J. Appl. Math. & Informatics Vol. **41***(2023), No. 4, pp. 781 - 799*
https://doi.org/10.14317/jami.2023.781

ON BIPOLAR $M-N-$MULTI $Q-$FUZZY SUBGROUPS

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Abstract. For any bipolar multi $Q-$fuzzy set δ of an universe set G, we redefined a normal, conjugate concepts, union and product operations of a bipolar $M-N-$multi $Q-$fuzzy subgroups and we discuss some of its properties. On the other hand, we introduce and define the level subsets positive $\beta-$cut and negative $\alpha-$cut of bipolar $M-N-$ multi $Q-$ fuzzy subgroup and discuss some of its related properties.

AMS Subject Classification : 03G25, 08A72, 03E72, 06F35.
Key words and phrases : Bipolar $Q-$fuzzy sets, multi–fuzzy sets, $M-N-$groups, bipolar $M-N-$multi$-Q-$fuzzy subgroup, bipolar $M-N-$multi $Q-$normal fuzzy subgroup, positive $\beta-$cut, negative $\alpha-$cut, conjugate.

1. Introduction

Fuzzy set concept was initially gave in 1965 by Zadeh [1]. Then it has become a vigorous research area in graph theory, engineering, economics, social science and medical science etc. In 1971, Rosenfeld [2] introduced fuzzy subgroups idea and the normal fuzzy subgroup was introduced by Wu [3] in 1981. Since then some authors discussed and introduced some properties of fuzzy subgroups [4, 5]. The membership degree of elements range over $[0, 1]$ in fuzzy sets. The degree of membership expresses the belongingness degree of elements to a fuzzy set. The membership degree 0 indicates that an element does not belong to fuzzy set while the membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set. On the interval $(0, 1)$ the degrees of membership indicate the partial membership to the fuzzy set. Some times, the degree of membership means that the satisfaction elements degree to constraint corresponding or some property to a fuzzy set. In 2004, Zhang et al [6] introduced a bipolar fuzzy sets concept as a generalization of fuzzy sets where the range of membership degree is increased to $[-1, 1]$. In a bipolar fuzzy set, the membership degrees on $(0, 1]$ means that elements some what satisfy the property, the

Received September 18, 2020. Revised February 8, 2023. Accepted May 4, 2023.
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membership degree 0 means that elements are irrelevant to the corresponding property and the membership degrees on $[-1, 0)$ means that elements some what satisfy the implicit counter property. Many interesting results of group, ring and graph theory have been obtained by embedding the ideas of bipolar fuzzy sets. For examples, bipolar fuzzy groups [7], bipolar fuzzy rings [8] and bipolar fuzzy graphs [9, 10].

Algebraic structures on bipolar $Q-$fuzzy set were firstly discussed by the definition of Arockia selvi et al [11]. Massa'deh and Fora [12] investigated some properties of bipolar $Q-$fuzzy $HX-$subgroups of a $HX-$ group. Anti bipolar $Q-$fuzzy normal semigroups is defined by Massa'deh [13], on the other hand Massa'deh and Ismail [14] applied this concept to H-ideals over i-Hemiring, Solairaju and Prasanna [15] studied new structure of fuzzy fully invariant bipolar $Q-$fuzzy regular lattices. In 2012, Massa'deh [16] introduced $M-N-$homomorphism and $M-N-$anti homomorphism over $M-N-$fuzzy subgroups, since then some authors discussed some properties on this concept. For examples (see [17, 18]). Ramakrishnan [19] introduced and studied a multi – fuzzy set theory in terms of multi – dimensional membership functions, multi – fuzzy set theory is an extension of fuzzy set theory.

Many interesting results of multi fuzzy set have been obtained by embedding the ideas of fuzzy sets. For examples, intuitionistic multi – anti fuzzy subgroups [20], bipolar multi – fuzzy sub algebra of Bg – Algebra [21].

In this paper, we introduce some basic properties and concepts of bipolar multi $Q-$fuzzy set. Furthermore, we study bipolar $M-N-$multi – $Q-$fuzzy subgroups with help of some properties of their positive $\beta-$cut and negative $\alpha-$cut sets, also we define the notion of union and product of bipolar $M-N-$multi – $Q-$fuzzy and discuss some of its properties on bipolar $M-N-$multi –$Q-$fuzzy subgroups. This paper is an attempt to combine the two notions: bipolar $Q-$fuzzy and multi – $Q-$fuzzy sets together by introducing two new concepts called bipolar multi – $Q-$fuzzy sets and bipolar $M-N-$ multi – $Q-$ fuzzy subgroups.

2. Preliminaries

Definition 2.1. [1] Let G be a non empty set. A fuzzy set is just a function $\delta : G \to [0, 1]$.

Definition 2.2. [19] If G is a non empty set. A multi–fuzzy set γ of G is defined as $\gamma = \{< g, \delta_\gamma(g) >; g \in G\}$ where $\delta_\gamma = (\delta_1, \delta_2, \delta_3, \ldots, \delta_r)$, that is $\delta_\gamma(g) = (\delta_1(g), \delta_2(g), \delta_3(g), \ldots, \delta_r(g))$ and $\delta_i : G \to [0, 1], \forall i = 1, 2, \ldots, r$, such that r is the finite dimension of γ. And note that, for all $i, \delta_i(g)$ is decreasingly ordered sequence of elements, that is, $\delta_1(g) \geq \delta_2(g) \geq \ldots \geq \delta_r(g), \forall g \in G$.

Definition 2.3. [6] If G is a non empty set and Q is a non empty arbitrary set. A bipolar $Q-$fuzzy set δ in $G \times Q$ is an object having the form $\delta = < g, q >; \delta^+(g, q), \delta^-(g, q); g \in G, q \in Q$ such that $\delta^+ : G \times Q \to [0, 1]$ and $\delta^- : G \times Q \to [-1, 0]$ are the mappings. The positive membership degree $\delta^+(g, q)$ denotes the satisfaction degree of an element (g, q) to the property corresponding to a bipolar $Q-$fuzzy set δ and the negative membership degree $\delta^{-}(g,q)$ denotes the satisfaction degree of an element (g,q) to some implicit counter property corresponding to a bipolar $Q-$fuzzy set δ. If $\delta^{+}(g,q)=(0,q)$ and $\delta^{-}(g,q)=(0,q)$, it is the situation that (g,q) is regarded as having only positive satisfaction for δ and if $\delta^{+}(g,q)=(0,q)$ and $\delta^{-}(g,q)\neq(0,q)$, it is the situation that (g,q) does not satisfy the property of δ. It is possible for an element (g,q) to be such that $\delta^{+}(g,q)\neq(0,q)$ and $\delta^{-}(g,q)\neq(0,q)$ when the membership function of the property overlaps that of its counter property one some portion of $G\times Q$. For the take of simplicity, we shall use the symbol $\delta=(\delta^{+},\delta^{-})$ for the bipolar $Q-$fuzzy set $\delta=\{<g,q>;\delta^{+}(g,q),\delta^{-}(g,q);g\in G,q\in Q\}$.

Definition 2.4. If G and Q are non empty sets, a bipolar multi $Q-$fuzzy set δ in G is defined as an an object having the form $\delta=\{<g,q>;\delta_i^{+}(g,q),\delta_i^{-}(g,q);g\in G,q\in Q\}$ such that $\delta_i^{+}:G\times Q\rightarrow[0,1]$ and $\delta_i^{-}:G\times Q\rightarrow[-1,0]$ are the mappings. The positive membership degree $\delta_i^{+}(g,q)$ denotes the satisfaction degree of an element (g,q) to the property corresponding to a bipolar $Q-$fuzzy set δ and the negative membership degree $\delta_i^{-}(g,q)$ denotes the satisfaction degree of an element (g,q) to some implicit counter property corresponding to a bipolar $Q-$fuzzy set δ. If $\delta_i^{+}(g,q)=(0,q)$ and $\delta_i^{-}(g,q)=(0,q)$, it is the situation that (g,q) is regarded as having only positive satisfaction for δ and if $\delta_i^{+}(g,q)=(0,q)$ and $\delta_i^{-}(g,q)\neq(0,q)$, it is the situation that (g,q) does not satisfy the property of δ. It is possible for an element (g,q) to be such that $\delta_i^{+}(g,q)\neq(0,q)$ and $\delta_i^{-}(g,q)(0,q)$ when the membership function of the property overlaps that of its counter property one some portion of $G\times Q$, where $i=1,2,\dots,r$.

Definition 2.5. [16] If M,N are left and right operator sets of group G respectively, if $(mg)n=m(gn)$ for all $g\in G,n\in N$ and $m\in M$. Then G is called $M-N-$group.

Definition 2.6. If $\delta_1=<g,q>;\delta_{1i}^{+}(g,q),\delta_{1i}^{-}(g,q);g\in G,q\in Q$ and $\delta_2=\{<g,q>;\delta_{2i}^{+}(g,q),$
$\delta_{2i}^{-}(g,q);g\in G,q\in Q\}$ are any two bipolar multi $Q-$fuzzy sets having the same dimension k of $G\times Q$. Then:

(1) $\delta_1\subseteq\delta_2$ if and only if $\delta_{1i}^{+}(g,q)\leq\delta_{2i}^{+}(g,q)$ and $\delta_{1i}^{-}(g,q)\geq\delta_{2i}^{-}(g,q)$ for all $g\in G$ and $q\in Q$.
(2) $\delta_1=\delta_2$ if and only if $\delta_{1i}^{+}(g,q)=\delta_{2i}^{+}(g,q)$ and $\delta_{1i}^{-}(g,q)=\delta_{2i}^{-}(g,q)$ for all $g\in G$ and $q\in Q$.
(3) $\delta_1\cap\delta_2=\{<g,q>;\delta_{1i}^{+}\cap\delta_{2i}^{+}(g,q),\delta_{1i}^{-}\cap\delta_{2i}^{-}(g,q);g\in G,q\in Q\}$ where $\delta_{1i}^{+}\cap\delta_{2i}^{+}(g,q)=\min\{\delta_{1i}^{+}(g,q),\delta_{2i}^{+}(g,q)\}$ and $\delta_{1i}^{-}\cap\delta_{2i}^{-}(g,q)$ $=\max\delta_{1i}^{-}(g,q),\delta_{2i}^{-}(g,q)$.
(4) $\delta_1\cup\delta_2=\{<g,q>;\delta_{1i}^{+}\cup\delta_{2i}^{+}(g,q),\delta_{1i}^{-}\cup\delta_{2i}^{-}(g,q);g\in G,q\in Q\}$ where $\delta_{1i}^{+}\cup\delta_{2i}^{+}(g,q)=\min\{\delta_{1i}^{+}(g,q),\delta_{2i}^{+}(g,q)\}$ and $\delta_{1i}^{-}\cup\delta_{2i}^{-}(g,q)$ $=\max\{\delta_{1i}^{-}(g,q),\delta_{2i}^{-}(g,q)\}$.

Here $\{\delta_{1i}^{+}(g,q),\delta_{2i}^{+}(g,q)\}$ represent the corresponding i^{th} position membership values of δ_1 and δ_2 respectively. Also $\{\delta_{1i}^{-}(g,q),\delta_{2i}^{-}(g,q)\}$ represent the corresponding i^{th} position non-membership values of δ_1 and δ_2 respectively.

Definition 2.7. If $\delta_1 = (\delta_{1i}^{+},\delta_{1i}^{-}),\delta_2 = (\delta_{2i}^{+},\delta_{2i}^{-})$ are bipolar multi $Q-$fuzzy subsets of the set G_1 and G_2 respectively. Then a product $\delta_1\times\delta_2 = ((\delta_{1i}\times\delta_{2i})^{+},(\delta_{1i}\times\delta_{2i})^{-})$ where $(\delta_{1i}\times\delta_{2i})^{+} : (G_1\times G_2)\times Q\to[0,1]$ and $(\delta_{1i}\times\delta_{2i})^{-} : (G_1\times G_2)\times Q\to[-1,0]$ are mapping defined by:

(1) $(\delta_{1i}\times\delta_{2i})^{+}((g_1,g_2),q) = \min\{\delta_{1i}^{+}(g_1,q),\delta_{2i}^{+}(g_2,q)\}$
(2) $(\delta_{1i}\times\delta_{2i})^{-}((g_1,g_2),q) = \max\{\delta_{1i}^{-}(g_1,q),\delta_{2i}^{-}(g_2,q)\}$, for all $g_1\in G_1, g_2\in G_2$ and $q\in Q$.

Definition 2.8. If G is $M-N-$group. A bipolar $Q-$fuzzy set δ in G is a bipolar $M-N-Q-$fuzzy subgroup of G, if for all $g_1,g_2G,q\in Q,m\in M$ and $n\in N$ the following axioms are satisfies:

(1) $\delta^{+}(m(g_1g_2)n,q)\geq\min\{\delta^{+}(mg_1n,q),\delta^{+}(mg_2n,q)\}$
(2) $\delta^{-}(m(g_1g_2)n,q)\leq\max\{\delta^{-}(mg_1n,q),\delta^{-}(mg_2n,q)\}$
(3) $\delta^{+}(mg_1^{-1}n,q)=\delta^{+}(mg_1n,q)$ and $\delta^{-}(mg_1^{-1}n,q)=\delta^{-}(mg_1n,q)$.

3. Main results

In this section we will study concepts of bipolar $M-N-$multi $Q-$fuzzy normal subgroups, positive $\beta-$cut, negative $\alpha-$cut and some operations on bipolar $M-N-$multi $Q-$fuzzy subgroups. We present some results

Definition 3.1. A bipolar multi $Q-$ fuzzy set $\delta=\{<g,q>;\delta_i^{+}(g,q),\delta_i^{-}(g,q);g\in G,q\in Q\}$ of an $M-N-$group G is called bipolar $M-N-$multi $Q-$fuzzy subgroup of G if it satisfies the following axioms:

(1) $\delta_i^{+}(m(g_1g_2)n,q)\geq\min\{\delta_i^{+}(mg_1n,q),\delta_i^{+}(mg_2n,q)\}$
(2) $\delta_i^{+}(mg_1^{-1}n,q)=\delta_i^{+}(mg_1n,q)$
(3) $\delta_i^{-}(m(g_1g_2)n,q)\leq\max\{\delta_i^{-}(mg_1n,q),\delta_i^{-}(mg_2n,q)\}$
(4) $\delta_i^{-}(mg_1^{-1}n,q)=\delta_i^{-}(mg_1n,q)$, For all $g_1,g_2\in G,q\in Q,m\in M$ and $n\in N$.

Lemma 3.2. *A bipolar multi $Q-$fuzzy set $\delta=\{<g,q>;\delta_i^{+}(g,q),\delta_i^{-}(g,q);g\in G,q\in Q\}$ of an $M-N-$group G is called bipolar $M-N-$multi $Q-$fuzzy subgroup of G if:*

(1) $\delta_i^{+}(m(g_1g_2^{-1})n,q)\geq\min\{\delta_i^{+}(mg_1n,q),\delta_i^{+}(mg_2n,q)\}$
(2) $\delta_i^{-}(m(g_1g_2^{-1})n,q)\leq\max\{\delta_i^{-}(mg_1n,q),\delta_i^{-}(mg_2n,q)\}$, *for all* $g_1,g_2\in G,q\in Q,m\in M$ *and* nN.

Proof. Straightforward. □

Theorem 3.3. *If δ is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of G. Then*

(1) $\delta_i^{+}(mgn,q)\leq\delta_i^{+}(men,q)$ *and* $\delta_i^{-}(mgn,q)\geq\delta_i^{-}(men,q)$ *for all* $g\in G,q\in Q,m\in M,n\in N$ *and e is the identity element of G.*

(2) *The subset* $H=\{g\in G, q\in Q, m\in M$ *and* $n\in N;\delta_i^+(mgn,q)=\delta_i^+(men,q)$ *and* $\delta_i^-(mgn,q)=\delta_i^-(men,q)\}$ *is* $M-N-$*subgroup of* G.

Proof. 1. If $g\in G, q\in Q, m\in M, n\in N$

$$\begin{aligned}\delta_i^+(men,q) &= \delta_i^+(m(gg^{-1})n,q)\\ &\geq\ min\{\delta_i^+(mgn,q),\delta_i^+(mg^{-1}n,q)\}\\ &\geq\ min\{\delta_i^+(mgn,q),\delta_i^+(mgn,q)\}\\ &= \delta_i^+(mgn,q).\end{aligned}$$

Then $\delta_i^+(men,q)\geq\delta_i^+(mgn,q)$.
Similarly

$$\begin{aligned}\delta_i^-(men,q) &= \delta_i^-(m(gg^{-1})n,q)\\ &\leq\ max\{\delta_i^-(mgn,q),\delta_i^-(mg^{-1}n,q)\}\\ &\leq\ max\{\delta_i^-(mgn,q),\delta_i^-(mgn,q)\}\\ &= \delta_i^-(mgn,q).\end{aligned}$$

Thus $\delta_i^-(men,q)\leq\delta_i^-(mgn,q)$, for all $g\in G, q\in Q, m\in M, n\in N$.

2. Since $H\neq\phi, e\in H$. If $g_1,g_2\in H$ then $\delta_i^+(mg_1n,q)=\delta_i^+(mg_2n,q)=\delta_i^+(men,q)$,
$\delta_i^-(mg_1n,q)=\delta_i^-(mg_2n,q)=\delta_i^-(men,q)$.

$$\begin{aligned}\delta_i^+(m(g_1g_2^{-1})n,q) &\geq \min\{\delta_i^+(mg_1n,q),\delta_i^+(mg_2^{-1}n,q)\}\\ &= \min\{\delta_i^+(mg_1n,q),\delta_i^+(mg_2n,q)\}\\ &= \min\{\delta_i^+(men,q),\delta_i^+(men,q)\}\\ &= \delta_i^+(men,q).\end{aligned}$$

That is,$\delta_i^+(m(g_1g_2^{-1})n,q)\geq\delta_i^+(men,q)$ and obviously $\delta_i^+(m(g_1g_2^{-1})n,q)\leq\delta_i^+(men,q)$. Hence $\delta_i^+(m(g_1g_2^{-1})n,q)=\delta_i^+(men,q)$. $(\star)$ Also

$$\begin{aligned}\delta_i^-(m(g_1g_2^{-1})n,q) &\leq \max\{\delta_i^-(mg_1n,q),\delta_i^-(mg_2^{-1}n,q)\}\\ &= \max\{\delta_i^-(mg_1n,q),\delta_i^-(mg_2n,q)\}\\ &= \max\{\delta_i^-(men,q),\delta_i^-(men,q)\}\\ &= \delta_i^-(men,q).\end{aligned}$$

That is,$\delta_i^-(m(g_1g_2^{-1})n,q)\leq\delta_i^-(men,q)$ and obviously $\delta_i^-(m(g_1g_2^{-1})n,q)\geq\delta_i^-(men,q)$. Hence $\delta_i^-(m(g_1g_2^{-1})n,q)=\delta_i^-(men,q)$. $(\star\star)$
Therefore by $(\star)$ and $(\star\star)$, $m(g_1g_2^{-1})n\in H$. Then H is $M-N-$ subgroup of G. □

Theorem 3.4. *If* δ *is a bipolar* $M-N-$ *multi* $Q-$ *fuzzy subgroup of* $M-N-$*group* G *with identity* e*, then:*
1. If $\delta_i^+(m(g_1g_2^{-1})n,q)=\delta_i^+(men,q)$ *then* $\delta_i^+(mg_1n,q)=\delta_i^+(mg_2n,q)$.
2. If $\delta_i^-(m(g_1g_2^{-1})n,q)=\delta_i^-(men,q)$ *then* $\delta_i^-(mg_1n,q)=\delta_i^-(mg_2n,q)$*, for all* $g_1,g_2\in G, q\in Q, m\in M$ *and* $n\in N$.

Proof. For all $g_1, g_2 \in G, q \in Q, m \in M$ and $n \in N$

$$\begin{aligned} 1.\delta_i^+(mg_1n, q) &= \delta_i^+(m(g_1e)n, q) \\ &= \delta_i^+(m(g_1g_2^{-1}g_2)n, q) \\ &\geq \min\{\delta_i^+(mg_1g_2^{-1}n, q), \delta_i^+(mg_2n, q)\} \\ &= \min\{\delta_i^+(men, q), \delta_i^+(mg_2n, q)\} \\ &= \delta_i^+(mg_2n, q). \end{aligned}$$

Thus $\delta_i^+(mg_1n, q) \geq \delta_i^+(mg_2n, q)$.
Now, $\delta_i^+(mg_2n, q) = \delta_i^+(mg_2^{-1}n, q)$, since δ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G.

$$\begin{aligned} \delta_i^+(mg_2n, q) &= \delta_i^+(m(eg_2^{-1})n, q) \\ &= \delta_i^+(m((g_1^{-1}g_1)g_2^{-1})n, q) \\ &= \delta_i^+(m(g_1^{-1}(g_1g_2^{-1}))n, q) \\ &\geq \min\{\delta_i^+(mg_1^{-1}n, q), \delta_i^+(m(g_1g_2^{-1})n, q)\} \quad lll \\ &= \min\{\delta_i^+(mg_1n, q), \delta_i^+(men, q)\} \\ &= \delta_i^+(mg_1n, q) \end{aligned}$$

Hence $\delta_i^+(mg_1n, q) = \delta_i^+(mg_2n, q)$.

$$\begin{aligned} 2.\delta_i^-(mg_1n, q) &= \delta_i^-(m(g_1e)n, q) \\ &= \delta_i^-(m(g_1g_2^{-1}g_2)n, q) \\ &\leq \max\{\delta_i^-(mg_1g_2^{-1}n, q), \delta_i^-(mg_2n, q)\} \\ &= \max\{\delta_i^-(men, q), \delta_i^-(mg_2n, q)\} \\ &= \delta_i^-(mg_2n, q). \end{aligned}$$

Thus $\delta_i^-(mg_1n, q) \leq \delta_i^-(mg_2n, q)$.
Now, $\delta_i^-(mg_2n, q) = \delta_i^-(mg_2^{-1}n, q)$, since δ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G.

$$\begin{aligned} \delta_i^-(mg_2n, q) &= \delta_i^-(m(eg_2^{-1})n, q) \\ &= \delta_i^-(m((g_1^{-1}g_1)g_2^{-1})n, q) \\ &= \delta_i^-(m(g_1^{-1}(g_1g_2^{-1}))n, q) \\ &\leq \max\{\delta_i^-(mg_1^{-1}n, q), \delta_i^-(m(g_1g_2^{-1})n, q)\} \\ &= \max\{\delta_i^-(mg_1n, q), \delta_i^-(men, q)\} \\ &= \delta_i^-(mg_1n, q) \end{aligned}$$

Hence $\delta_i^-(mg_1n, q) = \delta_i^-(mg_2n, q)$. □

Definition 3.5. A bipolar $M-N-$multi $Q-$fuzzy subgroup δ of G is called bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G if it satisfies

(1) $\delta_i^+(m(g_1g_2)n, q) = \delta_i^+(m(g_2g_1)n, q)$
(2) $\delta_i^-(m(g_1g_2)n, q) = \delta_i^-(m(g_2g_1)n, q)$, for all $g_1, g_2 \in G, q \in Q, m \in M$ and $n \in N$.

Theorem 3.6. *A bipolar $M-N-$multi $Q-$fuzzy subgroup δ of G is bipolar $M-N-$multi $Q-$normal fuzzy subgroup if the following axioms are satisfied*

(1) $\delta_i^+(m(g^{-1}hg)n,q)=\delta_i^+(mhn,q)$
(2) $\delta_i^-(m(g^{-1}hg)n,q)=\delta_i^-(mhn,q)$, *for all* $g\in G, h\in\delta, q\in Q, m\in M$ *and* $n\in N$.

Proof.

$$\begin{aligned}\delta_i^+(m(g^{-1}hg)n,q) &= \delta_i^+(m(g^{-1}(hg))n,q)\\ &= \delta_i^+(m((hg)g^{-1})n,q) \text{ Since } \delta \text{ is normal}\\ &= \delta_i^+(m(h(gg^{-1}))n,q)\\ &= \delta_i^+(m((he)n,q)\\ &= \delta_i^+(mhn,q).\end{aligned}$$

Also

$$\begin{aligned}\delta_i^-(m(g^{-1}hg)n,q) &= \delta_i^-(m(g^{-1}(hg))n,q)\\ &= \delta_i^-(m((hg)g^{-1})n,q) \text{ Since } \delta \text{ is normal}\\ &= \delta_i^-(m(h(gg^{-1}))n,q)\\ &= \delta_i^-(m((he)n,q)\\ &= \delta_i^-(mhn,q).\end{aligned}$$

□

Lemma 3.7. *If δ is a bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G, then for any $g_2\in G$ we have $\delta(m(g_2^{-1}g_1g_2)n,q)=\delta(m(g_2g_1g_2^{-1})n,q)$.*

Proof.

$$\begin{aligned}\delta_i^+(m(g_2^{-1}g_1g_2)n,q) &= \delta_i^+(m(g_1g_2^{-1}g_2)n,q)\\ &= \delta_i^+(m(g_1)n,q)\\ &= \delta_i^+(m(g_2g_2^{-1}g_1)n,q)\\ &= \delta_i^+(m(g_2g_1g_2^{-1})n,q).\end{aligned}$$

And

$$\begin{aligned}\delta_i^-(m(g_2^{-1}g_1g_2)n,q) &= \delta_i^-(m(g_1g_2^{-1}g_2)n,q)\\ &= \delta_i^-(m(g_1)n,q)\\ &= \delta_i^-(m(g_2g_2^{-1}g_1)n,q)\\ &= \delta_i^-(m(g_2g_1g_2^{-1})n,q).\end{aligned}$$

Therefore $\delta(m(g_2^{-1}g_1g_2)n,q)=\delta(m(g_2g_1g_2^{-1})n,q)$. □

Proposition 3.8. *Let δ be a bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G, then $g\delta g^{-1}$ is also bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G for all $g\in G$.*

Proof. Since δ is a bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G, then $g\delta g^{-1}$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G, for all $g\in G$. Now

$$\begin{aligned}g\delta_i^+g^{-1}(m(vwv^{-1})n,q) &= \delta_i^+(g^{-1}(m(vwv^{-1})n)g,q)\\ &= \delta_i^+(m(vwv-1)n,q)\\ &= \delta_i^+(mwn,q)\\ &= \delta_i^+(g(mwn)g^{-1},q)\\ &= g\delta_i^+g^{-1}(mwn,q)\end{aligned}$$

And

$$\begin{aligned} g\delta_i^- g^{-1}(m(vwv^{-1})n, q) &= \delta_i^-(g^{-1}(m(vwv^{-1})n)g, q) \\ &= \delta_i^-(m(vwv-1)n, q) \\ &= \delta_i^-(mwn, q) \\ &= \delta_i^-(g(mwn)g^{-1}, q) \\ &= g\delta_i^- g^{-1}(mwn, q) \end{aligned}$$

Therefore $g\delta g^{-1}$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G. □

Definition 3.9. If $\delta = \{< mgn, q >; \delta_i^+(mgn, q), \delta_i^-(mgn, q) >; g \in G, q \in Q, m \in M$ and $n \in N\}$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G. For $\alpha \in [-1, 0]^k$ and $\beta \in [0, 1]^k$ the set $U(\delta_i^+, \beta) = \{< mgn, q >; \delta_i^+(mgn, q) \geq \beta; g \in G, q \in Q, m \in M$ and $n \in N\}$ is called positive $\beta-$cut of δ and the set $L(\delta_i^-, \alpha) = \{< mgn, q >; \delta_i^-(mgn, q) \leq \alpha; g \in G, q \in Q, m \in M$ and $n \in N\}$ is called negative $\alpha-$cut of δ.

Simply, we denote by $\delta_{(\alpha,\beta)}$ to $(\alpha, \beta)-$level set of δ. This means that $\delta_{(\alpha,\beta)} = \{< mgn, q >; \delta_i^+(mgn, q) \geq \beta, \delta_i^-(mgn, q) \leq \alpha; g \in G, q \in Q, m \in M$ and $n \in N\}$.

Theorem 3.10. *Let δ be a bipolar $M-N-$multi $Q-$fuzzy subgroup of G, then the positive $\beta-$cut and negative $\alpha-$cut of δ are$M-N-$subgroup of G, where $\delta_i^+(men, q) \leq \beta, \delta_i^-(men, q) \geq \alpha$ such that e is the identity element in G.*

Proof. We know $\delta_i^+(men, q) \leq \beta, \delta_i^-(men, q) \geq \alpha$ then $men \in U(\delta_i^+, \beta), men \in L(\delta_i^-, \alpha)$ and $U(\delta_i^+, \beta) \neq \phi$ and $L(\delta_i^-, \alpha) \neq \phi$. Let $mg_1n, mg_2n \in U(\delta_i^+, \beta)$ then $\delta^+(mg_1n, q) \geq \beta$ and $\delta^+(mg_2n, q) \geq \beta$, hence $\forall i \delta_i^+(mg_1n, q) \geq \beta$ and $\delta_i^+(mg_2n, q) \geq \beta$.

$$\begin{aligned} \delta_i^+(m(g_1g_2^{-1})n, q) &\geq \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2^{-1}n, q)\} \\ &= \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2n, q)\} \\ &\geq \{\beta, \beta\} \\ &= \beta. \end{aligned}$$

Thus $m(g_1g_2)n \in U(\delta_i^+, \beta)$ and hence $U(\delta_i^+, \beta)$ is $M-N-$ subgroup in G. If $mg_1n, mg_2n \in L(\delta_i^-, \alpha)$ then $\delta^-(mg_1n, q) \leq \alpha$ and $\delta^-(mg_2n, q) \leq \alpha$, hence $\forall i \delta_i^-(mg_1n, q) \leq \alpha$ and $\delta_i^-(mg_2n, q) \leq \alpha$.

$$\begin{aligned} \delta_i^-(m(g_1g_2^{-1})n, q) &\leq \max\{\delta_i^-(mg_1n, q), \delta_i^+(mg_2^{-1}n, q)\} \\ &= \max\{\delta_i^-(mg_1n, q), \delta_i^-(mg_2n, q)\} \\ &\leq \{\alpha, \alpha\} \\ &= \alpha. \end{aligned}$$

Thus $m(g_1g_2)n \in L(\delta_i^-, \alpha)$ and hence $L(\delta_i^-, \alpha)$ is $M-N-$ subgroup in G. □

Lemma 3.11. *A bipolar multi $Q-$fuzzy set δ of G is a bipolar $M-N-$ multi $Q-$fuzzy subgroup if and only if each $U(\delta_i^+, \beta)$ and $L(\delta_i^-, \alpha)$ is $M-N-$ subgroup of G, for all $\alpha \in [-1, 0]^k$ and $\beta \in [0, 1]^k$.*

Proof. Straightforward. □

Corollary 3.12. *If δ is a bipolar $M-N-$ multi $Q-$normal fuzzy subgroup of G and, $\alpha \in [-1,0]^k$ and $\beta \in [0,1]^k$ then $U(\delta_i^+,\beta)$ and $L(\delta_i^-,\alpha)$ are normal $M-N-$ subgroup of G, where $\delta_i^+(men,q) \leq \beta, \delta_i^-(men,q) \geq \alpha$ such that e is the identity element in G.*

Proof. If $mhn \in U(\delta_i^+,\beta), mhnL(i-,\ ), gG, qQ, mMandnN..$ Then $\delta^+(mhn,q) \geq \beta, \delta^-(mhn,q) \leq \alpha$, that is, $\forall i \delta_i^+(mhn,q) \geq \beta$ and $\delta_i^-(mhn,q) \leq \alpha$. Since δ is a bipolar $M-N-$ multi $Q-$ normal fuzzy subgroup of G then $\delta_i^+(m(g-1hg)n,q) = \delta_i^+(mhn,q)$ and $\delta_i^-(m(g^{-1}hg)n,q) = \delta_i^-(mhn,q)$ therefore $\delta_i^+(m(g-1hg)n,q) \geq \beta_i$ and $\delta_i^-(m(g-1hg)n,q) \leq \alpha_i$ for all i thus $\delta^+(m(g^{-1}hg)n,q) \geq \beta$ and $\delta^-(m(g-1hg)n,q) \leq \alpha$ and we get $m(g^{-1}hg)n \in U(\delta_i^+,\beta), m(g^{-1}hg)n \in L(\delta_i^-,\alpha)$. Then $U(\delta_i^+,\beta)$ and $L(\delta_i^-,\alpha)$ are normal $M-N-$subgroup of $M-N-$ □

Theorem 3.13. *Let δ be a bipolar multi Q–fuzzy set of G is a bipolar $M-N-$ multi $Q-$fuzzy subgroup if and only if each $U(\delta_i^+,\beta)$ and $L(\delta_i^-,\alpha)$ is $M-N-$subgroup of G, for all $\alpha \in [-1,0]^k$ and $\beta \in [0,1]^k$.*

Proof. ($\Rightarrow$) Since δ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G, then by Lemma 3.11 each $U(\delta_i^+,\beta)$ and $L(\delta_i^-,\alpha)$ is $M-N-$subgroup of G. ($\Leftarrow$) If δ be a bipolar multi Q–fuzzy set of G such that each $U(\delta_i^+,\beta)$ and $L(\delta_i^-,\alpha)$ is $M-N-$subgroup. we need to show δ is a bipolar $M-N-$multi $Q-$fuzzy subgroup, we prove

1. $\delta^+(m(g_1g_2)n,q) \geq \min\{\delta^+(mg_1n,q), \delta^+(mg_2n,q)\}$ and $\delta^-(m(g_1g_2)n,q) \leq \max\{\delta^-(mg_1n,$
$q), \delta^-(mg_2n,q)\}$ for all $g_1,g_2 \in G, q \in Q, m \in M$ and $n \in N$. $\forall i$, let $\beta_i = \min\{\delta_i^+(mg_1n,q),$
$\delta_i^+(mg_2n,q)\}$ and $\alpha_i = \max\{\delta_i^-(mg_1n,q), \delta_i^-(mg_2n,q)\}$ we have $\delta_i^+(mg_1n,q) \geq \beta_i, \delta_i^+(mg_2n,q) \geq \beta_i$ and $\delta_i^-(mg_1n,q) \leq \alpha_i, \delta_i^-(mg_2n,q) \leq \alpha_i, \forall i$ then we have $\delta^+(mg_1n,q) \geq \beta, \delta^+(mg_2n,q) \geq \beta$ and $\delta^-(mg_1n,q) \leq \alpha, \delta^-(mg_2n,q) \leq \alpha$. That is $mg_1n, mg_2n \in U(\delta_i^+,\beta)$ and $mg_1n, mg_2n \in L(\delta_i^-,\alpha)$, thus $m(g_1g_2)n \in U(\delta_i^+,\beta)$ and $m(g_1g_2)n \in L(\delta_i^-,\alpha)$ since each $U(\delta_i^+,\beta), L(\delta_i^-,\alpha)$ is an $M-N-$subgroup by hypothesis. Therefore, for all i, we have $\delta_i^+(m(g_1g_2)n,q) \geq \beta_i = \min\{\delta_i^+(mg_1n,q), \delta_i^+(mg_2n,q)\}$ and $\delta_i^-(m(g_1g_2)n,q) \leq \alpha_i = \max\{\delta_i^-(mg_1n,q),$ $\delta_i^-(mg_2n,q)\}$ that is, $\delta^+(m(g_1g_2)n,q) \geq \min\{\delta^+(mg1n,q), \delta^+(mg_2nn,q)\}$ and $\delta^-(m(g_1g_2)n,q) \leq \max\{\delta^-(mg_1n,q), \delta^-(mg_2n,q)\}$.

2. If $g \in G, q \in Q, m \in M$ and $n \in N$, if $\delta_i^+(mgn,q) = \beta_i$ and $\delta_i^-(mgn,q) = \alpha_i$ for all i then $\delta_i^+(mgn,q) \geq \beta_i$ and $\delta_i^-(mgn,q) \leq \alpha_i$ therefore $\delta^+(mgn,q) \geq \beta$ and $\delta^-(mgn,q) \leq \alpha$. Thus $g \in L(\delta_i^-,\alpha)$ and $g \in U(\delta_i^+,\beta)$ but each $L(\delta_i^-,\alpha), U(\delta_i^+,\beta)$ is an $M-N-$subgroup of G, we have $g^{-1} \in L(\delta_i^-,\alpha)$ and $g^{-1}U(\delta_i^+,\beta)$ which implies that $\delta_i^+(mg^{-1}n,q) \geq \beta_i$ and $\delta_i^-(mg^{-1}n,q) \leq \alpha_i$ for all $i, \delta_i^+(mg^{-1}n,q) \geq \delta_i^+(mgn,q)$ and $\delta_i^-(mg^{-1}n,q) \leq \delta_i^-(mgn,q)$ for all i.
Therefore, for all $i, \delta_i^+(mgn,q) = \delta_i^+(m(g^{-1})^{-1}n,q) \geq \delta_i^+(mg^{-1}n,q) \geq \delta_i^+(mgn,q)$ which implies that $\delta_i^+(mg^{-1}n,q) = \delta_i^+(mgn,q)$ for all i and hence $\delta^+(mg^{-1}n,q) = \delta^+(mgn,q)$. Also for all $i, \delta_i^-(mgn,q) = \delta_i^-(m(g^{-1})^{-1}n,q) \leq \delta_i^-(mg^{-1}n,q)$

$\leq \delta_i^-(mgn, q)$ which implies that $\delta_i^-(mg^{-1}n, q) = \delta_i^-(mgn, q)$ for all i and hence $\delta^-(mg^{-1}n, q) = \delta^-(mgn, q)$. Therefore δ is bipolar $M - N-$ multi $Q-$fuzzy subgroup of G. □

Theorem 3.14. *Any $M-N-$subgroup H of an $M-N-$group G can be realized as a positive β-cut and negative α-cut $M-N-$subgroup of some bipolar $M-N-$multi $Q-$fuzzy subgroup of G.*

Proof. If δ is a $M-N-$multi $Q-$fuzzy subset and $g \in G, q \in Q, m \in M$ and $n \in N$. Define $\delta_i^+(mgn, q) = 0$, if $g \in H, \delta_i^+(mgn, q) = \beta_i$ if $g \in H$ and $\delta_i^-(mgn, q) = 0$, if $g \in H, \delta_i^-(mgn, q) = \alpha_i$ if $g \in H$, we need to show δ is bipolar $M-N-$multi $Q-$fuzzy subgroup of G. Let $g_1, g_2 \in G$.

1.Suppose that $mg_1n, mg_2n \in H$, then $m(g_1g_2)n \in H$ and $m(g_1g_2^{-1})n \in H, \delta_i^+(mg_1n, q) = \beta_i^+(mg2n, q) = 0$ this implies that

$$\begin{aligned}\delta_i^+(m(g_1g_2^{-1})n, q) &= 0\\ &\geq \min\{0, 0\}\\ &= \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2n, q)\}.\end{aligned}$$

Therefore $\delta_i^+(m(g_1g_2^{-1})n, q) \geq \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2n, q)\}$.
And $\delta_i^-(mg_1n, q) = \delta_i^-(mg_2n, q) = 0$ this implies that

$$\begin{aligned}\delta_i^-(m(g_1g_2^{-1})n, q) &= 0\\ &\leq \max\{0, 0\}\\ &= \max\{\delta_i^-(mg_1n, q), \delta_i^-(mg_2n, q)\}.\end{aligned}$$

Therefore $\delta_i^-(m(g_1g_2^{-1})n, q) \leq \max\{\delta_i^-(mg_1n, q), \delta_i^-(mg_2n, q)\}$.

2. If $mg_1n \in H, mg_2n \notin H$, then $m(g_1g_2)n \notin H$ and $m(g_1g_2^{-1})n \in H$, $\delta_i^+(mg_1n, q) = 0, \delta_i^+(mg_2n, q) = \beta_i$ this implies that

$$\begin{aligned}\delta_i^+(m(g_1g_2^{-1})n, q) &= \beta_i\\ &\geq \min\{0, \beta_i\}\\ &= \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2n, q)\}.\end{aligned}$$

Therefore $\delta_i^+(m(g_1g_2^{-1})n, q) \geq \min\{\delta_i^+(mg_1n, q), \delta_i^+(mg_2n, q)\}$. And $\delta_i^-(mg_1n, q) = 0, \delta_i^-(mg_2n, q) = \alpha_i$ this implies that

$$\begin{aligned}\delta_i^-(m(g_1g_2^{-1})n, q) &= \alpha_i\\ &\leq \max\{0, \alpha_i\}\\ &= \max\{\delta_i^-(mg_1n, q), \delta_i^-(mg_2n, q)\}.\end{aligned}$$

Therefore $\delta_i^-(m(g_1g_2^{-1})n, q) \leq \max\{\delta_i^-(mg_1n, q), \delta_i^-(mg_2n, q)\}$.

3. If $mg_1n, mg_2n \notin H$, then $m(g_1g_2)n \notin H$ or $m(g_1g_2^{-1})n \in H$, $\delta_i^+(mg_1n, q) = \delta_i^+(mg_2n, q) = \beta_i, \delta_i^-(mg_1n, q) = \delta_i^-(mg_2n, q) = \alpha_i$. Define

$$\begin{aligned}\delta_i^+(m(g_1g_2^{-1})n, q) &= 0; m(g_1g_2)n \in H\\ &= \beta_i; m(g_1g_2)n \notin H.\end{aligned}$$

And

$$\begin{aligned} \delta_i^-(m(g_1g_2^{-1})n,q) &= 0; m(g_1g_2)n \in H \\ &= \alpha_i; m(g_1g_2)n \notin H. \end{aligned}$$

Let $m(g_1g_2^{-1})n \in H$,

$$\begin{aligned} \delta_i^+(m(g_1g_2^{-1})n,q) &= \beta_i \\ &\geq \min\{\delta_i^+(mg_1n,q), \delta_i^+(mg_2n,q)\} \end{aligned}$$

Then $\delta_i^+(m(g_1g_2^{-1})n,q) \geq \min\{\delta_i^+(mg_1n,q), \delta_i^-(mg_2^{-1}n,q)\}$.

$$\begin{aligned} \delta_i^-(m(g_1g_2^{-1})n,q) &= 0 \\ &\leq \max\{0,0\} \\ &\max\{\delta_i^-(mg_1n,q), \delta_i^-(mg_2n,q)\}. \end{aligned}$$

Then $\delta_i^-(m(g_1g_2^{-1})n,q) \leq \max\{\delta_i^-(mg_1n,q), \delta_i^-(mg_2^{-1}n,q)\}$.
Therefore in all cases, δ bipolar $M-N-$multi $Q-$fuzzy subgroup of G. For this bipolar $M-N-$multi $Q-$fuzzy subgroup $U(\delta_i^+,\beta) = H = L(\delta_i^-,\alpha)$. □

Theorem 3.15. *Let δ_1, δ_2 be two bipolar $M-N-$multi $Q-$fuzzy subgroup of G, then $\delta_1 \cup \delta_2$ is a bipolar $M-N-$Multi $Q-$fuzzy subgroup of G.*

Proof. For all $g_1, g_2 \in G, q \in Q, m \in M$ and $n \in N$
$(\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_1g_2^{-1})n,q) = \min\{\delta_{1i}^+(m(g_1g_2^{-1})n,q), \delta_{2i}^+(m(g_1g_2^{-1})n,q)\}$
$\geq \min\{\min\{\delta_{1i}^+(mg_1n,q), \delta_{1i}^+(mg_2n,q)\}, \min\{\delta_{2i}^+(mg_1n,q), \delta_{2i}^+(mg_2n,q)\}\}$
$= \min\{\min\{\delta_{1i}^+(mg_1n,q), \delta_{2i}^+(mg_1n,q)\}, \min\{\delta_{1i}^+(mg_2n,q), \delta_{2i}^+(mg_2n,q)\}\}$
$= \min\{(\delta_{1i}^+ \cup \delta_{2i}^+)(mg_1n,q), (\delta_{1i}^+ \cup \delta_{2i}^+)(mg_2n,q)\}$.
And
$(\delta_{1i}^- \cup \delta_{2i}^-)(m(g_1g_2^{-1})n,q) = \max\{\delta_{1i}^-(m(g_1g_2^{-1})n,q), \delta_{2i}^-(m(g_1g_2^{-1})n,q)\}$
$\leq \max\{\max\{\delta_{1i}^-(mg_1n,q), \delta_{1i}^-(mg_2n,q)\}, \max\{\delta_{2i}^-(mg_1n,q), \delta_{2i}^-(mg_2n,q)\}\}$
$= \max\{\max\{\delta_{1i}^-(mg_1n,q), \delta_{2i}^-(mg_1n,q)\}, \max\{\delta_{1i}^-(mg_2n,q), \delta_{2i}^-(mg_2n,q)\}\}$
$= \max\{(\delta_{1i}^- \cup \delta_{2i}^-)(mg_1n,q), (\delta_{1i}^- \cup \delta_{2i}^-)(mg_2n,q)\}$.
Then $(\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_1g_2^{-1})n,q) \geq \min\{(\delta_{1i}^+ \cup \delta_{2i}^+)(mg_1n,q), (\delta_{1i}^+ \cup \delta_{2i}^+)(mg_2n,q)\}$ and $(\delta_{1i}^- \cup \delta_{2i}^-)(m(g_1g_2^{-1})n,q) \leq \max\{(\delta_{1i}^- \cup \delta_{2i}^-)(mg_1n,q), (\delta_{1i}^- \cup \delta_{2i}^-)(mg_2n,q)\}$.
Therefore $\delta_1 \cup \delta_2$ is a bipolar $M-N-$Multi $Q-$fuzzy subgroup of G. □

Corollary 3.16. *If δ_1, δ_2 are two bipolar $M-N-$Multi $Q-$normal fuzzy subgroup of G, then $\delta_1 \cup \delta_2$ is a bipolar $M-N-$Multi $Q-$normal fuzzy subgroup of G.*

Proof. By Theorem 3.15, we know $\delta_1 \cup \delta_2$ is a bipolar $M-N-$Multi $Q-$fuzzy subgroup of G, for all $g_1, g_2 \in G, q \in Q, m \in M$ and $n \in N$. We need to prove $(\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_2g_1g_2^{-1})n,q) = (\delta_{1i}^+ \cup \delta_{2i}^+)(mg_1n,q)$ and $(\delta_{1i}^- \cup \delta_{2i}^-)(m(g_2g_1g_2^{-1})n,q) = (\delta_{1i}^- \cup \delta_{2i}^-)(mg_1n,q)$.

$$\begin{aligned} (\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_2g_1g_2^{-1})n,q) &= \min\{\delta_{1i}^+(m(g_2g_1g_2^{-1})n,q), \delta_{2i}^+(m(g_2g_1g_2^{-1})n,q)\} \\ &= \min\{\delta_{1i}^+(mg_1n,q), \delta_{2i}^+(mg_1n,q)\} \\ &= (\delta_{1i}^+ \cup \delta_{2i}^+)(mg_1n,q). \end{aligned}$$

And

$$\begin{aligned}(\delta^-_{1i}\cup\delta^-_{2i})(m(g_2g_1g_2^{-1})n,q) &= \min\{\delta^-_{1i}(m(g_2g_1g_2^{-1})n,q),\delta^-_{2i}(m(g_2g_1g_2^{-1})n,q)\}\\ &= \min\{\delta^-_{1i}(mg_1n,q),\delta^-_{2i}(mg_1n,q)\}\\ &= (\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q).\end{aligned}$$

Hence $\delta_1\cup\delta_2$ is a bipolar $M-N-$Multi $Q-$normal fuzzy subgroup of G. □

Corollary 3.17. *If $(\delta_1\cup\delta_2)_i; i\in\Delta$ is a bipolar $M-N-$ multi $Q-$normal fuzzy is subgroup of G, then $\cup_{i\in\Delta}(\delta_1\cup\delta_2)_i$ is a bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G.*

Proof. Straightforward. □

Theorem 3.18. *If G is an $M-N-$ group and $(\delta_1\cup\delta_2)$ is a bipolar multi $Q-$fuzzy subset of G. Then $(\delta_1\cup\delta_2)$ is a bipolar $M-N-$multi $Q-$fuzzy is subgroup if and only if the level set $(\delta_1\cup\delta_2)_{(\alpha,\beta)}; \alpha\in[-1,0]^k$ and $\beta\in[0,1]^k$ are bipolar $M-N-$multi $Q-$fuzzy subgroup of G.*

Proof. ($\Rightarrow$) Suppose that $(\delta_1\cup\delta_2)$ is a bipolar $M-N-$multi $Q-$normal fuzzy is subgroup of G. Let $g_1,g_2\in(\delta^+_{1i}\cup\delta^+_{2i})_\beta$, then $(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q)\geq\beta_i$ and $(\delta^+_{1i}\cup\delta^+_{2i})(mg_2n,q)\geq\beta_i$

$$\begin{aligned}(\delta^+_{1i}\cup\delta^+_{2i})(m(g_1g_2^{-1})n,q) &\geq \min\{(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q),(\delta^+_{1i}\cup\delta^+_{2i})(mg_2^{-1}n,q)\}\\ &= \min\{(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q),(\delta^+_{1i}\cup\delta^+_{2i})(mg_2n,q)\}\\ &= \min\{\beta_i,\beta_i\}\\ &= \beta.\end{aligned}$$

Then $(\delta^+_{1i}\cup\delta^+_{2i})(m(g_1g_2^{-1})n,q)\geq\beta$.

Also Let $g_1,g_2\in(\delta^-_{1i}\cup\delta^-_{2i})_\alpha$, then $(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q)\leq\alpha_i$ and $(\delta^-_{1i}\cup\delta^-_{2i})(mg_2n,q)\leq\alpha_i$

$$\begin{aligned}(\delta^-_{1i}\cup\delta^-_{2i})(m(g_1g_2^{-1})n,q) &\leq \max\{(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q),(\delta^-_{1i}\cup\delta^-_{2i})(mg_2^{-1}n,q)\}\\ &= \max\{(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q),(\delta^-_{1i}\cup\delta^+_{2i})(mg_2n,q)\}\\ &= \max\{\alpha_i,\alpha_i\}\\ &= \alpha.\end{aligned}$$

Then $(\delta^-_{1i}\cup\delta^-_{2i})(m(g_1g_2^{-1})n,q)\leq\alpha$. Then $m(g_1g_2^{-1})n\in(\delta_1\cup\delta_2)_{(\alpha,\beta)}$

($\Leftarrow$) suppose that $(\delta_1\cup\delta_2)_{(\alpha,\beta)}$ is bipolar $M-N-$multi $Q-$fuzzy subgroup of G, if $g_1,g_2\in(\delta_1\cup\delta_2)_{(\alpha,\beta)}$ then $(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q)\geq\beta_i$, $(\delta^+_{1i}\cup\delta^+_{2i})(mg_2n,q)\geq\beta_i$ and $(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q)\leq\alpha_i$ and $(\delta^-_{1i}\cup\delta^-_{2i})(mg_2n,q)\leq\alpha_i$. Also $(\delta^+_{1i}\cup\delta^+_{2i})(m(g_1g_2^{-1})n,q)\geq\beta_i$ since $m(g_1g_2^{-1})n\in(\delta^+_{1i}\cup\delta^+_{2i})_\beta=\min\{\beta_i,\beta_i\}=\min\{(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q),(\delta^+_{1i}\cup\delta^+_{2i})(mg_2n,q)\}$. Thus $(\delta^+_{1i}\cup\delta^+_{2i})(m(g_1g_2)n,q)\geq\min\{(\delta^+_{1i}\cup\delta^+_{2i})(mg_1n,q),(\delta^+_{1i}\cup\delta^+_{2i})(mg_2n,q)\}$ and $(\delta^-_{1i}\cup\delta^-_{2i})(m(g_1g_2^{-1})n,q)\leq\alpha_i$ since $m(g_1g_2^{-1})n\in(\delta^-_{1i}\cup\delta^-_{2i})_\alpha=\max\{\alpha_i,\alpha_i\}=\max\{(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q),(\delta^-_{1i}\cup\delta^-_{2i})(mg_2n,q)\}$. Thus $(\delta^-_{1i}\cup\delta^-_{2i})(m(g_1g_2)n,q)\leq\max\{(\delta^-_{1i}\cup\delta^-_{2i})(mg_1n,q),$ $(\delta^-_{1i}\cup\delta^-_{2i})(mg_2n,q)\}$. Hence $(\delta_1\cup\delta_2)$ is a bipolar $M-N-$multi $Q-$fuzzy is subgroup of G.

□

Proposition 3.19. *If G is an $M-N-$ group and $(\delta_1 \cup \delta_2)$ is a bipolar multi $Q-$fuzzy subset of G. Then $(\delta_1 \cup \delta_2)$ is a bipolar $M-N-$ multi $Q-$normal fuzzy is subgroup if and only if the level set$(\delta_1 \cup \delta_2)_{(\alpha,\beta)}; \alpha \in [-1,0]^k$ and $\beta \in [0,1]^k$ is bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G.*

Proof. Since $(\delta_1 \cup \delta_2)$ is a bipolar $M-N-$multi $Q-$normal fuzzy is subgroup of G. If $g_1 \in G$ and $g_2 \in (\delta_1 \cup \delta_2)_{(\alpha,\beta)}$ then $(\delta_{1i}^+ \cup \delta_{2i}^+)(mg_2n, q) \geq \beta_i$ for $q \in Q, m \in M$ and $n \in N$. Now $(\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_1g_2g_1^{-1})n, q) = (\delta_{1i}^+ \cup \delta_{2i}^+)(mg_2n, q) \geq \beta_i$ and $(\delta_{1i}^- \cup \delta_{2i}^-)(m(g_1g_2g_1^{-1})n, q) = (\delta_{1i}^- \cup \delta_{2i}^-)(mg_2n, q) \leq \alpha_i$ since $(\delta_1 \cup \delta_2)$ is a bipolar $M-N-$ multi $Q-$normal fuzzy is subgroup of G, that is $(\delta_{1i}^+ \cup \delta_{2i}^+)(m(g_1g_2g_1^{-1})n, q) \geq \beta_i$ hence $(m(g_1g_2g_1^{-1})n, q) \in (\delta_{1i}^+ \cup \delta_{2i}^+)_\beta$ and $(\delta_{1i}^- \cup \delta_{2i}^-)(m(g_1g_2g_1^{-1})n, q) \leq \alpha_i$ therefore $(m(g_1g_2g_1^{-1})n, q) \in (\delta_{1i}^- \cup \delta_{2i}^-)_\alpha$. Thus $(\delta_1 \cup \delta_2)_{(\alpha,\beta)}$ is bipolar $M-N-$multi $Q-$normal fuzzy subgroup of G. □

Theorem 3.20. *If δ_1 and δ_2 are bipolar $M-N-$multi $Q-$fuzzy subgroups of G_1 and G_2 respectively. Then $\delta_1 \times \delta_2$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup.*

Proof. If $g, h \in G_1 \times G_2$ such that $g = (g_1, h_1), h = (g_2, h_2), q \in Q, m \in M$ and $n \in N$

1.$(\delta_{1i} \times \delta_{2i})^+(m(gh)n, q) = (\delta_{1i} \times \delta_{2i})^+(m((g_1, h_1)(g_2, h_2))n, q)$
$= (\delta_{1i} \times \delta_{2i})^+(m((g_1g_2, h_1h_2))n, q)$
$= \min\{\delta_{1i}^+(m(g_1g_2)n, q), \delta_{2i})^+(m(h_1h_2)n, q)\}$
$\geq \min\{\min\{\delta_{1i}^+(mg_1n, q), \{\delta_{1i}^+(mg_2n, q)\}, \min\{\delta_{2i}^+(mh_1n, q), \{\delta_{2i}^+(mh_2n, q)\}\}$
$= \min\{\min\{\delta_{1i}^+(mg_1n, q), \delta_{2i}^+(mh_1n, q)\}, \min\{\delta_{1i}^+(mg_2n, q), \delta_{2i}^+(mh_2n, q)\}\}$
$\min\{(\delta_{1i} \times \delta_{2i})^+(m(g_1, h_1)n, q), (\delta_{1i} \times \delta_{2i})^+(m(g_2, h_2)n, q)\}$
$= \min\{(\delta_{1i} \times \delta_{2i})^+(mgn, q), (\delta_{1i} \times \delta_{2i})^+(mhn, q)\}$.

2.$(\delta_{1i} \times \delta_{2i})^-(m(gh)n, q) = (\delta_{1i} \times \delta_{2i})^-(m((g_1, h_1)(g_2, h_2))n, q)$
$= (\delta_{1i} \times \delta_{2i})^-(m((g_1g_2, h_1h_2))n, q)$
$= \max\{\delta_{1i}^-(m(g_1g_2)n, q), \delta_{2i})^-(m(h_1h_2)n, q)\}$
$\leq \max\{\max\{\delta_{1i}^-(m(g_1)n, q), \{\delta_{1i}^-(mg_2n, q)\}, \max\{\delta_{2i}^-(mh_1n, q), \{\delta_{2i}^-(mh_2n, q)\}\}$
$= \max\{\max\{\delta_{1i}^-(mg_1n, q), \delta_{2i}^-(mh_1n, q)\}, \max\{\delta_{1i}^+(mg_2n, q), \delta_{2i}^+(mh_2n, q)\}\}$
$\max\{(\delta_{1i} \times \delta_{2i})^-(m(g_1, h_1)n, q), (\delta_{1i} \times \delta_{2i})^-(m(g_2, h_2)n, q)\}$
$= \max\{(\delta_{1i} \times \delta_{2i})^-(mgn, q), (\delta_{1i} \times \delta_{2i})^-(mhn, q)\}$.

3.$(\delta_{1i} \times \delta_{2i})^+(m(g^{-1})n, q) = (\delta_{1i} \times \delta_{2i})^+(m(g_1, h_1)^{-1`})n, q)$
$= (\delta_{1i} \times \delta_{2i})^+(m((g_1^{-1}, h_1^{-1}))n, q)$
$= \min\{\delta_{1i}^+(m(g_1^{-1})n, q), \delta_{2i})^+(m(h_1^{-1})n, q)\}$
$= \min\{\delta_{1i}^+(mg_1n, q), \delta_{2i}^+(mh_1n, q)\}$
$= (\delta_{1i} \times \delta_{2i})^+(m(g_1, h_1)n, q)$
$= (\delta_{1i} \times \delta_{2i})^+(mgn, q)$.

4.$(\delta_{1i} \times \delta_{2i})^-(m(g^{-1})n, q) = (\delta_{1i} \times \delta_{2i})^-(m(g_1, h_1)^{-1`})n, q)$
$= (\delta_{1i} \times \delta_{2i})^-(m((g_1^{-1}, h_1^{-1}))n, q)$
$= \max\{\delta_{1i}^-(m(g_1^{-1})n, q), \delta_{2i})^-(m(h_1^{-1})n, q)\}$
$= \max\{\delta_{1i}^-(mg_1n, q), \delta_{2i}^-(mh_1n, q)\}$
$= (\delta_{1i} \times \delta_{2i})^-(m(g_1, h_1)n, q)$
$= (\delta_{1i} \times \delta_{2i})^-(mgn, q)$.

Therefore $\delta_1 \times \delta_2$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup. $\square$

Theorem 3.21. *If δ_1 and δ_2 are bipolar $M-N-$multi $Q-$fuzzy subsets of G_1 and G_2 respectively. If e_1 and e_2 are the identity elements of G_1 and G_2 2 respectively and $\delta_1 \times \delta_2$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of $G_1 \times G_2$. Then one the following two axioms at least must hold.*
1. $\delta^+_{2i}(me_2n, q) \geq \delta^+_{1i}(mg_1n, q), \delta^-_{2i}(me_2n, q) \leq \delta^-_{1i}(mg_1n, q)$
2. $\delta^+_{1i}(me_1n, q) \geq \delta^+_{2i}(mg_2n, q), \delta^-_{1i}(me_1n, q) \leq \delta^-_{2i}(mg_2n, q)$. For all $g_1, g_2 \in G, q \in Q, m \in M$ and $n \in N$.

Proof. Suppose that none of the axioms (1) and (2) holds. Then we can find $h_1 \in G_1$ and $h_2 \in G_2$ such that $\delta^+_{2i}(me_2n, q) \leq \delta^+_{1i}(mh_1n, q), \delta^-_{2i}(me_2n, q) \geq \delta^-_{1i}(mh_1n, q)$ and $\delta^-_{1i}(me_1n, q) \leq \delta^+_{2i}(mh_2n, q), \delta^-_{1i}(me_1n, q) \geq \delta^-_{2i}(mh_2n, q)$.

$$\begin{aligned} 1.(\delta_{1i} \times \delta_{2i})^+(m(h_1, h_2)n, q) &= \min\{\delta^+_{1i}(mh_1n, q), \delta^+_{2i}(mh_2n, q)\} \\ &> \min\{\delta^+_{2i}(me_2n, q), \delta^+_{1i}(me_1n, q)\} \\ &= (\delta_{1i} \times \delta_{2i})^+(m(e_1, e_2)n, q) \end{aligned}$$

Then $(\delta_{1i} \times \delta_{2i})^+(m(h_1, h_2)n, q) > (\delta_{1i} \times \delta_{2i})^+(m(e_1, e_2)n, q)$.

$$\begin{aligned} 2.(\delta_{1i} \times \delta_{2i})^-(m(h_1, h_2)n, q) &= \max\{\delta^-_{1i}(mh_1n, q), \delta_{2i})^-(mh_2n, q)\} \\ &< \max\{\delta_{2i})^+(me_1n, q), \delta^+_{1i}(me_2n, q)\} \\ &= (\delta_{1i} \times \delta_{2i})^-(m(e_1, e_2)n, q). \end{aligned}$$

Then $(\delta_{1i} \times \delta_{2i})^-(m(h_1, h_2)n, q) < (\delta_{1i} \times \delta_{2i})^-(m(e_1, e_2)n, q)$.
Therefore the product $\delta_1 \times \delta_2$ is not a bipolar $M-N-$multi $Q-$fuzzy subgroup of $G_1 \times G_2$. Thus either (1) or (2) hold. $\square$

Theorem 3.22. *If δ_1 and δ_2 are bipolar $M-N-$multi $Q-$fuzzy subsets of G_1 and G_2 respectively. If e_1 and e_2 are the identity elements of G_1 and G_2 respectively and $\delta_1 \times \delta_2$ is a bipolar $M-N-$multi $Q-$fuzzy subgroup of $G_1 \times G_2$. Then*

1. If $\delta^+_{1i}(mg_1n, q) \leq \delta_{2i})^+(me_2n, q), \delta^-_{1i}(mg_1n, q) \geq \delta_{2i})^-(me_2n, q)$ then δ_1 is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G_1.

2. $\delta^+_{2i}(mg_2n, q) \leq \delta_{1i})^+(me_1n, q), \delta^-_{2i}(mg_2n, q) \geq \delta_{1i})^-(me_1n, q)$ then δ_2 is a bipolar $M-N-$multi $Q-$fuzzy subgroup of G_2.
For all $g_1 \in G_1, g_2 \in G_2, q \in Q, m \in M$ and $n \in N$.

Proof. 1. Let $\delta_1 \times \delta_2 = ((\delta_{1i} \times \delta_{2i})^+, (\delta_{1i} \times \delta_{2i})^-)$ be a bipolar $M-N-$multi $Q-$fuzzy subgroup of $G_1 \times G_2$ and $g_1, g_2 \in G_1$ then $(g_1, e_2), (g_2, e_2) \in G_1 \times G_2$, since $\delta^+_{1i}(mg_1n, q) \leq \delta_{2i})^+(me_2n, q), \delta^-_{1i}(mg_1n, q) \geq \delta_{2i})^-(me_2n, q)$ for all $g_1 \in G_1$ we get

$$\begin{aligned} &i.\delta^+_{1i}(mg_1g_2n, q) = \min\{\delta^+_{1i}(mg_1g_2n, q), \delta^+_{2i}(me_1e_2n, q)\} \\ &= (\delta_{1i} \times \delta_{2i})^+((mg_1g_2n, q), (me_1e_2n, q)) \\ &= (\delta_{1i} \times \delta_{2i})^+((m(g_1, e_2)(g_2, e_2)n, q)) \\ &\geq \min\{(\delta_{1i} \times \delta_{2i})^+(m(g_1, e_2)n, q), (\delta_{1i} \times \delta_{2i})^+(m(g_2, e_2)n, q)\} \\ &= \min\{\min\{\delta^+_{1i}(mg_1n, q), \delta^+_{2i}(me_2n, q)\}, \min\{\delta^+_{1i}(mg_2n, q), \delta^+_{2i}(me_2n, q)\}\} \\ &= \min\{\delta^+_{1i}(mg_1n, q), \delta^+_{1i}(mg_2n, q)\}. \end{aligned}$$

Then $\delta_{1i}^{+}(mg_1g_2n,q) \geq \; min\{\delta_{1i}^{+}(mg_1n,q),\delta_{1i}^{+}(mg_2n,q)\}$.

$$\begin{aligned}
&ii.\delta_{1i}^{-}(mg_1g_2n,q) = \max\{\delta_{1i}^{-}(mg_1g_2n,q),\delta_{2i}^{-}(me_1e_2n,q)\} \\
&= (\delta_{1i}\times\delta_{2i})^{-}((mg_1g_2n,q),(me_1e_2n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{-}((m(g_1,e_2)(g_2,e_2)n,q)) \\
&\leq \max\{(\delta_{1i}\times\delta_{2i})^{-}(m(g_1,e_2)n,q),(\delta_{1i}\times\delta_{2i})^{-}(m(g_2,e_2)n,q)\} \\
&= \max\{\max\{\delta_{1i}^{-}(mg_1n,q),\delta_{2i}^{-}(me_2n,q)\},\max\{\delta_{1i}^{-}(mg_2n,q),\delta_{2i}^{-}(me_2n,q)\}\} \\
&= \max\{\delta_{1i}^{-}(mg_1n,q),\delta_{1i}^{-}(mg_2n,q)\}.
\end{aligned}$$

Then $\delta_{1i}^{-}(mg_1g_2n,q) \leq \max\{\delta_{1i}^{-}(mg_1n,q),\delta_{1i}^{-}(mg_2n,q)\}$.

$$\begin{aligned}
&iii.\delta_{1i}^{+}(mg_1^{-1}n,q) = \min\{\delta_{1i}^{+}(mg_1^{-1}n,q),\delta_{2i}^{+}(me_1^{-1}n,q)\} \\
&= (\delta_{1i}\times\delta_{2i})^{+}((mg_1^{-1}n,q),(me_2^{-1}n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{+}((m(g_1,e_2)^{-1}n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{+}(m(g_1,e_2)n,q) \\
&= \min\{\delta_{1i}^{+}(mg_1n,q),\delta_{2i}^{+}(me_2n,q)\} \\
&= \delta_{1i}^{+}(mg_1n,q).
\end{aligned}$$

Then $\delta_{1i}^{+}(mg_1^{-1}n,q) = \delta_{1i}^{+}(mg_1n,q)$.

$$\begin{aligned}
&iv.\delta_{1i}^{-}(mg_1^{-1}n,q) = \max\{\delta_{1i}^{-}(mg_1^{-1}n,q),\delta_{2i}^{-}(me_1^{-1}n,q)\} \\
&= (\delta_{1i}\times\delta_{2i})^{-}((mg_1^{-1}n,q),(me_2^{-1}n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{-}((m(g_1,e_2)^{-1}n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{-}(m(g_1,e_2)n,q) \\
&= \max\{\delta_{1i}^{-}(mg_1n,q),\delta_{2i}^{-}(me_2n,q)\} \\
&= \delta_{1i}^{-}(mg_1n,q).
\end{aligned}$$

Then $\delta_{1i}^{-}(mg_1^{-1}n,q) = \delta_{1i}^{-}(mg_1n,q)$. Therefore δ_1 is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of G_1.

2. Let $\delta_1\times\delta_2 = ((\delta_{1i}\times\delta_{2i})^{+},(\delta_{1i}\times\delta_{2i})^{-})$ be a bipolar $M-N-$multi $Q-$fuzzy subgroup of $G_1\times G_2$ and $g_1,g_2\in G_2$ then $(e_1,g_1),(e_1,g_2)\in G_1\times G_2$, since $\delta_{2i}^{+}(mg_1n,q) \leq \delta_{1i})^{+}(me_1n,q), \delta_{2i}^{-}(mg_1n,q) \geq \delta_{1i})^{-}(me_1n,q)$ for all $g_1\in G_2, q\in Q, m\in M$ and $n\in N$ we get

$$\begin{aligned}
&i.\delta_{2i}^{+}(mg_1g_2n,q) = \min\{\delta_{1i}^{+}(me_1e_2n,q),\delta_{2i}^{+}(mg_1g_2n,q)\} \\
&= (\delta_{1i}\times\delta_{2i})^{+}((me_1e_2n,q),(mg_1g_2n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{+}((m(e_1,g_1)(e_1,g_2)n,q)) \\
&\geq \min\{(\delta_{1i}\times\delta_{2i})^{+}(m(e_1,g_1)n,q),(\delta_{1i}\times\delta_{2i})^{+}(m(e_1,g_2)n,q)\} \\
&= \min\{\min\{\delta_{1i}^{+}(me_1n,q),\delta_{2i}^{+}(mg_1n,q)\},\min\{\delta_{1i}^{+}(me_1n,q),\delta_{2i}^{+}(mg_2n,q)\}\} \\
&= \min\{\delta_{2i}^{+}(mg_1n,q),\delta_{1i}^{+}(mg_2n,q)\}.
\end{aligned}$$

Then $\delta_{2i}^{+}(mg_1g_2n,q) \geq \min\{\delta_{2i}^{+}(mg_1n,q),\delta_{2i}^{+}(mg_2n,q)\}$.

$$\begin{aligned}
&ii.\delta_{1i}^{-}(mg_1g_2n,q) = \max\{\delta_{1i}^{-}(mme_1e_2n,q),\delta_{2i}^{-}(mg_1g_2n,q)\} \\
&= (\delta_{1i}\times\delta_{2i})^{-}((me_1e_2n,q),(mg_1g_2n,q)) \\
&= (\delta_{1i}\times\delta_{2i})^{-}((m(e_1,g_1)(e_1,g_2)n,q)) \\
&\leq \max\{(\delta_{1i}\times\delta_{2i})^{-}(m(e_1,g_1)n,q),(\delta_{1i}\times\delta_{2i})^{-}(m(e_1,g_2)n,q)\} \\
&= \max\{\max\{\delta_{1i}^{-}(me_1n,q),\delta_{2i}^{-}(mg_1n,q)\},\max\{\delta_{1i}^{-}(me_1n,q),\delta_{2i}^{-}(mg_2n,q)\}\} \\
&= \max\{\delta_{1i}^{-}(mg_1n,q),\delta_{2i}^{-}(mg_2n,q)\}.
\end{aligned}$$

Then $\delta^-_{2i}(mg_1g_2n,q) \leq \max\{\delta^-_{2i}(mg_1n,q),\delta^-_{2i}(mg_2n,q)\}$.

$$\begin{aligned}
&iii.\delta^+_{2i}(mg_1^{-1}n,q) = \min\{\delta^+_{1i}(me_1^{-1}n,q),\delta^+_{2i}(mg_1^{-1}n,q)\}\\
&= (\delta_{1i}\times\delta_{2i})^+((me_1^{-1}n,q),(mg_1^{-1}n,q))\\
&= (\delta_{1i}\times\delta_{2i})^+((m(g_1,e_2)^{-1}n,q))\\
&= (\delta_{1i}\times\delta_{2i})^+(m(g_1,e_2)n,q)\\
&= \min\{\delta^+_{1i}(me_1n,q),\delta^+_{2i}(mg_1n,q)\}\\
&= \delta^+_{2i}(mg_1n,q).
\end{aligned}$$

Then $\delta^+_{2i}(mg_1^{-1}n,q) = \delta^+_{2i}(mg_1n,q)$.

$$\begin{aligned}
&iv.\delta^-_{2i}(mg_1^{-1}n,q) = \max\{\delta^-_{1i}(me_1^{-1}n,q),\delta^-_{2i}(mg_1^{-1}n,q)\}\\
&= (\delta_{1i}\times\delta_{2i})^-((me_1^{-1}n,q),(mg_1^{-1}n,q))\\
&= (\delta_{1i}\times\delta_{2i})^-((m(e_1,g_1)^{-1}n,q))\\
&= (\delta_{1i}\times\delta_{2i})^-(m(e_1,g_1)n,q)\\
&= \max\{\delta^-_{1i}(me_1n,q),\delta^-_{2i}(mg_1n,q)\}\\
&= \delta^-_{2i}(mg_1n,q).
\end{aligned}$$

Then $\delta^-_{2i}(mg_1^{-1}n,q) = \delta^-_{2i}(mg_1n,q)$. Therefore δ_2 is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of G_2. □

Corollary 3.23. *If δ_1 and δ_2 are bipolar $M-N-$ multi $Q-$fuzzy subsets of G_1 and G_2 respectively, such that $\delta_1\times\delta_2$ is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of $G_1\times G_2$. Then either δ_1 is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of G_1 or δ_2 is a bipolar $M-N-$ multi $Q-$fuzzy subgroup of G_2.*

Proof. Straightforward. □

Proposition 3.24. *If δ_1 and δ_2 are bipolar $M-N-$ multi $Q-$normal fuzzy subgroups of G_1 and G_2 respectively. Then $\delta_1\times\delta_2$ is a bipolar $M-N-$ multi $Q-$ normal fuzzy subgroup.*

Proof. By Theorem 3.20, $\delta_1\times\delta_2$ is a bipolar $M-N-$ multi $Q-$normal fuzzy subgroup.
Now, for $(g_1,g_2),(h_1,h_2)\in G_1\times G_2$.

$$\begin{aligned}
&(\delta_{1i}\times\delta_{2i})^+(m((g_1,g_2)(h_1,h_2))n,q) = (\delta_{1i}\times\delta_{2i})^+(m((g_1h_1,g_2h_2))n,q)\\
&= \min\{\delta^+_{1i}(m(g_1h_1)n,q),\delta^+_{2i}(m(g_2h_2)n,q)\}\\
&= \min\{\delta^+_{1i}(m(h_1g_1)n,q),\delta^+_{2i}(m(h_2g_2)n,q)\}\\
&= (\delta_{1i}\times\delta_{2i})^+(m((h_1,h_2)(g_1,g_2))n,q).
\end{aligned}$$

And

$$\begin{aligned}
&(\delta_{1i}\times\delta_{2i})^-(m((g_1,g_2)(h_1,h_2))n,q) = (\delta_{1i}\times\delta_{2i})^-(m((g_1h_1,g_2h_2))n,q)\\
&= \max\{\delta^-_{1i}(m(g_1h_1)n,q),\delta^-_{2i}(m(g_2h_2)n,q)\}\\
&= \max\{\delta^-_{1i}(m(h_1g_1)n,q),\delta^-_{2i}(m(h_2g_2)n,q)\}\\
&= (\delta_{1i}\times\delta_{2i})^-(m((h_1,h_2)(g_1,g_2))n,q).
\end{aligned}$$

Therefore $\delta_1\times\delta_2$ is a bipolar $M-N-$multi $Q-$normal fuzzy subgroup $G_1\times G_2$. □

Definition 3.25. A bipolar $M-N-$multi $Q-$fuzzy subgroup δ_1 of G is called conjugate to bipolar $M-N-$multi $Q-$fuzzy subgroup δ_2 of G if there exists $g_2 \in G$ such that for all $g_1 \in G, q \in Q, m \in M$ and $n \in N$.
1. $\delta^+_{1i}(mg_1n, q) = \delta^+_{2i}(m(g_2^{-1}g_1g_2)n, q)$
2.$\delta^+_{1i}(mg_1n, q) = \delta^+_{2i}(m(g_2^{-1}g_1g_2)n, q)$.

Theorem 3.26. *If a bipolar $M-N-$multi $Q-$fuzzy subgroup δ_1 of G_1 is a conjugate to $M-N-$multi $Q-$fuzzy subgroup δ_2 of G_1 and a bipolar $M-N-$multi $Q-$fuzzy subgroup λ_1 of G_2 is a conjugate to $M-N-$multi $Q-$fuzzy subgroup λ_2 of G_2. Then the bipolar $M-N-$multi $Q-$fuzzy subgroup $\delta_1 \times \lambda_1$ of $G_1 \times G_2$ is conjugate to bipolar $M-N-$multi $Q-$fuzzy subgroup $\delta_2 \times \lambda_2$ of $G_1 \times G_2$.*

Proof. Since δ_1 is a conjugate to $M-N-$multi $Q-$fuzzy subgroup δ_2 of G_1, then there exist $h_1 \in G_1$ such that for all $g_1 \in G_1, q \in Q, m \in M$ and $n \in N$
1. $\delta^+_{1i}(mg_1n, q) = \delta^+_{2i}(m(h_1^{-1}g_1h_1)n, q)$
2. $\delta^-_{1i}(mg_1n, q) = \delta^+_{2i}(m(h_1^{-1}g_1h_1)n, q)$.
Also, λ_1 is a conjugate to$M-N-$multi $Q-$fuzzy subgroup λ_2 of G_2, then there exist $h_2 \in G_2$ such that for all $g_2 \in G_2, q \in Q, m \in M$ and $n \in N$ 1. $\lambda^+_{1i}(mg_2n, q) = \lambda^+_{2i}(m(h_2^{-1}g_2h_2)n, q)$
2. $\lambda^-_{1i}(mg_2n, q) = \lambda^-_{2i}(m(h_2^{-1}g_2h_2)n, q)$.
If there exist $(h_1, h_2) \in G_1 \times G_2$ such that for all $(g_1, g_2) \in G_1 \times G_2, q \in Q, m \in M$ and $n \in N$.

$$\begin{aligned}
&(\delta_{1i} \times \lambda_{1i})^+(m(g_1g_2)n, q) = \min\{\delta^+_{1i}(mg_1n, q), \lambda_{1i})^+(mg_2n, q)\}\\
&= \min\{\delta^+_{2i}(m(h_1^{-1}g_1h_1)n, q), \lambda_{2i})^+(m(h_2^{-1}g_2h_2)n, q)\}\\
&= (\delta_{1i} \times \lambda_{1i})^+(m(h_1^{-1}g_1h_1, h_2^{-1}g_2h_2)n, q)\\
&= (\delta_{1i} \times \lambda_{1i})^+m((h_1^{-1}, h_2^{-1})(g_1, g_2)(h_1, h_2)n, q)\\
&= (\delta_{1i} \times \lambda_{1i})^+(m((h_1, h_2)^{-1}(g_1, g_2)(h_1, h_2)n, q).
\end{aligned}$$

And

$$\begin{aligned}
&(\delta_{1i} \times \lambda_{1i})^-(m(g_1g_2)n, q) = \max\{\delta^-_{1i}(mg_1n, q), \lambda_{1i})^-(mg_2n, q)\}\\
&= \max\{\delta^-_{2i}(m(h_1^{-1}g_1h_1)n, q), \lambda_{2i})^-(m(h_2^{-1}g_2h_2)n, q)\}\\
&= (\delta_{1i} \times \lambda_{1i})^-(m(h_1^{-1}g_1h_1, h_2^{-1}g_2h_2)n, q)\\
&= (\delta_{1i} \times \lambda_{1i})^-m((h_1^{-1}, h_2^{-1})(g_1, g_2)(h_1, h_2)n, q)\\
&= (\delta_{1i} \times \lambda_{1i})^-(m((h_1, h_2)^{-1}(g_1, g_2)(h_1, h_2)n, q).
\end{aligned}$$

Therefore $\delta_1 \times \lambda_1$ of $G_1 \times G_2$ is conjugate to bipolar $M-N-$ multi $Q-$fuzzy subgroup $\delta_2 \times \lambda_2$ of $G_1 \times G_2$. □

Conclusion We summarized the basic notions of bipolar $M-N-$ multi $Q-$fuzzy subgroups and then presented a detailed theoretical study of generalized some properties and operations of bipolar$M-N-$ multi $Q-$ fuzzy subgroups. This work can be extended to the theorems and properties of different concepts of positive $\beta-$cut, negative $\alpha-$cut, union and product of bipolar $M-N-$ multi $Q-$ fuzzy subgroups.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The authors are thankful to the referees for their valuable comments and suggestions for improving the article.

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