\cos(kz), \end{aligned} \tag{30}$$

$$\begin{aligned} \nabla\times^{\gamma}\bar{E} &= \cos\big(\frac{\gamma\pi}{2}\big)\begin{bmatrix}\nabla^{\gamma}_{yz} & -\partial^{\gamma}_{x} & -\partial^{\gamma}_{x}\\ -\partial^{\gamma}_{y} & \nabla^{\gamma}_{zx} & -\partial^{\gamma}_{y}\\ -\partial^{\gamma}_{z} & -\partial^{\gamma}_{z} & \nabla^{\gamma}_{xy}\end{bmatrix}\begin{bmatrix}-\hat{x}2iE_0\sin(kz)\\ 0\\ 0\end{bmatrix} \\ &\quad + \sin\big(\frac{\gamma\pi}{2}\big)\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial^{\gamma}_{x} & \partial^{\gamma}_{y} & \partial^{\gamma}_{z}\\ -2iE_0\sin(kz) & 0 & 0\end{vmatrix}, \\ \nabla\times^{\gamma}\eta\bar{H} &= \cos\big(\frac{\gamma\pi}{2}\big)\begin{bmatrix}\nabla^{\gamma}_{yz} & -\partial^{\gamma}_{x} & -\partial^{\gamma}_{x}\\ -\partial^{\gamma}_{y} & \nabla^{\gamma}_{zx} & -\partial^{\gamma}_{y}\\ -\partial^{\gamma}_{z} & -\partial^{\gamma}_{z} & \nabla^{\gamma}_{xy}\end{bmatrix}\begin{bmatrix}0\\ \hat{y}2E_0\cos(kz)\\ 0\end{bmatrix} \\ &\quad + \sin\big(\frac{\gamma\pi}{2}\big)\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial^{\gamma}_{x} & \partial^{\gamma}_{y} & \partial^{\gamma}_{z}\\ 0 & 2E_0\cos(kz) & 0\end{vmatrix}, \end{aligned} \tag{31}$$

$$\begin{aligned} \nabla\times^{\gamma}\bar{E} &= -2iE_0\sin\big(\frac{\gamma\pi}{2}\big)\{\hat{y}\partial^{\gamma}_{z}[\sin(kz)]\} = -2iE_0\sin\big(\frac{\gamma\pi}{2}\big)\hat{y}k^{\gamma}\sin k(z+\frac{\gamma\pi}{2}), \\ \nabla\times^{\gamma}\eta\bar{H} &= -2E_0\sin\big(\frac{\gamma\pi}{2}\big)\{\hat{x}\partial^{\gamma}_{z}[\cos(kz)]\} = -2E_0\sin\big(\frac{\gamma\pi}{2}\big)\hat{x}k^{\gamma}\cos k(z+\frac{\gamma\pi}{2}). \end{aligned} \tag{32}$$

$$\begin{aligned} \bar{E}_f &= -2ie^{\frac{-i\gamma\pi}{2}}E_0\sin\big(\frac{\gamma\pi}{2}\big)\hat{y}\sin k(z+\frac{\gamma\pi}{2}), \\ \eta\bar{H}_f &= -2E_0 e^{\frac{-i\gamma\pi}{2}}\sin\big(\frac{\gamma\pi}{2}\big)\hat{x}\cos k(z+\frac{\gamma\pi}{2}). \end{aligned} \tag{33}$$

Using eqn (33) we get the impedance as

$$Z_f = \frac{E_{f-\hat{y}}}{H_{f-\hat{x}}} = \frac{i}{\eta}\tan k(z+\frac{\gamma\pi}{2}). \tag{34}$$

At $z=0$ we have $Z_f = \frac{i}{\eta}\tan(\frac{\gamma\pi}{2})$.

For $\gamma = 0, Z_f = 0$ implies that a flat surface in x-y plane located at $z = 0$ is a perfect electric conductor (PEC).
For $\gamma = 1, Z_f = \infty$ the surface is a perfect magnetic conductor (PMC). For $0 < \gamma < 1$ the surface is between PEC and PMC and we get fractional fields of the standing wave.

9. Conclusion

In this paper we have given elaborate formulations for fractional curl and fractional fields of different forms of electromagnetic wave propagation. As new definition has been used for these formulations hence this mechanism has added a new dimension to fractional operator. We have proved that the surface behaves as PEC at $\gamma = 0$, as PMC at $\gamma = 1$ and between PEC and PMC when $0 < \gamma < 1$. These derivations will be useful in study of polarization, reflection, impedance, boundary conditions where fractional solutions have applications.

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