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UNIQUENESS OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS AND CERTAIN DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we explore the uniqueness property between the transcendental meromorphic functions and differential polynomial. With the notion of weighted sharing, we generalised the many previous results on uniqueness property. Here we discussed the uniqueness of $[P(f)(\alpha f^m + \beta)^s]^{(k)} - \eta(z)$ and $[P(g)(\alpha g^m + \beta)^s]^{(k)} - \eta(z)$. Meanwhile, we generalised the result of Harina P. waghamore and Rajeshwari S[1].

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1. Introduction

Throughout this article, we refer a meromorphic function as one that exists in the complex plane. We take for granted that readers are familiar with the common notations used in [2], [3], [4] Nevanlinna's value distribution theory of meromorphic functions, as described in. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$, possibly outside of a set of finite linear measure.

For any constant a we define,

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

where $\overline{N}\left(r, \dfrac{1}{f-a}\right)$ is the reduced counting function which counts zeros of $f(z) - a$ in $|z| \le r$, counted only once, and $a \in \mathbb{C} \cup \{\infty\}$.

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Let f and g be a two non-constant meromorphic function. If $f-a$ and $g-a$, assume the same zeros with the same multiplicities, then we call that f and g share the value a CM (Counting Multiplicities), we call that f and g share the value a IM (Ignoring Multiplicity), if we do not consider the multiplicities.

In 1967, Hayman and Clunie proved the following result.

Theorem 1.1. [5] *Let $f(z)$ be a transcendental entire function, $n \geq 1$ a positive integer. Then $f^n f^{'} = 1$ has infinitely many solutions.*

In 1998, W Yuefei and F Mingliang proved the following result.

Theorem 1.2. [6] *Let $f(z)$ be a transcendental entire function, n, k be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.*

In 2002, C. Y. Fang and M. L. Fang proved the following result.

Theorem 1.3. [7] *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let $n(\geq 8)$ be a positive integer. If $[f^n(z)(f(z)-1)]f^{'}(z)$ and $[g^n(z)(g(z)-1)]g^{'}(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

The following example shows that Theorem 1.3 is not valid when f and g are two meromorphic functions.

Example 1.1. $f = \dfrac{(n+2)(h-h^{n+2})}{(n+1)(1-h^{n+2})}$, $g = \dfrac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, where $h = e^z$. Then $[f^n(z)(f(z-1))]f^{'}(z)$ and $[g^n(z)(g(z-1))]g^{'}(z)$ share 1 CM, but $f(z) \not\equiv g(z)$.

In 2004, Lin and Yi generalized the above results and obtained the following results.

Theorem 1.4. [8] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with $\Theta(\infty, f) > \dfrac{2}{n+1}$, and let $n(\geq 12)$ be a positive integer. If $[f^n(z)(f(z)-1)]f^{'}(z)$ and $[g^n(z)(g(z)-1)]g^{'}(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2007, Bhoosnurmath and Dyavanal proved the following result

Theorem 1.5. [9] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \dfrac{3}{n+1}$, and let n, k be two positive integer with $n > 3k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2008, Liu proved the following result.

Theorem 1.6. [10] *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integer sunch that $n > 5k+4m+9$. If $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1 IM, then either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$.*

In 2015, Abhijith Banerjee proved the following Theorems.

Theorem 1.7. [11] *Let f and g be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f)+\Theta(\infty, g) > \frac{1}{4}$. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq 3k+9$ or if $l = 1$ and $n \geq 4k+10$ or if $l = 0$ and $n \geq 9k+18$, then $f \equiv g$.*

Theorem 1.8. [11] *Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af+b)]^{(k)} - P$ and $[g^n(ag+b)]^{(k)} - P$ share $(0, l)$ where $P(\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq 2k+6$ or if $l = 1$ and $n \geq \frac{5k}{2} + 7$ or if $l = 0$ and $n \geq 5k+12$, then $f \equiv g$.*

In 2017, Harina P. Waghamore and Rajeshwari S. Proved the following result.

Theorem 1.9. [1] *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{1}{4}$. Suppose for two nonzero constants a and b, $[f^n(af^m+b)]^{(k)} - P$ and $[g^n(ag^m+b)]^{(k)} - P$ share $(0, l)$ where $P(\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq \frac{3k+8}{s} + m$ or if $l = 1$ and $n \geq \frac{4k+9}{s} + \frac{3m}{2}$ or if $l = 0$ and $n \geq \frac{9k+14}{s} + 4m$, then $f \equiv g$.*

Theorem 1.10. [1] *Let f and g be two transcendental entire functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose for two nonzero constants a and b, $[f^n(af^m+b)]^{(k)} - P$ and $[g^n(ag^m+b)]^{(k)} - P$ share $(0, 1)$ where $P(\equiv 0)$ is a polynomial. If $l \geq 2$ and $n \geq \frac{2k+5}{s} + m$ or if $l = 1$ and $n \geq \frac{5k+10}{2s} + 4m$ or if $l = 0$ and $n \geq \frac{5k+8}{s} + 4m$, then $f \equiv g$.*

The purpose of the paper is to bring all the above results under a single umbrella. To this end, we consider a more generalized differential polynomial generated by a meromorphic function and significantly improve all the above results.
Throughout the paper let us denote by $P(z)$ the following n degree polynomial:

$$P(z) = \sum_{j=1}^{n} a_j z^j = a_n \prod_{j=1}^{s} (z - d_{l_j})^{l_j}, \tag{1}$$

where $a_1, ..., a_n (\neq 0) \in \mathbb{C}$ and $d_{l_j} (j = 1, 2, ..., s)$ are distinct and $l_1, l_2, ..., l_s, n \in \mathbb{N}$ such that $\sum\limits_{j=1}^{s} l_j = n$. Clearly, $P(0) = 0$.

We denote by n_1 and n_2 respectively be the number of simple and multiple zeros of $P(z)$,
where the zeros of $P(z)$ contributing to n_2 have been counted ignoring multiplicities. Throughout the paper we will use $\eta(z) = az + b$, where $|a| + |b| \neq 0$.

Motivation: The Theorem 1.9 and 1.10 motivate us to think that, whether there exists a similar result, if $[f^n(af^m + b)]^{(k)} - P$ and $[g^n(ag^m + b)]^{(k)} - P$ in Theorem 1.9 and 1.10 is replaced by $[P(f)(\alpha f^m + \beta)^s]^{(k)} - \eta(z)$ and $[P(g)(\alpha g^m + \beta)^s]^{(k)} - \eta(z)$. In this paper, we prove significant result which generalizes Theorem 1.9 and 1.10.

2. Main Results

Theorem 2.1. *Let f and g be two transcendental meromorphic functions, s be a non-negative integers, and let n, m, k be three positive integers and α, β be two constants with $|\alpha| + |\beta| \neq 0$. Suppose that $[P(f)(\alpha f^m + \beta)^s]^{(k)} - \eta(z)$ and $[P(g)(\alpha g^m + \beta)^s]^{(k)} - \eta(z)$ share $(0, l)$. If*
$l \geq 2$ and

$$n > (k + 4) + 2n_2(k + 2) + 2n_1 + ms, \tag{2}$$

or, $l = 1$ and

$$n > \frac{3k}{2} + \frac{9}{2} + \left(\frac{5k}{2} + \frac{9}{2}\right)n_2 + \frac{5n_1}{2} + \frac{3ms}{2}, \tag{3}$$

or, $l = 0$ and

$$n > 4k + 7 + (5k + 7)n_2 + 5n_1 + 4ms, \tag{4}$$

then one of the following two cases holds.
$(i)[P(f)(\alpha f^m + \beta)^s]^{(k)}[P(g)(\alpha g^m + \beta)^s]^{(k)} = \eta^2(z)$,
$(ii)P(f)(\alpha f^m + \beta)^s = P(g)(\alpha g^m + \beta)^s$ or $f \equiv tg$, for a constant t satisfying $t^{\chi_n} = 1$,
where

$$\chi_n = \begin{cases} 1, & \sum_{j=1}^{n-1} |a_{n-j}| \neq 0 \ ; \\ d_1, & a_j = 0, \forall j = 1, 2, \cdots, n-1 \ , \end{cases}$$

$d_1 = GCD(ms + n, ...m(s - i) + n, .., n), i = 0, 1, ..., s.$

Putting $s = 0$ and $P(z) = z^n$ in Theorem 2.1, we obtain the following corollary

Corollary 2.1. *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers. Suppose that $(f^n)^{(k)} - \eta(z)$ and $(g^n)^{(k)} - \eta(z)$ share $(0, l)$. If*
$l \geq 2$ and

$$n > 3k + 8, \tag{5}$$

or, $l = 1$ and

$$n > 4k + 9, \tag{6}$$

or, $l = 0$ *and*

$$n > 9k + 14, \tag{7}$$

then $f = tg$*, for a constant* t *satisfying* $t^n = 1$.

Putting $s = 1$ and $P(z) = z^n$ in Theorem 2.1, we obtain the following corollary

Corollary 2.2. *Let* f *and* g *be two transcendental meromorphic functions, and let* n*,* m*,* k *be a three positive integers and* α*,* β *be a two constant sunch that* $|\alpha| + |\beta| \neq 0$*. Suppose taht* $[f^n(\alpha f^m + \beta)]^k - \eta(z)$ *and* $[g^n(\alpha g^m + \beta)]^k - \eta(z)$ *share* $(0, l)$*. If*
$l \geq 2$ *and*

$$n > 3k + m + 8, \tag{8}$$

or $l = 1$ *and*

$$n > 4k + \frac{3m}{2} + 9, \tag{9}$$

or $l = 0$ *and*

$$n > 9k + 4m + 14, \tag{10}$$

then one of the following two cases holds.
(i) *when* $\alpha\beta = 0$*, then* $f = tg$*, for a constant* t *satisfying* $t^{n+m} = 1$*,*
(ii) *when* $\alpha\beta \neq 0$ *and* $k \geq 2$*, then* $f \equiv tg$*,* t *is a constant satisfying* $t^d = 1$*.*

Putting $m = 1$, $\alpha = 1$, $\beta = -1$ and $P(z) = z^n$ in Theorem 2.1, we obtain the following corollary

Corollary 2.3. *Let* f *and* g *be two transcendental meromorphic functions,* s *be a non-negative integer and* n*,* $(k \geq 2)$ *be two positive integer. Suppose that* $[f^n(f-1)^s]^{(k)} - \eta(z)$ *and* $[g^n(g-1)^s]^{(k)} - \eta(z)$ *share* $(0, l)$*. If*
$l \geq 2$ *and*

$$n > 3k + s + 8, \tag{11}$$

or $l = 1$ *and*

$$n > 4k + \frac{3s}{2} + 9, \tag{12}$$

or $l = 0$ *and*

$$n > 9k + 4s + 14, \tag{13}$$

then either $f \equiv g$ *or* $f^n(f-1)^s \equiv g^n(g-1)^s$*.*

Theorem 2.2. *Let* f *and* g *be two transcendental entire functions,* s *be a non-negative integer, and let* n*,* m*,* k *be three positive integers and* α, β *be two constants with* $|\alpha| + |\beta| \neq 0$*. Suppose that* $[P(f)(\alpha f^m + \beta)^s]^{(k)} - \eta(z)$ *and* $[P(g)(\alpha g^m + \beta)^s]^{(k)} - \eta(z)$ *share* $(0, l)$*. If*
$l \geq 2$ *and*

$$n > 2k + 4 + ms, \tag{14}$$

or, $l = 1$ *and*

$$n > \frac{5k}{2} + \frac{9}{2} + \frac{3ms}{2}, \tag{15}$$

or, $l = 0$ *and*

$$n > 5k + 7 + 4ms, \tag{16}$$

then one of the following two cases holds.
$(i)[P(f)(\alpha f^m + \beta)^s]^{(k)}[P(g)(\alpha g^m + \beta)^s]^{(k)} = \eta^2(z)$,
$(ii)P(f)(\alpha f^m + \beta)^s = P(g)(\alpha f^m + \beta)^s$ *0r* $f \equiv tg$, *for a constant* t *satisfying* $t^{\chi_n} = 1$,
where

$$\chi_n = \begin{cases} 1, & \sum_{j=1}^{n-1} |a_{n-j}| \neq 0 \ ; \\ d_1, & a_j = 0, \forall j = 1, 2, \cdots, n-1 \ , \end{cases}$$

$d_1 = GCD(ms + n, ...m(s - i) + n, .., n), i = 0, 1, ..., s.$

Putting $s = 0$ and $P(z) = z^n$ in Theorem 2.2, we obtain the following corollary

Corollary 2.4. *Let* f *and* g *be two transcendental entire functions, and let* n, k *be two positive integers. Suppose that* $(f^n)^{(k)} - \eta(z)$ *and* $(g^n)^{(k)} - \eta(z)$ *share* $(0, l)$. *If*
$l \geq 2$ *and*

$$n > 2k + 4, \tag{17}$$

or, $l = 1$ *and*

$$n > \frac{5k}{2} + \frac{9}{2}, \tag{18}$$

or, $l = 0$ *and*

$$n > 5k + 7, \tag{19}$$

then $f = tg$, *for a constant* t *satisfying* $t^n = 1$.

Putting $s = 1$ and $P(z) = z^n$ in Theorem 2.2, we obtain the following corollary

Corollary 2.5. *Let* f *and* g *be two transcendental entire functions, and let* n, m, k *be a three positive integers and* α, β *be a two constant sunch that* $|\alpha| + |\beta| \neq 0$. *Suppose taht* $[f^n(\alpha f^m + \beta)]^k - \eta(z)$ *and* $[g^n(\alpha g^m + \beta)]^k - \eta(z)$ *share* $(0, l)$. *If* $l \geq 2$ *and*

$$n > 2k + m + 4, \tag{20}$$

or $l = 1$ *and*

$$n > \frac{5k}{2} + \frac{3m}{2} + \frac{9}{2}, \tag{21}$$

or $l = 0$ *and*

$$n > 5k + 4m + 7, \tag{22}$$

then one of the following two cases holds.
(i) *when* $\alpha\beta = 0$, *then* $f = tg$, *for a constant* t *satisfying* $t^{n+m} = 1$,
(ii) *when* $\alpha\beta \neq 0$ *and* $k \geq 2$, *then* $f \equiv tg$, t *is a constant satisfying* $t^d = 1$.

Putting $m = 1$, $\alpha = 1$, $\beta = -1$ and $P(z) = z^n$ in Theorem 2.2, we obtain the following corollary

Corollary 2.6. *Let f and g be two transcendental entire functions, s be a non-negative integer and n, k $(k \geq 2)$ be two positive integers. Suppose that $[f^n(f-1)^s]^{(k)} - \eta(z)$ and $[g^n(g-1)^s]^{(k)} - \eta(z)$ share $(0, l)$. If*
$l \geq 2$ and

$$n > 2k + s + 4, \tag{23}$$

or $l = 1$ and

$$n > \frac{5k}{2} + \frac{3s}{2} + \frac{9}{2}, \tag{24}$$

or $l = 0$ and

$$n > 5k + 4s + 7, \tag{25}$$

then either $f \equiv g$ or $f^n(f-1)^s \equiv g^n(g-1)^s$.

3. Auxiliary Definitions

Definition 3.1. [12] A meromorphic function $b(z)$ $(\not\equiv 0, \infty)$ defined in $\mathbb{C}$ is called a "small function" with respect to $f(z)$ if $T(r, b(z)) = S(r, f)$.

Definition 3.2. [12] Let k be a positive integer, for any constant a in the complex plane $\mathbb{C}$.
We denote
(i) by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\geq k$.
(ii)by $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of $f(z)$ with multiplicity $\leq k$.

Definition 3.3. Let a be an any value in the extended complex plane and let k be an arbitrary non-negative integer. we define

$$\Theta(a, f) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

$$\delta_k(a, f) = 1 - \lim_{r\to\infty} \sup \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Remark 3.1. By Definition 3.3 we have

$$0 \leq \delta_k(a, f) \leq \delta_{k-1}(a, f) \leq \delta_1(a, f) \leq \theta(a, f) \leq 1.$$

4. Preliminary Lemmas

We now prove several lemmas which will play key roles in proving the main results of the paper. Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1}\right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1}\right)$$

Lemma 4.1. [13] *Suppose that f is a non-constant meromorphic function and let $a_0, a_1, ..., a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 4.2. [14] *Let F, G be two non-constant meromorphic functions such that they share $(1, 0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Lemma 4.3. [15] *Let F, G be two non-constant meromorphic functions sharing $(1, l)$, where $0 \leq l \leq \infty$. Then*

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_E^{1)}(r,1;F)+\left(l-\frac{1}{2}\right)\overline{N}_*(r,1;F,G) \leq \frac{1}{2}[N(r,1;F)+N(r,1;G)].$$

Lemma 4.4. [9] *Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 4.5. [16] *Let $f(z)$ be a non-constant meromorphic function and s, k be any two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{s+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Clearly $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right)$.

Lemma 4.6. *Let F, G be two non-constant meromorphic functions such that they share $(1, l)$. Then*

$$\begin{aligned}\overline{N}_*(r,1;F,G) \leq & \frac{l}{l+1}\{\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\} \\ & + S(r,F) + S(r,G).\end{aligned}$$

Proof. The proof can be carried out in the line of the proof of Lemma 2.6 in [17]. □

Lemma 4.7. [2] *Let f be a non-constant meromorphic function, k be a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned}T(r,f) \leq & \overline{N}(r,\infty;f) + N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ \leq & \overline{N}(r,\infty;f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f),\end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ *is a counting function of those zeros of* $f^{(k+1)}$ *in* $|z|<r$ *which are not zeros of* $f(f^{(k)}-c)$ *in* $|z|<r$.

Lemma 4.8. [14] *If* $H\equiv 0$, *then* F, G *share* $(1,\infty)$. *If further* F, G *share* $(\infty,0)$ *then* F, G *share* (∞,∞).

Lemma 4.9. [18] *Let* f *and* g *be two transcendental meromorphic functions and* $H\not\equiv 0$. *Let for two integers* $k(\geq 1)$ *and* $l(\geq 0)$, *if* $f^{(k)}-Q$, $g^{(k)}-Q$ *share* $(0,l)$, *where* $Q\not\equiv 0$ *is a polynomial. Then*

$$\frac{1}{2}[T(r,f)+T(r,g)]\leq\left(\frac{k}{2}+2\right)\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]+N_{k+2}(r,0;f)$$
$$+N_{k+2}(r,0;g)-\left(l-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g),$$

where $F=\dfrac{f^{(k)}}{Q}$ *and* $G=\dfrac{g^{(k)}}{Q}$.

Lemma 4.10. [18] *Let* f *and* g *be two transcendental meromorphic functions and* F, G *be defined as in Lemma 2.10* [18] . *Then either* $f^{(k)}g^{(k)}\equiv Q^2$ *or* $f\equiv g$, *whenever* f *and* g *satisfies one of the following conditions.*
(i) $l\geq 2$ *and*

$$\left(\frac{k}{2}+2\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)>k+5, \tag{26}$$

(ii) $l=1$ *and*

$$\begin{aligned}&\left(\frac{3k}{4}+\frac{9}{4}\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\&+\frac{1}{4}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\}>\frac{3k}{2}+6,\end{aligned} \tag{27}$$

(iii) $l=0$ *and*

$$\begin{aligned}&\left(2k+\frac{7}{2}\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\&+\frac{3}{2}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\}>4k+11.\end{aligned} \tag{28}$$

Lemma 4.11. *Let* f *and* g *be two transcendental entire functions and* F, G *be defined as in Lemma 2.10* [18]. *Then either* $f^{(k)}g^{(k)}\equiv Q^2$ *or* $f\equiv g$, *whenever* f *and* g *satisfies one of the following conditions.*
(i) $l\geq 2$ *and*

$$\delta_{k+2}(0,f)+\delta_{k+2}(0,g)>1, \tag{29}$$

(ii) $l=1$ *and*

$$\delta_{k+2}(0,f)+\delta_{k+2}(0,g)+\frac{1}{4}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\}>\frac{3}{2}, \tag{30}$$

(iii) $l=0$ *and*

$$\delta_{k+2}(0,f)+\delta_{k+2}(0,g)+\frac{3}{2}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\}>4. \tag{31}$$

Proof. Let f and g be an entire function, we have $\Theta(\infty,f)=1$ and $\Theta(\infty,g)=1$. Using the same argument as above Lemma 4.10, we can easily obtain the following Lemma. □

5. Proof of The Main Results

Theorem 2.1.

Proof. We set functions F_1 and G_1 as follows.

$$F_1=\frac{F^{(k)}}{\eta(z)}, G_1=\frac{G^{(k)}}{\eta(z)},$$

where $F=P(f)(\alpha f^m+\beta)^s$ and $G=P(g)(\alpha g^m+\beta)^s$.
We have from Lemma 4.10

$$\Delta_1=\left(\frac{k}{2}+2\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}+\delta_{k+2}(0,f)+\delta_{k+2}(0,g),$$

$$\begin{aligned}\Delta_2=&\left(\frac{3k}{4}+\frac{9}{4}\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\&+\frac{1}{4}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\},\end{aligned}$$

and

$$\begin{aligned}\Delta_3=&\left(2k+\frac{7}{2}\right)\{\Theta(\infty,f)+\Theta(\infty,g)\}\delta_{k+2}(0,f)+\delta_{k+2}(0,g)\\&+\frac{3}{2}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\}.\end{aligned}$$

Using Lemma 4.1, we have

$$\begin{aligned}\Theta(\infty,F)=&1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\infty;F)}{T(r,F)}\\\geq&1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\infty;P(f)(\alpha f^m+\beta)^s)}{(n+ms)T(r,F)+0(1)}\\\geq&1-\frac{1}{n+ms}.\end{aligned}\tag{32}$$

Similarly,

$$\Theta(\infty,G)\geq 1-\frac{1}{n+ms}. \tag{33}$$

Since

$$\begin{aligned}\delta_{k+2}(0,F) =&1-\overline{\lim_{r\to\infty}}\frac{N_{k+2}(r,0;F)}{T(r,F)}\\ \geq&1-\overline{\lim_{r\to\infty}}\frac{N_{k+2}(r,0;P(f))+N_{k+2}(r,0;(\alpha f^m+\beta)^s)}{(n+ms)T(r,f)+0(1)}\\ \geq&1-\frac{n_2(k+2)+n_1+ms}{n+ms}.\end{aligned} \tag{34}$$

Similarly,

$$\delta_{k+2}(0,G)\geq 1-\frac{n_2(k+2)+n_1+ms}{n+ms}. \tag{35}$$

Since

$$\begin{aligned}\delta_{k+1}(0,F) =&1-\overline{\lim_{r\to\infty}}\frac{N_{k+1}(r,0;F)}{T(r,F)}\\ \geq&1-\overline{\lim_{r\to\infty}}\frac{N_{k+1}(r,0;P(f))+N_{k+1}(r,0;(\alpha f^m+\beta)^s)}{(n+ms)T(r,f)+0(1)}\\ \geq&1-\frac{n_2(k+1)+n_1+ms}{n+ms}.\end{aligned} \tag{36}$$

Similarly,

$$\delta_{k+1}(0,G)\geq 1-\frac{n_2(k+1)+n_1+ms}{n+ms}. \tag{37}$$

Case 1. Let $l\geq 2$.
From the inequalities (32)-(35), we get

$$\Delta_1=k+6-\frac{k+4+2n_2(k+2)+2n_1+2ms}{n+ms}.$$

Since $n>k+4+2n_2(k+2)+2n_1+ms$, we get $\Delta_1>k+5$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,2)$, then by Lemma 4.10 we deduce that either $F^{(k)}G^{(k)}\equiv\eta^2(z)$ or $F\equiv G$.
Let $F\equiv G$, i.e.,

$$P(f)(\alpha f^m+\beta)^s=P(g)(\alpha g^m+\beta)^s. \tag{38}$$

Now we set

$$h=\frac{f}{g}. \tag{39}$$

If h is a non-constant meromorphic function, then we get (38).
Suppose h is a constant. Then from (39), we get

$$\begin{aligned}&[a_nf^n+a_{n-1}f^{n-1}+...+a_1z]\left[(\alpha f^m)^s+\binom{s}{1}(\alpha f^m)^{s-1}\beta+...+\binom{s}{s}\beta^s\right]\\ =&[a_ng^n+a_{n-1}g^{n-1}+...+a_1z]\left[(\alpha g^m)^s+\binom{s}{1}(\alpha g^m)^{s-1}\beta+...+\binom{s}{s}\beta^s\right],\end{aligned}$$

i.e.,

$$\sum_{i=0}^{s}\binom{s}{i}\beta^i[a_n g^{m(s-i)+n}(h^{m(s-i)+n}-1)+a_{n-1}g^{m(s-i)+n-1}(h^{m(s-i)+n-1}-1)$$
$$+...+a_1 g^{m(s-i)+1}(h^{m(s-i)+1}-1)]=0,$$

which implies $h^{\chi_n}=1$,
where

$$\chi_n=\begin{cases}1, & \sum_{j=1}^{n-1}|a_{n-j}|\neq 0\ ;\\ d_1, & a_j=0,\forall j=1,2,\cdots,n-1\ ,\end{cases}$$

$d_1=GCD(ms+n,...,m(s-i)+n,...,n)$, $i=0,1,...,s$. Therefore, $f=tg$, for a constant t satisfying $t^{X_n}=1$.

Case 2. Let $l=1$.
From the inequalities (32)-(37), we get

$$\Delta_2=\frac{3k}{2}+7-\frac{\frac{3k}{2}+\frac{9}{2}+\left(\frac{5k}{2}+\frac{9}{2}\right)n_2+\frac{5n_1}{2}+\frac{5ms}{2}}{n+ms}.$$

Since $n>\frac{3k}{2}+\frac{9}{2}+\left(\frac{5k}{2}+\frac{9}{2}\right)n_2+\frac{5n_1}{2}+\frac{3ms}{2}$, we get $\Delta_2>\frac{3k}{2}+6$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,1)$, then by Lemma 4.10 we deduce that either $F^{(k)}G^{(k)}\equiv\eta^2(z)$ or $F\equiv G$. Proceeding in the same manner as done in Case 1, we get the conclusion.

Case 3. Let $l=0$.
From the inequalities (32)-(37), we get

$$\Delta_3=4k+12-\frac{4k+7+(5k+7)n_2+5n_1+5ms}{n+ms}.$$

Since $n>4k+7+(5k+7)n_2+5n_1+4ms$, we get $\Delta_3>4k+11$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,0)$, then by Lemma 4.10 we deduce that either $F^{(k)}G^{(k)}\equiv\eta^2(z)$ or $F\equiv G$. Proceeding in the same manner as done in Case 1, we get the conclusion. □

Theorem 2.2.

Proof. We set functions F_1 and G_1 as follows

$$F_1=\frac{F^{(k)}}{\eta(z)},G_1=\frac{G^{(k)}}{\eta(z)},$$

where $F=P(f)(\alpha f^m+\beta)^s$ and $G=P(g)(\alpha g^m+\beta)^s$.
We have from Lemma 4.11

$$\Delta_1=\delta_{k+2}(0,f)+\delta_{k+2}(0,g),$$

$$\Delta_2=\delta_{k+2}(0,f)+\delta_{k+2}(0,g)+\frac{1}{4}\{\delta_{k+1}(0,f)+\delta_{k+1}(0,g)\},$$

$$\Delta_3 = \delta_{k+2}(0,f) + \delta_{k+2}(0,g) + \frac{3}{2}\{\delta_{k+1}(0,f) + \delta_{k+1}(0,g)\}.$$

Using Lemma 4.1, we have

$$\begin{aligned}\delta_{k+2}(0,F) =& 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+2}(r,0;F)}{T(r,F)} \\ \geq& 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+2}(r,0;P(f)) + N_{k+2}(r,0;(\alpha f^m+\beta)^s)}{(n+ms)T(r,f)+0(1)} \\ \geq& 1 - \frac{k+2+ms}{n+ms}.\end{aligned} \tag{40}$$

Similarly,

$$\delta_{k+2}(0,G) \geq 1 - \frac{k+2+ms}{n+ms}. \tag{41}$$

Since

$$\begin{aligned}\delta_{k+1}(0,F) =& 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r,0;F)}{T(r,F)} \\ \geq& 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r,0;P(f)) + N_{k+1}(r,0;(\alpha f^m+\beta)^s)}{(n+ms)T(r,f)+0(1)} \\ \geq& 1 - \frac{k+1+ms}{n+ms}.\end{aligned} \tag{42}$$

Similarly,

$$\delta_{k+1}(0,G) \geq 1 - \frac{k+1+ms}{n+ms}. \tag{43}$$

Case 1. Let $l \geq 2$.
From the inequalities (40)-(41), we get

$$\Delta_1 = 2 - \frac{2k+4+2ms}{n+ms}.$$

Since $n > 2k+4+ms$, we get $\Delta_1 > 2$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,2)$, then by Lemma 4.11 we deduce that either $F^{(k)}G^{(k)} \equiv \eta^2(z)$ or $F \equiv G$.
Let $F \equiv G$, i.e.,

$$P(f)(\alpha f^m+\beta)^s = P(g)(\alpha g^m+\beta)^s. \tag{44}$$

Now we set

$$h = \frac{f}{g}. \tag{45}$$

If h is a non-constant meromorphic function, then we get (44).
Suppose h is a constant. Then from (45), we get

$$\begin{aligned}&[a_n f^n + a_{n-1}f^{n-1} + ... + a_1 z]\left[(\alpha f^m)^s + \binom{s}{1}(\alpha f^m)^{s-1}\beta + ... + \binom{s}{s}\beta^s\right] \\ =&[a_n g^n + a_{n-1}g^{n-1} + ... + a_1 z]\left[(\alpha g^m)^s + \binom{s}{1}(\alpha g^m)^{s-1}\beta + ... + \binom{s}{s}\beta^s\right],\end{aligned}$$

i.e.,

$$\sum_{i=0}^{s}\binom{s}{i}\beta^i[a_n g^{m(s-i)+n}(h^{m(s-i)+n}-1)+a_{n-1}g^{m(s-i)+n-1}(h^{m(s-i)+n-1}-1)$$
$$+\ldots+a_1 g^{m(s-i)+1}(h^{m(s-i)+1}-1)]=0,$$

which implies $h^{\chi_n}=1$,
where

$$\chi_n=\begin{cases}1, & \sum_{j=1}^{n-1}|a_{n-j}|\neq 0\ ;\\ d_1, & a_j=0, \forall j=1,2,\cdots,n-1\ ,\end{cases}$$

$d_1=GCD(ms+n,...,m(s-i)+n,...,n)$, $i=0,1,...,s$. Therefore, $f=tg$, for a constant t satisfying $t^{\chi_n}=1$.

Case 2. Let $l=1$.
From the inequalities (40)-(43), we get

$$\Delta_2=\frac{5}{2}-\frac{\frac{5k}{2}+\frac{9}{2}+\frac{5ms}{2}}{n+ms}.$$

Since $n>\frac{5k}{2}+\frac{9}{2}+\frac{3ms}{2}$, we get $\Delta_2>\frac{3}{2}$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,1)$, then by Lemma 4.11 we deduce that either $F^{(k)}G^{(k)}\equiv\eta^2(z)$ or $F\equiv G$. Proceeding in the same manner as done in Case 1, we get the conclusion.

Case 3. Let $l=0$.
From the inequalities (40)-(43), we get

$$\Delta_3=5-\frac{5k+7+5ms}{n+ms}.$$

Since $n>5k+7+4ms$, we get $\Delta_3>4$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(0,0)$, then by Lemma 4.11 we deduce that either $F^{(k)}G^{(k)}\equiv\eta^2(z)$ or $F\equiv G$. Proceeding in the same manner as done in Case 1, we get the conclusion. □

6. Conclusion

When the two transcendental meromorphic functions of the type $[P(f)(\alpha f^m+\beta)^s]^{(k)}-\eta(z)$ and $[P(g)(\alpha g^m+\beta)^s]^{(k)}-\eta(z)$ share the value zero with the weight l. Then there exists a uniqueness between the functions with respect to the sharing values and it's conditions.

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