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IDENTICAL THEOREM OF APPROXIMATION UNBOUNDED FUNCTIONS BY LINEAR OPERATORS†

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Abstract. The aim of this paper, investigated of weighted space which contained the unbounded functions which is to be approximated by linear operators in terms some Well-known approximation tools such as the modulus of smoothness and K-functional. The characteristics of the identical theorem between modulus of smoothness and K-functional are consider. In addition to the establish the direct, converse and identical theorem by using some linear operators in terms modulus Ditzian-Totik.

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1. Introduction

In algebraic structure, the approximation theory is a vital tool in estimates for the differences between two linear operators, and this theory has received broad interest among researchers for the purpose of new contribution [1, 2, 3, 4, 5]. Latest, since the weighted spaces are more interesting than the space of measurable functions, which are denoted by L_p- spaces, there is grow demand in problem of approximation in weighted spaces. For arranged efficiently we presented from [6] some basic concepts and definitions which related main results. There are many researchers have used linear operators, in particular Baskakov operator see [6, 10, 11, 12, 13, 15, 16, 17] to obtain approximate results in several spaces . In (1995) Guta [9] prove direct theorem by modified Baskakov type operators in terms Ditizain – Totik modulus of smoothness, in (1998) Agrawal [7] introduced a sequences of linear operators to approximate unbounded functions in $C_\alpha[0, \infty)$, in (2006) Finta [8] discuss direct approximation for discrete type

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operators in terms a modified K-functional as applications he proved direct theorems for Szász - Mirakjan type operators , Lupa operators and Baskakov type operators, in (2013) Naragan [16] interduced definitions and properties of q-Baskakov- Beta- Stancu operators and he gave some approximation properties and asymptotic formulae for these operators, in (2017) Rao [17] verify of generalized Baskakov operators and study the degree of approximation by modulus of continuity, order of approximation for the derivative of functions and proved direct theorem in terms K-functional and Ditzian-Totik modulus of smoothness. Recently, Alaa Auad and Abdulsattar [20] established the concept the existence and uniqueness for best approximation in linear k-normed spaces, proved the mapping form k-normed space into finite dimensional subspace of k-normed space is continuous , bounded compact subset of linear k-normed is proximal and characterization of best uniform approximation in same space. In this paper, we will approximate the unbounded functions in weighted space by using some linear operators , the most important one is Baskakov operator. Introduction is here

2. Preliminaries and Notes

Let $X = [0, \infty), 1 \leq p < \infty$, the space $L_p(X)$ of all measurable functions on X with any function P in this space with equipped the norm

$$\|\rho\|_p = (\int_x |\rho(x)|^p dx)^{\frac{1}{p}} < \infty,$$

is the set of all weighted functions as $\mu : X \to \mathbb{R}^+$ is an almost everywhere positive function which is locally integrable.
Consider $L_{(p,\mu)}(X)$ the weighted space of all unbounded functions, where μ is weighted function,$1 \leq p < \infty$ and every function ρ belong to the space$L_{(p,\mu)}(X)$ has the following

$$\|\rho\|_{p,\mu}^p = \int_x |\rho(x).\mu(x)|^p dx < \infty \tag{1}$$

The modulus of smoothness of order
$k \in \mathbb{N}$ of the function $\rho \in L_{(p,\mu)}(X)$ is defined by

$$\Omega_k(\rho, h_{p,\mu}) = \underbrace{sup}_{|\delta \leq h|} \{\|\Delta_\delta^k \rho(.)\|_{p,\mu} \quad , h > 0\} \tag{2}$$

where $\Delta_\delta^k \rho(t)$ is called the k^{th} difference with step δ at the point t and defined by

$$\Delta_\delta^k \rho(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \rho(t + j\delta).$$

And the k^{th} Ditzian- Totik modulus of smoothness of ρ in $L_{(p,\mu)}(X)$ is defined by

$$\Omega_k^{\vartheta}(\rho,h)_{p,\mu}=\underbrace{sup}_{|\delta\leq h|}\{\|\Delta_{\delta\vartheta}^k\rho(.)\|_{p,\mu}\quad,h>0\} \tag{3}$$

Where

$$\Delta_{\delta\vartheta}^k\rho(t)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\rho(t+j\delta\vartheta(t)).$$

Let δ_k be a subspace of $L_{(p,\mu)}(X)$ and ρ belong to the $L_{(p,\mu)}(X)$ with $\rho\in\delta_k$. Then the K-functional of the function ρ is defined by

$$K_k(\rho,\delta)_{p,\mu}=inf\{\|\rho-\varrho\|_{p,\mu}+\delta^k\|D^k(\varrho)\|_{p,\mu}\} \tag{4}$$

If $\mathbb{P}_k$ the subspace of $L_{(p,\mu)}(X)$ with algebraic polynomial of degree k, then best approximation of $\rho\in L_{(p,\mu)}(X)$ is defined by

$$\varepsilon_{p,\mu}=inf\{\|\rho-p_k\|_{p,\mu}\quad,\varrho_k\in\mathbb{P}_k\} \tag{5}$$

For the function ρ in the space $L_{(p,\mu)}(X)$, the class of Baskakov linear operators are defined see [9] as

$$G_r(\rho,t)=\sum_{r=0}^{\infty}\rho(\frac{r}{n})g_{n,r}(t). \tag{6}$$

Where

$$g_{n,r}(t)=\binom{n+r-1}{r}\frac{t^r}{(1+t)^{n+r}}\quad n,r\in\mathbb{N}.$$

Now, we using defined the Kantorovich linear operator [see [14]] as following

$$\mathrm{B}(\rho,t)=(n-1)\sum_{r=0}^{\infty}\quad G_{n,r}(t)\int_{r/(n-1)}^{(r+1)/(n-1)}\rho(x)dx, \tag{7}$$

to approximate the functions in weighted space $L_{(p,\mu)}(X)$. For $n,k\in\mathbb{N}$, such that $n\geq 2k$, the linear combinations of the Baskakov-Kantorovich linear operator given by

$$P_m(\rho,t)=\sum_{j=0}^{2k-1}C_j(m,k)\mathrm{B}_{mj}(\rho,t). \tag{8}$$

where the coefficients $C_j(m,k)$ dependent of m,k and satisfy the following conditions:

$$m\leq m_0\leq m_1\leq...\leq m_{2k-1}\leq C_m,\quad\sum_{j=0}^{2k-1}C_j(m,k)=1. \tag{9}$$

$$\sum_{j=0}^{2k-1}C_j(m,k)\mathbf{B}_{mj}((x-t)^m;t)=0\quad,m=1,2,...,2k-1 \tag{10}$$

3. Auxiliary lemmas

In this section we will present some lemmas that we need in the proofs of the main results:

Lemma 3.1. *The operator* $\mathbf{B}(\rho,t)$ *which defined in equation(7) satisfying* $\mathbf{B}(1,t)=1$.
$\mathbf{B}((u-t)^{2k},t)\leq\frac{Ch_n^{2k}(t)}{(n)^k}$, *where* $h_n^{2k}(t)=max\{\theta^{2n}(t),\frac{1}{n^k}\}$,

$$\theta(t)=\sqrt{t(t+1)},n,k\in\mathbb{N}\quad \textit{and C positive constant}$$

The proof of this it's clear.

Lemma 3.2. [14] *If* $\mu(t)=t^\alpha(t+1)^\beta and\alpha,\beta\in\mathbb{R}, then\frac{\mu(t)}{\mu(z)}\leq 2^{|\beta|}[(\frac{t}{z})^\alpha+(\frac{t}{z})^\beta]$.

Lemma 3.3. *Let* $\rho\in L_{p,\mu}(X),\mu(t)=t^\alpha(t+1)^\beta,\alpha,\beta$ *are natural numbers and* $0\leq\alpha,\beta<n-1$. *Then*

$$\|\mathbf{B}(\rho,.)\|_{p,\mu}\leq C\|\rho\|_{p,\mu}$$

Proof : *From equation (7),lemma 3.1 and lemma 3.2 we have:*

$$|\mathrm{B}(\rho,t)\mu(t)|=|\sum_{r=0}^{\infty}\quad G_{n,r}(t)\mu(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)}\rho(x)dx|,$$

$$\leq\sum_{r=0}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}[(\frac{t(n-1)}{r+1})^\alpha+(\frac{t(n-1)}{r+1})^\beta]\int_{r/(n-1)}^{(r+1)/(n-1)}|\mu(t)\rho(x)|dx,$$

we set

$$\Gamma_1=\sum_{r=0}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\alpha+\int_{r/(n-1)}^{(r+1)/(n-1)}|\mu(t)\rho(x)|dx,$$

$$\Gamma_2=\sum_{r=0}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\beta+\int_{r/(n-1)}^{(r+1)/(n-1)}|\mu(t)\rho(x)|dx.$$

Since,

$$\|\Gamma_1\|_{\rho,\mu}=\{\int_x|\sum_{r=0}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\alpha+\int_{r/(n-1)}^{(r+1)/(n-1)}|\mu(t)\rho(x)dx|^pdt\}^{\frac{1}{p}},$$

$$\leq C\{\int_x C\sum_{r=0}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\alpha+\int_{r/(n-1)}^{(r+1)/(n-1)}C|\mu(t)\rho(x)dx|^pdt\}^{\frac{1}{p}},$$

$$=C\{\int_x C\sum_{r=\alpha}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\alpha+\int_{(r-\alpha)/(n-1)}^{(r-\alpha+1)/(n-1)}C|\mu(t)\rho(x)dx|^pdt\}^{\frac{1}{p}},$$

$$=C\{\int_x C\sum_{r=\alpha}^{\infty}\quad G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^\alpha+\int_{(r-\alpha)/(n-1)}^{(r-\alpha+1)/(n-1)}C|\mu(t)\rho(x)dx|^p+$$

$$C\sum_{r=\alpha}^{\infty} G_{n,r}(t)(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^{\alpha}+\int_{max(0,\frac{r-\alpha}{n-1})}^{max(0,\frac{r-\alpha+1}{n-1})} C|\mu(t)\rho(x)dx|^{p}dt\}^{\frac{1}{p}},$$

$$\leq C\{\int_{x} C\sum_{r=\alpha}^{\infty} G_{n,r}(t)C(n-1)2^{|\beta|+1}(\frac{t(n-1)}{r+1})^{\alpha}+\int_{max(0,\frac{r-\alpha}{n-1})}^{max(0,\frac{r-\alpha+1}{n-1})} C|\mu(t)\rho(x)dx|^{p}dt\}^{\frac{1}{p}},$$

$$\leq max(C)\{\int_{x} C|\mu(t)\rho(x)^{p}|dx\}^{\frac{1}{p}} = C\|\rho\|_{p,\mu}$$

Thus,

$$\|\Gamma_1\|_{p,\mu} \leq C\|\rho\|_{p,\mu} \tag{11}$$

Similarly, we can to prove

$$\|\Gamma_2\|_{p,\mu} \leq C\|\rho\|_{p,\mu} \tag{12}$$

From equation (11) and equation (12), we obtain the proof of lemma.

Lemma 3.4. [18] *Let* $\rho \in L_{P([0,\infty))}, n \geq kandn, k \in \mathbb{N}$. *Then*

(1) $\|P_{n,k}^{(2k)}(\rho,.)\|_p \leq Cn^k\|\rho\|_p$.

(2) $\|P_{n,k}^{(2k)}(\rho,.)\|_p \leq Cn^k\|\rho^{2k}\|_p$.

Lemma 3.5. [19] *Let* $\rho \in L_{P([0,\infty))}, \delta > 0, k \in \mathbb{N}$
and there exists positive constant C_1, C_2 *Then*

$$C_1\Omega_k^{\vartheta}(\rho,\delta)_p \leq K_k(\rho,\delta^k)_p \leq C_2\Omega_k^{\vartheta}(\rho,\delta)_p$$

4. Main results

Our results in this section three theorems, direct, inverse and equivalent theorem which we will prove in terms Ditzian- Totik modulus of smoothness.

Theorem 4.1. *Let* $\rho \in L_{p,\mu}(X)$, $1 \leq p \leq \infty, \alpha, \beta, r, n$ *are natural numbers such that* $0 \leq \alpha, \beta < n-1$ *and* $\vartheta(t) = \sqrt{t(t+1)}$. *Then*

$$\varepsilon_k(\rho)_{\rho,\mu} \leq \|\rho - P_{n,k}(\rho)\| \leq C\Omega_k^{\vartheta}(\rho,h)_{p,\mu}.$$

Proof. We define the space $Q = \{\ell : \ell^{2k-1} \in C[0,1), \mu^{(2k)}, \ell^{(2k)} \in L_{(p,\mu)}(X)\}$ and

$$\ell(a) = \sum_{j=0}^{2k-1} \frac{1}{j}(a-t)^j\ell^j(t) + F_{2k}(\ell,a,t)$$

where

$$F_{2k}(\ell,a,t) = \frac{1}{(2k-1)!}\int_t^{\alpha}(a-x)^{2k-1}\ell^{2k}(u)du, \quad t \in [0,1]$$

From equation 10, we obtain

$$\mathbb{B}_{n,k}(\ell,t) - \ell(t) = \mathbb{B}_{n,k}(f_{2k}(h,a,t);t).$$

We need to approximate the operator $\mathbb{B}_{n,k}(f_{2k}(\ell,a,t);t)$ in the space $L_{\rho,\mu}(X)$. Since,

$$\|\mathbb{B}_{n,k}(f_{2k}(\ell,a,t);t).\|_{p,\mu} = \|\sum_{r=0}^{\infty} \quad G_{n,r}(t)(t)(n-1)$$
$$\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{1}{(2k-1)!}\int_{t}^{\alpha}(a-x)^{(2k)}\ell(x)dx)dt\|_{p,\mu}$$
$$\leq \sum_{r=0}^{\infty} \quad G_{n,r}(t)(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{1}{(2k-1)!}.\frac{(x-t)}{\vartheta^{2k-2}(t)}.(\frac{1}{t(t+1)}+$$
$$(\frac{1}{t(x+1)})(\frac{1}{\mu(x)}+\frac{1}{\mu(t)})dx\|h^{2k}\ell^{2k(.)}\|_{p,\mu}$$
$$\cong (I_1+I_2+I_3+I_4)\|\gamma(h^{2k}\ell^{(2k)})(.)\|_{p,\mu}$$

From lemma 3.1, we have

$$I_1 = \sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r)/(n-1)} \frac{1}{(2k-1)!}.\frac{(x-t)^{2k}}{\vartheta^{2k}(t)}.\frac{\mu(t)}{\mu(x)}dx$$
$$\leq \frac{1}{(2k-1)!}\vartheta^{2k}(t)$$
$$(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{\mu(t)}{\mu(x)}dx)^{\frac{1}{2}}.$$
$$(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} (x-t)^{2k}dx)^{\frac{1}{2}}$$
$$\leq \frac{Ch^{2k}(t)}{\vartheta^{2k}(2k-1)!} = \frac{C}{n^k}.$$

Similarly, we can evaluate I_2 and I_3 for each of them less than or equal $\frac{C}{n^k}$. Lastly, we approximate I_4 also by using lemma 3.1.

$$I_4 = \frac{\mu(t)}{\vartheta^{2k-1}(t)(2k-1)!}\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{(x-t)^{2k}}{\vartheta(t)}t(1+x)dx$$
$$\leq \frac{1}{\vartheta^{2k-2}(t)}(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} (x-t)^{4k}dx)\frac{1}{2}.$$
$$(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{1}{(1+x)^2}dx)\frac{1}{2}$$
$$\leq \frac{1}{\vartheta^{2k-2}(t)}.\frac{Ch^{2k}2^{\frac{1}{2}}}{1+t} = \frac{C}{n^k}.$$

From lemma 3.3, we obtain:

$$\|\mathbb{B}_{n,k}(F_{2k}(\ell,a,t);t)\|_{p,\mu} \leq \frac{C}{n^k}\|h^{2k}\ell^{2k}(.)\|_{p,\mu}.$$

We set $h^{2k} = \frac{1}{n^k}$ for $0 \leq t < \infty$.

$$\|\mathbb{B}_{n,k}(F_{2k}(\ell, a, t); t)\|_{p,\mu} \leq \sum_{r=0}^{\infty} \quad G_{n,r}(t)(t)(n-1)$$

$$\int_{r/(n-1)}^{(r+1)/(n-1)} (x-t)^{2k}(\frac{\mu(t)}{\mu(x)} + 1)dx|\vartheta(\mu(t)h^{2k}\upsilon^{(2k)}(t)|$$

From lemma 3.1, we obtain:

$$\frac{\mu(t)n^k}{2k-1!}\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{(x-t)^{2k}}{(\mu(x)}dx \leq \frac{(2n)^k}{(2k-1)!}$$

$$(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{(x-t)^{2k}}{\mu(x)}dx)^{\frac{1}{2}}$$

$$(\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{\mu^2(t)}{\mu^2(x)}dx)\frac{1}{2} = \frac{C}{n^k}.$$

And,

$$\frac{(2n)^k}{(2k-1)!}\sum_{r=0}^{\infty} \quad G_{n,r}(t)(n-1)\int_{r/(n-1)}^{(r+1)/(n-1)} \frac{(x-t)^{2k}}{\mu(x)}dx$$
$$= \frac{(2n)^k}{(2k-1)!}\mathbb{B}_{n,k}((\upsilon - t^{2k}), t) \leq \frac{C}{n^k}$$

Implies,

$$\|\mathbb{B}_{n,k}(F_{2k}(\nu, .); .)\|_{p,\mu}\| \leq \frac{C}{n^k}\|h^{2k}\nu^{2k}\|_{p,\mu}.$$

Consequently,

$$\|P_{n,k}(F_{2k}(\nu, .); .)\|_{p,\mu}\| \leq \sum_{j=0}^{2k-1}\| \quad \mathbb{B}_{n,j}(F_{2k}(\nu, .); .)\|_{p,\mu}. \leq \frac{C}{n^k}\sum_{j=0}^{2k-1} C_j(n,k)\|h^{2k}\nu^{2k}\|_{p,\mu}.$$

So,

$$\mathcal{E}_k(\rho)_{\rho,\mu} \leq \|\rho - P_{n,k}(\rho)\|_{\rho,\mu} \leq \|\rho - \varrho - P_{n,k}(\rho - \varrho)\|_{p,\mu} + \|\varrho - P_{n,k}(\rho)\|_{p,\mu}$$

$$C\|\rho - \varrho\|_{p,\mu} + \frac{C}{n^k}\|h^{2k}\varrho^{2k}\|_{p,\mu} \leq C\Omega_k^{\vartheta}(\rho, h)_{p,\mu}.$$

The proof of the theorem is finished. □

Theorem 4.2. *(Inverse theorem) Let* $\rho \in L_{(p,\mu)}(X)$, $1 \leq p < \infty, \vartheta(t) = (t(1+t))^{\frac{1}{2}}$ *,and* $n, k \in \mathbb{N}$ *such that* $n \geq k$ *. Then*

$$\Omega_k^{\vartheta}(\rho, \delta)_{p,\mu} \leq \frac{C}{n^k}\sum_{i=0}^{n} i^{k-1}\mathcal{E}_i(\rho, \delta)_{p,\mu}, \qquad \delta = \frac{1}{\sqrt{n^k}}.$$

Proof. From equation 4, lemma 3.4 and lemma 3.5, we obtain

$$\Omega_k^{\vartheta}(\rho,\delta)_{p,\mu} \leq K_k(\rho,\delta)_{p,\mu} \leq \|\rho - P_{n,k}(\rho)\|_{\rho,\mu} + \delta^k \|D^k(P_{n,k}(\rho))\|_{p,\mu}$$
$$\leq \frac{C}{n^k}\sum_{i=1}^{n} i^{k-1}\|\rho - P_{n,k}(\rho)\|_{p,\mu} \leq \frac{C}{n^k}\sum_{i=1}^{n} i^{k-1}\mathcal{E}_i(\rho,\delta)_{p,\mu}.$$

□

Theorem 4.3. *(Identical theorem) Let $\rho \in L_{(p,\mu)}(X)$, $1 \leq p < \infty, \vartheta(t) = (t(1+t))^{\frac{1}{2}}$,and $\delta, k \in \mathbb{N}$. Then*

$$\mathcal{E}(\rho,\delta)_{p,\mu} = O(\frac{1}{\sqrt{\delta}}), \quad \delta \to \infty \cong \Omega_k^{\theta}(\rho,\delta)_{p,\mu} = O(\delta), \quad \delta \to 0.$$

Proof. From direct theorem and inverse theorem conclude the identical theorem. □

5. Conclusion

In this work, we established some theorems to approximate the unbounded functions in weighted space by using some linear operators, and we investigated can obtain of the identical theorem from the direct and inverse theorems in terms dizian-Totik modulus of smoothness. Also for future studies, we recommend applying these tools to other mathematical concepts [21, 22, 23, 24, 25, 26, 27].

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